Zero Coprime System Equivalence of Singular 2-D Linear Models

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***Abstract*— The connection between the polynomial matrix descriptions (PMD's) of the various singular 2-D linear models is considered. It is shown that the transformation of zero coprime system equivalence provides the basis for the reduction of any given singular 2-D general PMD to a singular Roesser form. The exact form of the transformation is established.**

I. INTRODUCTION

In recent years, the field of multidimensional linear systems theory has attracted many researchers [2], [3]. The reason is the wide application in areas such as image processing, linear multipass processes, iterative learning control systems, lumped and distributed networks etc. Local state space models for 2-D systems have been proposed by Roesser [17], Fornasini and Marchesini [7] and Attasi [1]. However these models are suitable only for representing causal transfer functions. To overcome the problem of causality, singular versions of these models have been proposed by Kaczorek [10], [11] and have been shown in [12] to be equivalent in the sense that they can all be embedded in the 2-D singular general model. In this paper, the PMD's corresponding to the various singular 2-D models are considered and the exact form of the transformation linking them is established.

II. SINGULAR 2-D LINEAR MODELS

The singular 2-D general model (GM) is given by [11] :

$$\begin{aligned} Ex(i+1,j+1) &= A_2x(i+1,j)+A_1x(i,j+1)+ \\ & A_0x(i,j)+B_2u(i+1,j)+ \\ & B_1u(i,j+1)+B_0u(i,j) \end{aligned} \tag{1}$$

where $x(i,j)\in\mathbb{R}^n$ is the local state vector at the point (i,j), $u(i,j)\in\mathbb{R}^m$ is the input vector, $y(i,j)\in\mathbb{R}^l$ is the output vector, $E,A_0,A_1,A_2,B_0,B_1,B_2,C,D$ are real matrices of appropriate dimensions and E is singular.

Taking the 2-D z-transform of (1) and assuming zero boundary conditions, i.e. $x(i,0)=0$ and $x(0,j)=0$ for all $i,j=0,1,\ldots,$ gives

$$P_{GM}(s,z)\begin{bmatrix} \bar{x}(s,z) \\ -\bar{u}(s,z) \end{bmatrix} = \begin{bmatrix} 0 \\ -\bar{y}(s,z) \end{bmatrix} \tag{2}$$

where $P_{GM}(s,z)$ is the polynomial system matrix associated with the system (1) and is given by:

$$P_{GM}(s,z)=\begin{bmatrix} szE-sA_2-zA_1-A_0 & sB_2+zB_1+B_0 \\ -C & D \end{bmatrix} \tag{3}$$

The system matrix in (3) describes a number of singular 2-D systems as special cases. In particular setting $B_1=B_2=0$ in (3), gives rise to the system matrix

$$P_{FM1}(s,z)=\begin{bmatrix} szE-sA_2-zA_1-A_0 & B_0 \\ -C & D \end{bmatrix} \tag{4}$$

which is associated with the first singular Fornasini-Marchesini model (FM1). Alternatively when $B_0=0$ in (3), the resulting system matrix is that of the second singular Fornasini-Marchesini model (FM2),

$$P_{FM2}(s,z)=\begin{bmatrix} szE-sA_2-zA_1-A_0 & sB_2+zB_1 \\ -C & D \end{bmatrix} \tag{5}$$

The system matrix associated with the singular Attasi model (AM) is obtained from (4) by setting $A_0=-A_1A_2=-A_2A_1$, i.e.

$$P_{AM}(s,z)=\begin{bmatrix} szE-sA_2-zA_1+A_1A_2 & B_0 \\ -C & D \end{bmatrix} \tag{6}$$

A different type of singular model, where the state is divided into a horizontal and vertical states is the singular Roesser model (RM):

$$\begin{aligned} E\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} \\ &+\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}u(i,j) \\ y(i,j) &= \begin{bmatrix} C_1 & C_2 \end{bmatrix}\begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} \\ &+Du(i,j) \end{aligned} \tag{7}$$

where $x^h(i,j)\in\mathbb{R}^{n_1}$ is the horizontal state vector, $x^v(i,j)\in\mathbb{R}^{n_2}$ is the vertical state vector, $u(i,j)\in\mathbb{R}^m$ is the input vector, $y(i,j)\in\mathbb{R}^l$ is the output vector, $E,A_{11},A_{12},A_{21},A_{22},B_1,B_2,C_1,C_2,D$ are real matrices of appropriate dimensions and E is singular.

Taking the 2-D z-transform of (7) and assuming zero boundary conditions, i.e. $x^h(0,j)=0$ and $x^v(i,0)=0$ for all $i,j=0,1,\dots,$ yields the corresponding system matrix

$$P_{RM}=\left[\begin{array}{cc} sE_2+zE_1-A & B \\ -C & D \end{array}\right] \tag{8}$$

where

$$E=\left[\begin{array}{cc} E_{11} & E_{12} \\ E_{21} & E_{22} \end{array}\right], E_1=\left[\begin{array}{cc} 0 & E_{12} \\ 0 & E_{22} \end{array}\right], E_2=\left[\begin{array}{cc} E_{11} & 0 \\ E_{21} & 0 \end{array}\right],$$

$$A=\left[\begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array}\right], B=\left[\begin{array}{c} B_1 \\ B_2 \end{array}\right], C=\left[\begin{array}{cc} C_1 & C_2 \end{array}\right]. \tag{9}$$

III. ZERO COPRIME SYSTEM EQUIVALENCE

Consider the 2-D polynomial system matrix in the general form:

$$P(s,z)=\left[\begin{array}{cc} T(s,z) & U(s,z) \\ -V(s,z) & W(s,z) \end{array}\right] \tag{10}$$

where $T(s,z), U(s,z), V(s,z)$ and $W(s,z)$ are respectively $r\times r$, $r\times n$, $m\times r$ and $m\times n$ polynomial matrices with $T(s,z)$ invertible.

The transfer function of (10) is given by:

$$G(s,z)=V(s,z)T^{-1}(s,z)U(s,z)+W(s,z) \tag{11}$$

Definition 1: Two polynomial matrices $P_1(s,z)$ and $S_1(s,z)$ of appropriate dimensions, are said to be zero left coprime if the matrix $\left[\begin{array}{cc} P_1(s,z) & S_1(s,z) \end{array}\right]$ has full rank for all $(s,z)\in\mathbb{C}^2$.

Similarly, $P_2(s,z)$ and $S_2(s,z)$, of appropriate dimensions, are said to be zero right coprime if the matrix $\left[\begin{array}{cc} P_2^T(s,z) & S_2^T(s,z) \end{array}\right]^T$ has full rank for all $(s,z)\in\mathbb{C}^2$.

Definition 2: Let $\mathbb{P}(m,n)$ denote the class of polynomial system matrices having the same number of inputs n and outputs m and which can be brought by a suitable trivial expansion to the same size. Then Two polynomial system matrices $P_1(s,z)$ and $P_2(s,z)\in\mathbb{P}(m,n)$, are said to be zero coprime system equivalent if they are related by the following:

$$S_1(s,z)P_1(s,z)=P_2(s,z)S_2(s,z) \tag{12}$$

where

$$S_1(s,z)=\left[\begin{array}{cc} M(s,z) & 0 \\ X(s,z) & I_m \end{array}\right], \tag{13}$$

$$S_2(s,z)=\left[\begin{array}{cc} N(s,z) & Y(s,z) \\ 0 & I_n \end{array}\right] \tag{14}$$

$P_1(s,z), S_1(s,z)$ are zero left coprime, $P_2(s,z), S_2(s,z)$ are zero right coprime and $M(s,z)$, $N(s,z)$, $X(s,z)$ and $Y(s,z)$ are polynomial matrices of appropriate dimensions.

The transformation of zero coprime system equivalence is an extension of Fuhrmann's [8] strict system equivalence. This transformation can be regarded as the classical unimodular transformation introduced by Rosenbrock [18] coupled with trivial expansion or deflation of the system matrices. The transformation of zero coprime system equivalence has been shown by [13] to provide the connection between all least order polynomial realizations of a given 2-D transfer function matrix. Boudellioua and Chentouf [5], [4] showed that it forms the basis of a reduction procedure of a given 2-D polynomial system matrix to a singular 2-D general form. In [6], a zero coprime system equivalence transformation was established between regular Fornasini-Marchesini PMD's and corresponding regular Roesser PMD's. while Pugh *et al.* [14] used the transformation to reduce a given 2-D polynomial system matrix to a singular Roesser form. In all these works, the nature of the equivalence is similar and has been shown in [14] to be generated by a trivial expansion and elementary row and column operations.

Furthermore zero coprime system equivalence has been shown to preserve important system matrix properties as illustrated by the following results.

Lemma 1: (Pugh *et al.*, [16]) Suppose that two polynomial matrices $P(s,z)$ and $Q(s,z)\in\mathbb{P}(m,n)$, are related by zero coprime system equivalence and let $\Phi_1^{[P]},\Phi_2^{[P]},\dots,\Phi_h^{[P]}$, where $h=\min(r^{[P]}+m, r^{[P]}+n)$, denote the invariant polynomials of $P(s,z)$ and $\Phi_1^{[Q]},\Phi_2^{[Q]},\dots,\Phi_k^{[Q]}$, where $k=\min(r^{[Q]}+m, r^{[Q]}+n)$, denote the invariant polynomials of $Q(s,z)$, then

$$\Phi_{h-i}^{[P]}=c_i\Phi_{k-i}^{[Q]} \tag{15}$$

for $i=0,1,\dots,\max(k-1,h-1)$, where

$$\Phi_j^{[P]}=1, \Phi_j^{[Q]}=1 \tag{16}$$

for any $j<1$, $c_i\in\mathbb{R}\backslash\{0\}$.

Lemma 2: (Pugh *et al.* [15]) Suppose that two polynomial matrices $P(s,z)$ and $Q(s,z)\in\mathbb{P}(m,n)$, are related by zero coprime equivalence and let $\mathcal{I}_j^{[P]}$ for $j=1,\dots,h=\min(r^{[P]}+m, r^{[P]}+n)$ denote the ideal generated by the $j\times j$ minors of $P(s,z)$ and $\mathcal{I}_i^{[Q]}$, for $i=1,\dots,k=\min(r^{[Q]}+m, r^{[Q]}+n)$ denote the ideal generated by the $i\times i$ minors of $Q(s,z)$. Then

$$\mathcal{I}_{h-i}^{[P]}=\mathcal{I}_{k-i}^{[Q]}, i=0,\dots,\bar{h} \tag{17}$$

where

$$\bar{h}=\min(h-1,k-1)$$

and for any $i>h$,

$$\mathcal{I}_{h-i}^{[P]}=\langle 1\rangle \text{ or } \mathcal{I}_{k-i}^{[Q]}=\langle 1\rangle \text{ in case } i<h \text{ or } i<k.$$

The ith order invariant zeros of $P(s,z)$ are the elements of the variety $\mathcal{V}_{\mathbb{R}}(\mathcal{I}^{[P]})$ defined by the ideal $\mathcal{I}_i^{[P]}$ generated by the ith order minors of $P(s,z)$.

Lemma 3: (Johnson, [9], Pugh *et al.* [15]) The transformation of *zero coprime system equivalence* preserves the transfer function of $P_i(s,z)$, the invariant polynomials in the sense of Lemma 1, and the invariant zeros in the sense of Lemma 2 of the matrices:

$$T_i(s,z), P_i(s,z), \left[\begin{array}{cc} T_i(s,z) & U_i(s,z) \end{array}\right], \left[\begin{array}{c} T_i(s,z) \\ -V_i(s,z) \end{array}\right]. \tag{18}$$

In the following we establish the exact form of the connection between the polynomial system matrix in the singular general form (3) and the corresponding singular Roesser form in (8). We will show that the transformation involved is that of zero coprime system equivalence. The connections between all other singular models and the singular Roesser model are deduced as special cases.

IV. Connection between the models

Theorem 1: Given an arbitrary $(r+m)\times(r+n)$ singular 2-D general system matrix $P_{GM}(s,z)$ in the form (3), then $P_{GM}(s,z)$ is zero coprime system equivalent to a singular 2-D Roesser system matrix $\bar{P}_{RM}(s,z)$ in the form (8).

Proof: Consider the following system transformation:

$$S_1(s,z)P_{GM}(s,z)=\bar{P}_{RM}(s,z)S_2(s,z) \tag{19}$$

where,

$$S_1(s,z)=\left[\begin{array}{c|c} 0 & 0\\ I_r & 0\\ 0 & 0\\ 0 & 0\\ \hline 0 & I_m\end{array}\right],\ S_2(s,z)=\left[\begin{array}{c|c} sI_r & 0\\ I_r & 0\\ 0 & I_n\\ C & -D\\ \hline 0 & I_n\end{array}\right], \tag{20}$$

and the resulting singular Roesser system matrix $\bar{P}_{RM}(s,z)$ is given by

$$\left[\begin{array}{cccc|c} I_r & -sI_r & 0 & 0 & 0\\ zE-A_1 & -zA_2-A_0 & sB_2+zB_1+B_0 & 0 & 0\\ 0 & -C & D & I_m & 0\\ 0 & 0 & -I_n & 0 & I_n\\ \hline 0 & 0 & 0 & -I_m & 0\end{array}\right]. \tag{21}$$

The matrices $\bar{E}_1,\bar{E}_2,\bar{A},\bar{B},\bar{C}$ and $\bar{D}$ corresponding to (8) are:

$$\bar{E}_1=\begin{bmatrix} 0 & 0 & 0 & 0\\ E & -A_2 & B_1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}, \tag{22}$$

$$\bar{E}_2=\begin{bmatrix} 0 & -I_r & 0 & 0\\ 0 & 0 & B_2 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}, \tag{23}$$

$$\bar{A}=\begin{bmatrix} -I_r & 0 & 0 & 0\\ A_1 & A_0 & -B_0 & 0\\ 0 & C & -D & -I_m\\ 0 & 0 & I_n & 0\end{bmatrix}, \tag{24}$$

$$\bar{B}^T=\begin{bmatrix} 0 & 0 & 0 & I_n\end{bmatrix}, \tag{25}$$

$$\bar{C}=\begin{bmatrix} 0 & 0 & 0 & I_m\end{bmatrix}, \tag{26}$$

and

$$\bar{D}=0. \tag{27}$$

Clearly the transformation in (19) is in the required form (12), so it remains to prove the equality and the zero coprimeness of the matrices. In fact, it can be easily verified that the LHS and RHS of (19) both yield the matrix

$$\begin{bmatrix} 0 & 0\\ szE-sA_2-zA_1-A_0 & sB_2+zB_1+B_0\\ 0 & 0\\ 0 & 0\\ C & -D\end{bmatrix} \tag{28}$$

The zero left coprimeness of $P_{RM}(s,z)$ and $S_1(s,z)$ follows from the fact that the matrix

$$\begin{bmatrix} P_{RM}(s,z) & S_1(s,z)\end{bmatrix} \tag{29}$$

has a highest order minor

$$\begin{vmatrix} I_r & 0 & 0 & 0 & 0\\ zE-A_1 & 0 & 0 & I_r & 0\\ 0 & I_m & 0 & 0 & 0\\ 0 & 0 & I_n & 0 & 0\\ 0 & -I_m & 0 & 0 & I_m\end{vmatrix}=1 \tag{30}$$

obtained by deleting the second and third block columns. Similarly the zero right coprimeness of $P_{FM}(s,z)$ and $S_2(s,z)$ follows from the fact that the matrix

$$\begin{bmatrix} P_{GM}(s,z)\\ S_2(s,z)\end{bmatrix} \tag{31}$$

has a highest order minor

$$\begin{vmatrix} I_r & 0\\ 0 & I_n\end{vmatrix}=1 \tag{32}$$

obtained by deleting all the block rows except the 4th and 5th. ∎

V. Conclusions

This paper established the connection between the PMD's associated with the various singular 2-D linear models. The transformation of zero coprime system equivalence turned out to provide the link between such models. The results highlight the importance of this transformation in 2-D polynomial systems theory.

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