An equivalent reduction of a 2-D symmetric polynomial matrix

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***Abstract*— A new family of companion forms for polynomials and polynomial matrices has recently been developed in [4] and [1] respectively. The application of these new companion forms to polynomial matrices with symmetries has been examined in [2]. In this work we extend the results presented in [2] to the case of 2-D polynomial matrices and thus provide a new linearization of a 2-D polynomial matrix that preserves both the symmetric structure and the structural invariants, of the original 2-D polynomial matrix.**

I. INTRODUCTION

In [4] a new family of companion forms associated to a polynomial matrix has been presented. [1] extends these results to regular polynomial matrices by providing a new family of companion forms. The members of this new family of companion matrices are strictly equivalent to the known *first* and *second companion forms* of the original polynomial matrix presented initially in [6] and thus preserves both its finite and infinite elementary divisor structure. The application of this new family of companion forms to the special case of polynomial matrices with symmetries was given in [2]. A completely different systematic approach of generating a class of matrix pencils that share the same finite and infinite elementary divisor structure of a given polynomial matrix and additional special properties like symmetry was given in [12].

The reduction of an arbitrary two-variable polynomial matrix to pencil form was first studied by [14]. The reduction algorithm presented in [14], is a two stage algorithm that does not give a priori the form of either the resulting 2-D matrix pencil or the transformation linking it to the original polynomial matrix. [15] and [3] later provide an exact 2-D matrix pencil reduction in singular Roesser form and Fornasini-Marchesini form respectively and provide the transformation linking this pencil to the original 2-D polynomial matrix. The matrix pencil provided by [3] is analog to the first companion form presented in [6] for 1-D polynomial matrices, while the respective Roesser model presented in [15] comes from a modification of the first companion form presented in [6]. In the special case of 2-D symmetric polynomial matrices, none of these models gives rise to 2-D symmetric pencils.

The aim of this work is to determine a two-stage algorithm, easily applicable in a computer symbolic environment like MATHEMATICA, for the reduction of a 2-D symmetric polynomial matrix to a zero-coprime equivalent 2-D symmetric matrix pencil. This new pencil keeps invariant the zero structure of the original polynomial matrix due to the properties of the zero coprime equivalence transformation. The reduction procedure is then adapted to create a system reduction procedure for a 2-D polynomial system matrix. Illustrative examples for the symmetric reduction of 2-D quadratic matrix pencils are given.

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II. PRELIMINARY RESULTS

Let $F[s,z]$ (resp.$F[s]$) denote the ring of 2-D (resp. 1-D) polynomials in the indeterminates s,z (resp. s) over the field F ($\mathbb{R}$ or $\mathbb{C}$). Denote also by $F^{r\times m}[s,z]$ (resp.$F^{r\times m}[s]$) the set of $r\times m$ matrices with elements in $F[s,z]$ (resp.$F[s]$). An equivalence relation of particular importance for 2-D systems is the zero coprime equivalence defined below. Let $\mathcal{P}(m,l)$, be the set of $(r+m)\times(r+l)$ 2-D polynomial matrices where $r>\max(-m,-l)$.

Definition 1: [15] If there exist polynomial matrices $L(s,z), R(s,z)$ such that $L(s,z)P_1(s,z) = P_2(s,z)R(s,z)$ where the compound matrices $\begin{pmatrix} L(s,z) & P_2(s,z)\end{pmatrix}$ and $\begin{pmatrix} P_1(s,z)^T & R(s,z)^T\end{pmatrix}^T$ have full rank $\forall(s,z)\in\mathbb{C}^2$ then $P_1(s,z), P_2(s,z)\in\mathcal{P}(m,l)$ are said to be zero-coprime equivalent (ZC-E).

Zero-coprime equivalence is an extension of Fuhrmann's strict system equivalence [5] into the 2-D setting and is an equivalence relation in the sense of being reflexive, transitive and symmetric [11], [7]. The system counterpart transformation is given below.

Definition 2: [15] Let two 2-D systems described by the Rosenbrock system matrices of the form

$$P_i(s,z) = \begin{pmatrix} T_i(s,z) & U_i(s,z) \\ -V_i(s,z) & W_i(s,z)\end{pmatrix}, i=1,2$$

If there exist polynomial matrices $L(s,z), R(s,z), X(s,z), Y(s,z)$ such that

$$\begin{pmatrix} L & 0 \\ X & I\end{pmatrix} P_1 = P_2 \begin{pmatrix} R & Y \\ 0 & I\end{pmatrix}$$

where the compound matrices $\begin{pmatrix} L(s,z) & T_2(s,z)\end{pmatrix}$ and $\begin{pmatrix} T_1(s,z)^T & R(s,z)^T\end{pmatrix}^T$ have full rank $\forall(s,z)\in\mathbb{C}^2$ then $P_1(s,z), P_2(s,z)\in\mathcal{P}(m,l)$ are said to be zero-coprime system equivalent (ZC-SE).

ZC-E (ZC-SE) has been shown to preserve important system matrix properties as illustrated by the following results.

Lemma 3: [13] Suppose that two polynomial matrices $P_1(s,z)$ and $P_2(s,z) \in \mathcal{P}(m,l)$, are related by zero coprime equivalence and let $\Phi_1^{[P_1]}, \Phi_2^{[P_1]}, ..., \Phi_h^{[P_1]}$, where $h = \min\left(r^{[P_1]} + m, r^{[P_1]} + l\right)$ and $r^{[P]} = rank_{\mathbb{R}}[P]$, denote the *invariant polynomials* of $P_1(s,z)$ and $\Phi_1^{[P_2]}, \Phi_2^{[P_2]}, ..., \Phi_k^{[P_2]}$, where $k = \min\left(r^{[P_2]} + m, r^{[P_2]} + l\right)$, denote the invariant polynomials of $P_2(s,z)$, then

$$\Phi_{h-i}^{[P_1]} = c_i \Phi_{k-i}^{[P_2]}$$

for $i = 0, 1, ..., \max(k-1, h-1)$, where

$$\Phi_j^{[P_1]} = 1, \Phi_j^{[P_2]} = 1$$

for any $j < 1, c_i \in \mathbb{R} \backslash \{0\}$.

Lemma 4: [16] Suppose that two polynomial matrices $P_1(s,z)$ and $P_2(s,z) \in \mathcal{P}(m,l)$, are related by zero coprime equivalence and let $I_j^{[P_1]}$ for $j = 1, ..., h = \min\left(r^{[P_1]} + m, r^{[P_1]} + l\right)$ denote the ideal generated by the $j \times j$ minors of $P_1(s,z)$ and $I_i^{[P_1]}$ for $i = 1, ..., k = \min\left(r^{[P_2]} + m, r^{[P_2]} + l\right)$ denote the ideal generated by the $i \times i$ minors of $P_2(s,z)$. Then

$$I_{h-i}^{[P_1]} = I_{k-i}^{[P_2]}, i = 0, 1, ..., \bar{h}$$

where $\bar{h} = \min(k-1, h-1)$ and

$$I_{h-i}^{[P_1]} = \langle 1 \rangle \text{ or } I_{k-i}^{[P_2]} = \langle 1 \rangle \text{ in case } i < h \text{ or } i < k$$

The ith order *invariant zeros* of $P(s,z)$ are the elements of the variety $V_{\mathbb{R}}\left(I^{[P]}\right)$ defined by the ideal $I_i^{[P]}$ generated by the ith order minors of $P(s,z)$.

Lemma 5: [16] The transformation ZC-SE preserves the transfer function of $P_i(s,z), i = 1, 2$, the invariant polynomials in the sense of Lemma 3, and the invariant zeros in the sense of Lemma 4 of the matrices

$$T_i(s,z), P_i(s,z), \left[\begin{array}{cc} T_i(s,z) & U_i(s,z) \end{array}\right], \left[\begin{array}{c} T_i(s,z) \\ -V_i(s,z) \end{array}\right]$$

Thus, in order to prove that two polynomial matrices (or 2-D systems) exhibit the same invariant polynomials or the same zero-structure, it is enough to prove that there exists a ZC-E (ZC-SE) transformation that connects these polynomial matrices (or polynomial system matrices). Note, that the zero-structure of 2-D systems is strongly connected with the structural properties of 2-D systems such as behavior, controllability, observability [20], [21] e.t.c.. We shall try in the next section to reduce any 2-D polynomial (system) matrix to a ZC-E 2-D matrix pencil (2-D singular model) by using the above transformations in order to preserve the zero-structure and the invariant polynomials of the original polynomial (system) matrix. An extra restriction will be to preserve the symmetry of the original polynomial matrix. This will be helpful if we are interested in the sequel to apply known numerical algorithms for symmetric 2-D matrix pencils.

III. 2-D Polynomial Matrix Reduction Procedure

In this section we shall present a two-stage algorithm for the reduction of a 2-D polynomial matrix to a 2-D matrix pencil that will preserve the zero structure and all the symmetries of the original polynomial matrix. Consider the two variable polynomial matrix:

$$T(s,z) = \underbrace{\left(T_{p,q} z^q + \cdots + T_{p,0}\right)}_{T_p(z)} s^p + \qquad (1)$$
$$+ \cdots + \underbrace{\left(T_{0,q} z^q + T_{0,q-1} z^{q-1} + \cdots + T_{0,0}\right)}_{T_0(z)} \in \mathbb{R}[s,z]^{r \times r}$$

Without loss of generality we assume that p and q are always odd numbers (otherwise we add extra zero leading terms). Let

$$\begin{aligned}
A_p(z) &= diag\{T_p(z), I_{rp}\} \\
A_k(z) &= \begin{pmatrix} I_{r(p-k-1)} & 0 & 0 \\ 0 & C_k(z) & 0 \\ 0 & 0 & I_{r(k-1)} \end{pmatrix} \\
C_k &= \begin{pmatrix} -T_k(z) & I_r \\ I_r & 0 \end{pmatrix}, k = 1, 2, ..., p-1 \\
A_0(z) &= diag\left\{I_{r(p-1)}, -T_0(z)\right\}
\end{aligned}$$

Then

$$P_s(s,z) = \begin{pmatrix} sT_p + T_{p-1} & T_{p-2} & \cdots & T_0 \\ -I_r & sI_r & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -I_r & sI_r \end{pmatrix}$$

where $P_s(s,z) \in \mathbb{R}[s,z]^{pr \times pr}$, will be the first companion form of the regular two-variable polynomial matrix $T(s,z)$ in terms of s.

Theorem 6: The polynomial matrix $T(s,z) \in \mathbb{R}[s,z]^{r \times r}$ defined in (1) and the matrix pencil $P_s(s,z) \in \mathbb{R}[s,z]^{pr \times pr}$ are ZC-E.

Proof: Consider the identity

$$\underbrace{\begin{pmatrix} I_r \\ 0_{(q-1)r,r} \end{pmatrix}}_{M(s,z)} T(s,z) =$$

$$= \underbrace{\begin{pmatrix} sT_p + T_{p-1} & T_{p-2} & \cdots & T_0 \\ -I_r & sI_r & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -I_r & sI_r \end{pmatrix}}_{P_s(s,z)} \underbrace{\begin{pmatrix} s^{q-1} I_r \\ s^{q-2} I_r \\ \vdots \\ I_r \end{pmatrix}}_{N(s,z)}$$

The first compound matrix $\left(\begin{array}{cc} M(s,z) & P_s(s,z) \end{array}\right)$ i.e.

$$\begin{pmatrix} I_r & sT_p + T_{p-1} & T_{p-2} & \cdots & T_0 \\ 0 & -I_r & sI_r & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -I_r & sI_r \end{pmatrix}$$

contains the submatrix

$$L(s) := \begin{pmatrix} I_r & sT_p + T_{p-1} & T_{p-2} & \cdots \\ 0 & -I_r & sI_r & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & \cdots & -I_r \end{pmatrix}$$

which has constant determinant and thus $M(s,z)$ and $P_s(s,z)$ are zero coprime. The second compound matrix

$$\begin{pmatrix} T(s,z) \\ -N(s,z) \end{pmatrix} = \begin{pmatrix} T(s,z) \\ -s^{q-1}I_r \\ -s^{q-2}I_r \\ \vdots \\ -I_r \end{pmatrix}$$

contains an $r \times r$ constant minor i.e. $\det[-I_r]$, and thus $T(s,z)$ and $N(s,z)$ are zero coprime. Therefore according to Definition 1, $P_s(s,z)$ and $T(s,z)$ are ZC-E. ■

According to [1] the first companion form of $T(s,z)$ is given by

$$P_s(s,z) = sA_p(z) - A_{p-1}(z)A_{p-2}(z)\cdots A_0(z)$$

whereas a companion form of $T(s,z)$ of particular importance is

$$R_s(s,z) = sA_{odd}(z) - A_{even}(z)$$

where

$$A_{even}(z) = A_0(z)A_2(z)\cdots A_{p-1}(z)$$
$$A_{odd}(z) = A_p(z)A_{p-2}^{-1}\cdots A_3^{-1}(z)A_1^{-1}(z)$$

Note that if we do not get the assumption that p is odd, and use according to [1] the following companion form for p even

$$R_s(s,z) = sA_{odd}(z) - A_{even}(z)$$

where

$$A_{odd}(z) = A_{p-1}^{-1}(z)\cdots A_3^{-1}(z)A_1^{-1}(z)$$
$$A_{even}(z) = A_0(z)A_2(z)\cdots A_{p-2}(z)A_p^{-1}(z)$$

we need the restriction that $\det(A_p(z)) \neq 0$ or equivalently $\det(T_p(z)) \neq 0$ (the matrices $A_i(z), i = 1,..,p-1$ are unimodular, and thus are always invertible). However, even in the case where this restriction is satisfied we cannot use the suggested companion form since the matrix $A_p(z)^{-1}$ or $T_p(z)^{-1}$ involved in the matrix pencil might be rational in terms of z. Therefore, we keep the companion form suggested in [1] for p odd, and in the case where p is even we add a zero leading term in the two variable polynomial matrix $T(s,z)$ i.e. $T_{p+1}(z) = 0$ and use the companion form for $p+1$ odd. The matrix pencil $R_s(s,z)$ in terms of s has the appealing property of keeping the same zero-structure of the original polynomial matrix as given by the following Theorem.

Theorem 7: Let $T(s,z)$ be a self-adjoint two-variable polynomial matrix described in (1). Then the companion form $R_s(s,z)$ of $T(s,z)$ is a self-adjoint polynomial matrix pencil which is zero coprime equivalent to $P_s(s,z)$ and therefore to $T(s,z)$.

Proof: Note that the general form of $R_s(s,z) \in \mathbb{R}^{l\times l}[s,z]$ will be the following

$$R_s(s,z) :=$$
$$= \begin{pmatrix} \ddots & \vdots & \vdots & \vdots \\ \cdots & T_2(z)+sT_3(z) & -I_r & 0 \\ \cdots & -I_r & 0 & sI_r \\ \cdots & 0 & sI_r & T_0(z)+sT_1(z) \end{pmatrix}$$

where $l = rp$. A transformation that connects $P_s(s,z)$ with $R_s(s,z)$ is the following

$$M(s,z)P_s(s,z) = R_s(s,z)N(s,z)$$

where

$$M_{p,p+1-i} := \begin{cases} T_i(z) + sT_{i+1}(z) + \cdots + s^{i-1}T_{2i-1}(z), \\ \quad i = 1,2,...,p-1 \\ I_r, i = p \end{cases}$$
$$M_{p-(2i+1),p-i}(s,z) = I_r$$
$$M_{k,l} = 0_{r,r}, \forall (k,l) \neq (p-(2i+1), p-i) \,\&k \neq p$$

and

$$N_{p-(2i+1),p-i}(s,z) = I_r, i = 0,1,..,(p-1)/2$$
$$N_{p-k-(2i+1),p-k-i} := \begin{cases} I_r \\ k = 0, i = 0,1,..,(p-1)/2 \\ s^k\left(T_{2(k+i)+2} + sT_{2(k+i)+3}\right) \\ k = 1,..,p-2(i+1) \\ i = 0,1,..,(p-3)/2 \end{cases}$$
$$N_{k,l} = 0_{r,r}, \forall (k,l) \neq (p-k-(2i+1), p-k-i),$$
$$k = 0,..,p-2(i+1), i = 0,1,..,(p-3)/2$$

and $T_i(z) = 0, i > p$. Note that the $l \times l$ minor of the compound matrix $\begin{pmatrix} M(s,z) & R_s(s,z) \end{pmatrix}$ consisting of the 1rst block-column of $M(s,z)$ and the 1rst-(last-1)th block-columns of $R_s(s,z)$ has determinant constant (± 1), and therefore the compound matrix $\begin{pmatrix} M(s,z) & R_s(s,z) \end{pmatrix}$ has full row rank $\forall (s,z) \in \mathbb{C}^2$. Similarly, the $l \times l$ minor of the compound matrix $\begin{pmatrix} P_s(s,z)^T & N(s,z)^T \end{pmatrix}^T$ consisting of the 2nd until the last block row of $P_s(s,z)$ and the last block row of $N(s,z)$ has determinant constant (± 1), and therefore the compound matrix $\begin{pmatrix} P_s(s,z)^T & N(s,z)^T \end{pmatrix}^T$ has full row rank $\forall (s,z) \in \mathbb{C}^2$. Thus the above transformation is a zero-coprime equivalence transformation. ■

Example 8: Consider the two-variable polynomial matrix

$$T(s,z) = Ks^2 + Mz^2 \in \mathbb{R}^{r\times r}[s,z]$$

where K, M are symmetric real matrices. This polynomial matrix is associated with the 2-D discrete-time AR-representation

$$Ku(x+2h,y) + Mu(x,y+2k) = 0$$

Rewrite $T(s,z)$ as follows

$$T(s,z) = \underbrace{0_{r,r}}_{T_3(z)}s^3 + \underbrace{K}_{T_2(z)}s^2 + \underbrace{0}_{T_1(z)}s + \underbrace{Mz^2}_{T_0(z)}$$

The alternative first companion form (by taking $T_3(z)=0$) defined in [15] of the matrix $T(s,z)$ will be the following

$$P_s^2(s,z) = \begin{pmatrix} T_2(z) & T_1(z) & T_0(z) \\ -I_r & sI_r & 0 \\ 0 & -I_r & sI_r \end{pmatrix} = \begin{pmatrix} K & 0 & Mz^2 \\ -I_r & sI_r & 0 \\ 0 & -I_r & sI_r \end{pmatrix}$$

which is not in symmetric form. Since $p=2$ is even, we define the matrices

$$A_3(z) = diag\{T_3(z), I_{2r}\}, A_2(z) = \left(\begin{array}{cc|c} -K & I_r & 0 \\ I_r & 0 & 0 \\ \hline 0 & 0 & I_r \end{array}\right)$$

$$A_1(z) = \left(\begin{array}{c|cc} I_r & 0 & 0 \\ \hline 0 & 0 & I_r \\ 0 & I_r & 0 \end{array}\right), A_0(z) = diag\left\{I_{2r}, -Mz^2\right\}$$

Then the companion form will be the following

$$R_s(s,z) = s\left(A_3(z) A_1^{-1}(z)\right) - \left(A_0(z) A_2(z)\right) = \begin{pmatrix} K & -I_r & 0 \\ -I_r & 0 & sI_r \\ 0 & sI_r & Mz^2 \end{pmatrix}$$

and the zero-coprime equivalence that connects the matrix pencils $R_s(s,z)$ and $P_s^2(s,z)$ will be the following

$$\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & I_r \\ I_r & K & 0 \end{pmatrix}}_{M_1(s,z)} \underbrace{\begin{pmatrix} K & 0 & Mz^2 \\ -I_r & sI_r & 0 \\ 0 & -I_r & sI_r \end{pmatrix}}_{P_s^2(s,z)} = \underbrace{\begin{pmatrix} K & -I_r & 0 \\ -I_r & 0 & sI_r \\ 0 & sI_r & Mz^2 \end{pmatrix}}_{R_s(s,z)} \underbrace{\begin{pmatrix} 0 & I_r & 0 \\ 0 & K & 0 \\ 0 & 0 & I_r \end{pmatrix}}_{N_1(s,z)}$$

Let now

$$R_s(s,z) = \underbrace{(R_{q,1}s + R_{q,0})}_{L_q(s)} z^q + \underbrace{(R_{q-1,1}s + R_{q-1,0})}_{L_{q-1}(s)} z^{q-1} + \cdots + \underbrace{(R_{0,1}s + R_{0,0})}_{L_0(s)} \quad (2)$$

where we have already assumed that q is odd. Applying now, the same techniques with the ones defined above we define

$$B_q(s) = diag\{L_q(s), I_{rq}\}$$

$$B_k(s) = \begin{pmatrix} I_{rp(q-k-1)} & 0 & 0 \\ 0 & D_k(s) & 0 \\ 0 & 0 & I_{rp(k-1)} \end{pmatrix}$$

$$D_k(s) = \begin{pmatrix} -L_k(s) & I_{rp} \\ I_{rp} & 0 \end{pmatrix}, k = 1, 2, ..., q-1$$

$$B_0(s) = diag\left\{I_{rp(q-1)}, -L_0(s)\right\}$$

Then

$$P_z(s,z) = \begin{pmatrix} zL_q + L_{q-1} & L_{q-2} & \cdots & L_0 \\ -I_{rp} & sI_r & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -I_{rp} & sI_r \end{pmatrix} = zB_q(s) - B_{q-1}(s) B_{q-2}(s) \cdots B_0(s)$$

will be the first companion form of the regular two-variable polynomial matrix $R_s(s,z)$ in terms of s. A new companion form of $R_s(s,z)$ and thus of $T(s,z)$ is given by

$$R_z(s,z) = zB_{odd}(s) - B_{even}(s)$$

where

$$B_{even}(s) = B_0(s) B_2(s) \cdots B_{q-1}(s)$$
$$B_{odd}(s) = B_q(s) B_{q-2}^{-1}(s) \cdots B_1^{-1}(s)$$

By using similar lines with the the proof of Theorem 7 and using the transitivity property of ZC-E we can show the following:

Theorem 9: The companion form $R_z(s,z)$ of $T(s,z)$ is a self-adjoint matrix pencil which is zero coprime equivalent to $P_z(s,z)$ and therefore to $T(s,z)$.

Note that the general form of $R_z(s,z) \in \mathbb{R}^{l\times l}[s,z]$ will be the following

$$R_z(s,z) = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots \\ \cdots & L_2 + zL_3 & -I_l & 0 \\ \cdots & -I_l & 0 & zI_l \\ \cdots & 0 & zI_l & L_0 + zL_1 \end{pmatrix}$$

where $l = rpq$ (since p, q odd), and

$$L_0 + zL_1 = (R_{0,1}s + R_{0,0}) + z(R_{1,1}s + R_{1,0})$$

$$L_i + zL_{i+1} = (R_{i,1}s + R_{i,0}) + z(R_{i+1,1}s + R_{i+1,0})$$

Example 10: Consider the two-variable polynomial matrix $T(s,z)$ presented in Example 8 and the symmetric two-variable matrix pencil

$$R_s(s,z) = \underbrace{0_{3r,3r}}_{L_3(s)} z^3 + \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & M \end{pmatrix}}_{L_2(s)} z^2 + \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{L_1(s)} z + \underbrace{\begin{pmatrix} K & -I_r & 0 \\ -I_r & 0 & sI_r \\ 0 & sI_r & 0 \end{pmatrix}}_{L_0(s)}$$

Define the matrices

$$B_3(s) = diag\left\{L_3(s), I_{3\times r\times(3-1)}\right\}$$

$$B_2(s) = \left(\begin{array}{cc|c} -L_2(s) & I_{3r} & 0 \\ I_{3r} & 0 & 0 \\ \hline 0 & 0 & I_{3r} \end{array}\right)$$

$$B_1(s) = \left(\begin{array}{c|cc} I_{3r} & 0 & 0 \\ \hline 0 & -L_1(s) & I_{3r} \\ 0 & I_{3r} & 0 \end{array}\right)$$

$$B_0(s) = diag\left\{I_{3\times r\times(3-1)}, -L_0(s)\right\}$$

Then the companion form of $T(s,z)$ will be the following symmetric matrix pencil

$$R_z(s,z) = zB_3(s)B_1^{-1}(s) - B_0(s)B_2(s) = \\ = \begin{pmatrix} L_2(s) & -I_{3r} & 0_{3r,3r} \\ -I_{3r} & 0_{3r,3r} & zI_{3r} \\ 0_{3r,3r} & zI_{3r} & L_0(s) \end{pmatrix}$$

IV. 2-D Reduction of Polynomial System Matrices

The reduction procedure described above can be easily extended to the system matrix case following similar lines with [15]. Let the 2-D polynomial system matrix be the following:

$$P(s,z) = \begin{pmatrix} T(s,z) & U(s,z) \\ -V(s,z) & W(s,z) \end{pmatrix} \in \mathbb{R}[s,z]^{(r+l)\times(r+m)}$$

or its normalized form

$$\mathcal{P}(s,z) = \left(\begin{array}{lll|l} T(s,z) & U(s,z) & 0 & 0 \\ -V(s,z) & W(s,z) & I_l & 0 \\ 0 & -I_m & 0 & I_m \\ \hline 0 & 0 & -I_l & 0 \end{array}\right) \quad (3)$$

$$=: \begin{pmatrix} \mathcal{T}(s,z) & \mathcal{U} \\ -\mathcal{V} & 0 \end{pmatrix} \in \mathbb{R}[s,z]^{(r+2l+m)\times(r+2m+l)}$$

and let

$$R_z(s,z) = szE_1 + zA_1 + sA_2 + A_0$$

be the reduced matrix pencil of $\mathcal{T}(s,z)$ according to the algorithm presented in the previous section. Then we have the following:

Theorem 11: The 2-D polynomial system matrix $\mathcal{P}(s,z)$ is zero-coprime equivalent to a singular model of the form

$$\mathcal{P}_{SM}(s,z) = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{R_z(s,z)} & \begin{matrix} 0 \\ \mathcal{U} \end{matrix} \\ \hline 0 & -\mathcal{V} & 0_{l\times m} \end{array}\right)$$

Proof: Note that

$$R_z(s,z) := \begin{pmatrix} A_0(s,z) & B_0(z) \\ -C_0(z) & L_0(s) + zL_1(s) \end{pmatrix}$$

The polynomial matrix

$$A_0(s,z) = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots \\ \cdots & 0 & zI_l & 0 \\ \cdots & zI_l & L_2(s) + zL_3(s) & -I_l \\ \cdots & 0 & -I_l & 0 \end{pmatrix}$$

is unimodular, and therefore its inverse is a polynomial matrix. The following transformation

$$\underbrace{\left(\begin{array}{c|c} 0 & 0 \\ I_r & 0 \\ \hline 0 & I_l \end{array}\right)}_{M(s,z)} \mathcal{P}(s,z) =$$

$$= \mathcal{P}_{SM}(s,z) \underbrace{\left(\begin{array}{l|l} -A_0(s,z)^{-1}B_0(z) & 0 \\ I_l & 0 \\ \hline 0 & I_m \end{array}\right)}_{N(s,z)}$$

is a ZC-S equivalence transformation, since the compound matrix $\begin{pmatrix} M(s,z) & \mathcal{P}_{SM}(s,z) \end{pmatrix}$ includes the matrix

$$\begin{pmatrix} 0 & 0 & A_0(s,z) \\ I_{r+l+m} & 0 & -C_0(z) \\ 0 & I_l & 0 \end{pmatrix}$$

that has constant determinant (since $A_0(s,z)$ is a unimodular matrix), and the compound matrix $\begin{pmatrix} \mathcal{P}(s,z)^T & N(s,z)^T \end{pmatrix}^T$ includes the unit matrix, that has also constant determinant. ■

V. Conclusions

A two-stage algorithm, easily implementable in a computer symbolic environment, has been provided for the reduction of a 2-D symmetric polynomial matrix to a zero coprime equivalent 2-D symmetric matrix pencil. The results has also been adapted to 2-D system matrices. The main advantage of this reduction procedure is that we can use existing robust numerical algorithms for 2-D matrix pencils in order to compute structural invariants of 2-D symmetric polynomial matrices, while the main disadvantage is the size of the matrices that we use. An implementation of this algorithm in the package *MATHEMATICA* accompanied with one example is given in the appendix.

References

[1] E.N. Antoniou, S. Vologiannidis, 2004, A new family of companion forms of polynomial matrices, Electronic Journal of Linear Algebra, Vol. 11, pp 78-87.

[2] E.N. Antoniou, S. Vologiannidis, 2006, Linearizations of polynomial matrices with symmetries and their applications, Electronic Journal of Linear Algebra, Vol.15, pp.107-114.

[3] M.S. Boudellioua and B. Chentouf, 2003, A pencil equivalent of a general 2-D polynomial matrix, 11th Mediterranean Conference on Control and Automation, June 18-20, 2003, Rhodes, Greece.

[4] Miroslav Fiedler, 2003, A note on companion matrices. Linear Algebra Appl., 372:325–331.

[5] P.A. Fuhrmann, 1977, On strict system equivalence and similarity, International Journal of Control, 25, pp.5-10.

[6] I. Gohberg, P. Lancaster, and L. Rodman. Matrix polynomials, 1982, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York. Computer Science and Applied Mathematics.

[7] D.S. Johnson, 1993, Coprimeness in multidimensional system theory and symbolic computation, Ph.D. Thesis, Loughborough University of Technology, Loughborough, U.K..

[8] N. Karampetakis and S. Vologiannidis, 2003, Infinite elementary divisor structure-preserving transformations for polynomial matrices, International Journal of Applied Mathematics and Computer Science, Vol.13, No.4, pp.101-111.

[9] N. Karampetakis, S. Vologiannidis and A.I. Vardulakis, 2003, On a new equivalence for polynomial matrices, International Journal of Control, Vol.77, No.6, pp.584-597.

[10] N. P. Karampetakis, 2004, On the solution space of discrete time AR-representations over a finite time horizon, Linear Algebra and its Applications, 382, 83-116.

[11] B. C. Levy, 2-D polynomial and rational matrices and their applications for the modelling of 2-D dynamical systems, Ph.D. Thesis, Stanford University, U.S.A..

[12] D. Steven Mackey, Niloufer Mackey, Christian Mehl, Volker Mehrmann, 2006, Vector spaces of linearizations for matrix polynomials, to appear in SIAM J. Matrix Analysis Aplications.

[13] A.C. Pugh, S.J. McInerney, M. Hou and G.E. Hayton, 1996, A transformation for 2-D systems and its invariants, Proceedings of the 35th IEEE Conference on decision and Control, Kobe (Japan).

[14] A.C. Pugh, S. J. McInerney, M.S. Boudellioua and G.E. Hayton, 1998, Matrix pencil equivalents of a general 2-D polynomial matrix, Int. J. Control, Vol.71, pp.1027-1050.

[15] A.C. Pugh, S. J. McInerney and E.M.O. El-Nabrawy, Equivalence and reduction of 2-D systems, IEEE Transactions on Circuits and Systems - II:Express Briefs, Vol.52, No.5, May 2005.

[16] A. C. Pugh, S. J. McInerney and E. M. O. El-Nabrawy, 2005, Zero structures of n-D systems, International Journal of Control, Vol. 78, No. 4, 277–285.

[17] H. H. Rosenbrock, 1970, State Space and Multivariable Theory, New York:Nelson-Wiley.

[18] Liansheng Tan and A. C. Pugh, 2002, Spectral structures of the generalized companion form and applications. Systems Control Lett., 46(2):75–84.

[19] A. I. G. Vardulakis and E. Antoniou. Fundamental equivalence of discrete-time AR representations. Internat. J. Control, 76(11):1078–1088, 2003.

[20] P. Zaris, J. Wood and E. Rogers, 2001, Controllable and uncontrollable poles and zeros of n-D systems", Mathematics of Control, Signals and Systems, 14, pp. 281–298.

[21] E. Zerz, Primeness of multivariate polynomial matrices, 1996, Systems and Control Letters, 29, pp. 139–145.

VI. Appendix

For the sake of completeness we present the following code in the programming language of *Mathematica*, for the computation of the reduced matrix pencil in terms of the variable s, based on the reduction algorithm presented in the above work.

```
In[1]:=<<LinearAlgebra`
In[2]:= f[A1_, s_] := Module[{n, r, T, B, i, k, EE, AA, j, tmp},
n = Max[Exponent[A1, s]];
r = Length[A1];
If[Mod[n, 2] == 0,
T[i_] := Coefficient[A1, s, i]; T[n + 1] := ZeroMatrix[r]; n = n + 1,
T[i_] := Coefficient[A1, s, i]];
B[0] := BlockMatrix[Table[If[i == j && j ≠ n, IdentityMatrix[r], If[i ≠ j, ZeroMatrix[ r], -T[0]]], {i, 1, n}, {j, 1, n}]];
B[n] := BlockMatrix[Table[If[i == j == 1, T[n], If[ i == j, IdentityMatrix[r], ZeroMatrix[r]]], {i, 1, n}, {j, 1, n}]];
MkI[ll_ , mm_] := Module[{i, j, mat},
mat = mm;
For[i = ll[[1, 1]], i ≤ ll[[1, 2]],
For[j = ll[[2, 1]], j ≤ ll[[2, 2]],
mat[[i, j]] = If[Abs[j - i] == ll[[1, 2]] - ll[[1,
1]] + 1, 1, 0]; j++]; i++];
Return[mat]];
MkZ[ll_ , mm_] := Module[{i, j, mat},
mat = mm;
For[i = ll[[1, 1]], i ≤ ll[[1, 2]],
For[j = ll[[2, 1]], j ≤ ll[[2, 2]],
mat[[i, j]] = 0; j++]; i++];
Return[mat]];
B[k_] := Block[{tmp, i, j, n1 = n, r1 = r},
    tmp = BlockMatrix[Table[If[i == j == n1 - k, -T[k], If[i == j, IdentityMatrix[r1], ZeroMatrix[r1]]]], {i, 1, n1}, {j, 1, n1}]];
    If[r1 ¿ 1, tmp = MkZ[{{(n1 - k)r1 +1, (n1 - k + 1)r1}, {(n1 - k)r1 + 1, (n1 - k + 1)r1}}, tmp],
tmp[[{(n1 - k)r1 + 1, (n1 - k + 1)r1}, {(n1 - k)r1 +1, (n1 - k + 1)r1}]] = 0];
If[r1 ¿ 1, tmp = MkI[{{(n1 - k - 1)r1 + 1, (n1 - k)r1}, {(n1 - k)r1 + 1, (n1 - k + 1)r1}}, tmp],
tmp[[{(n1 - k - 1)r1 + 1, (n1 - k) r1}, {(n1 - k)r1 + 1, (n1 - k + 1)r1}]] = 1];
If[r1 ¿ 1, tmp = MkI[{{(n1 - k)r1 + 1, (n1 - k + 1)r1}, {(n1 - k - 1)r1 + 1, (n1 - k)r1}}, tmp], tmp[[{(n1 - k)r1 + 1, (n1 - k + 1)r1}, {(n1 - k - 1)r1 + 1, (n1 - k)r1}]] = 1];
    tmp];
EE = B[n];
For[i = n - 2, i ≥ 1, EE = EE.Inverse[B[i]]; i = i - 2];
AA = B[0];
For[i = 2, i ≤ n - 1, AA = AA.B[i]; i = i + 2];
Return[s EE - AA]]
```

Example 12: First we define the matrix $A1 = \left[Ks^2 + Mz^2\right]$

$$\begin{aligned}\mathbf{In}[3] &:= \mathtt{A1} = \{\{\mathtt{K\ s\hat{}2 + M\ z\hat{}2}\}\} \\ \mathbf{Out}[3] &:= \{\{K\ s\hat{}2 + M\ z\hat{}2\}\}\end{aligned}$$

Then we reduce the matrix $A1$ to a matrix pencil in terms of s

$$\begin{aligned}\mathbf{In}[4] &:= \mathtt{f[A1,s]} \\ \mathbf{Out}[4] &:= \{\{K,-1,0\},\{-1,0,s\},\{0,s,M\ z\hat{}2\}\}\end{aligned}$$

and finally we reduce the last result to a matrix pencil in terms of z

$\mathbf{In}[5]$:=$\mathtt{f[\%,z]}$

$\mathbf{Out}[5]$:=

$\{\{0,0,0,-1,0,0,0,0,0\},\{0,0,0,0,-1,0,0,0,0\},$
$\{0,0,M,0,0,-1,0,0,0\},\{-1,0,0,0,0,0,z,0,0\},$
$\{0,-1,0,0,0,0,0,z,0\},\{0,0,-1,0,0,0,0,0,z\},$
$\{0,0,0,z,0,0,K,-1,0\},\{0,0,0,0,z,0,-1,0,s\},$
$\{0,0,0,0,0,z,0,s,0\}\}$

Note that the above reduction function $f[]$ may also be used for polynomial matrices with more than two variables.

$$\begin{aligned}\mathbf{In}[7] &:= \mathtt{A1} = \{\{\mathtt{K\ s\hat{}2 + M\ z\hat{}2 + N\ w\hat{}2}\}\}; \\ \mathbf{In}[8] &:= \mathtt{f[A1,s]}; \\ \mathbf{In}[9] &:= \mathtt{f[\%,z]}; \\ \mathbf{In}[10] &:= \mathtt{f[\%,w]//MatrixForm}\end{aligned}$$