On the realization theory of polynomial matrices and the algebraic structure of pure generalized state space systems.

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***Abstract*— We review the realization theory of polynomial (transfer function) matrices via "pure" generalized state space system models. The concept of an *irreducible* at infinity generalized state space realization of a polynomial matrix is defined and the mechanism of the "cancellations" of "decoupling zeros at infinity" is closely examined. The difference between the concepts of *irreducibility* and *minimally* of generalized state space realizations of polynomial (transfer function) matrices is pointed out and the associated concepts of *dynamic* and *non dynamic* variables appearing in generalized state space realizations are also examined Finally the isomorphism between the zeros at infinity of the "infinite pole pencil" and the Rosenbrock system matrix of an irreducible at infinity generalized state space realization of a polynomial matrix and the *pole and zero structure at infinity* of such a polynomial matrix is examined**

I. Introduction

Motivated by the classical state space realization theory of proper rational transfer function matrices of linear time invariant multivariable systems in this paper we review the realization theory of polynomial (transfer function) matrices corresponding to linear, time invariant, *"pure" generalized state space systems*. The concepts of *decoupling zeros at infinity* and of a *reducible at infinity* generalized state space realization of a polynomial matrix as opposed to those of finite decoupling zeros and reducibility at finite frequencies of state space realizations of proper rational matrices are defined and the mechanism of the "cancellations" of decoupling zeros at infinity during the formation of the polynomial transfer function matrix from a *reducible at infinity* generalized state space realization of such a polynomial matrix is closely examined. The difference between the concepts of *irreducibility at infinity* and *minimality* of generalized state space realizations of polynomial (transfer function) matrices is pointed out and the associated concepts of *dynamic* and *non dynamic* variables appearing in such generalized state space realizations are also reviewed.

In what follows the time variable t can be considered either as continuous or discrete, taking values respectively in the sets of reals $\mathbb{R}$ or the integers $\mathbb{Z}$. Correspondingly the variable s can be considered as denoting either the Laplace variable in the Laplace transform $\mathcal{L}\{x(t)\} = X(s) := \int_{-\infty}^{\infty} x(t)\, e^{-st}dt$ of a continuous time function $x(t) : \mathbb{R} \to \mathbb{R}$ or the "z" transform variable in the $\mathcal{Z}$ transform $Z\{x(t)\} = X(z) = \sum_{t=-\infty}^{\infty} x(t)\, z^{-t}$ of a discrete time function $x(t) : \mathbb{Z} \to \mathbb{R}$. $\mathbb{C}$ denotes the field of complex numbers. By $\mathbb{R}(s)^{p\times m}$, $\mathbb{R}_{pr}(s)^{p\times m}$ and $\mathbb{R}[s]^{p\times m}$ we denote respectively the sets of $p \times m$ rational, proper rational and polynomial matrices with real coefficients and indeterminate $s \in \mathbb{C}$. A polynomial matrix

$$A(s) = A_q s^q + A_{q-1}s^{q-1} + \ldots + A_0 \in \mathbb{R}[s]^{p\times m}, \quad (1)$$

is called *regular* iff $p = m$ and $\det A(s) \neq 0$ for almost every $s \in \mathbb{C}$, while in every other case it is called *singular*. If $q = 1$ then $A(s) = A_1 s + A_0 \in \mathbb{R}[s]^{p\times m}$ is called a *matrix pencil* [6]. The (finite) zeros of $A(s)$ are defined as the roots of the equation $\det A(s) = 0$, equivalently $\lambda_i \in \mathbb{C}$ is a (finite) *zero* of $A(s)$ iff $rank_{\mathbb{C}} A(\lambda_i) < r$. $\delta_M[.]$ denotes the McMillan degree of a rational matrix i.e. the total number of its poles in $\mathbb{C}\cup\{\infty\}$. Every rational matrix $A(s) \in \mathbb{R}(s)^{p\times m}$ with $rank_{\mathbb{R}(s)} A(s) = r \leq \min(p, m)$ is *biproperly equivalent* [1] to its Smith-McMillan form at $s = \infty$

$$S^{\infty}_{A(s)} = diag\left[\overbrace{\underbrace{s^{q_1}, ..., s^{q_k}}_{k}, I_{v-k}}^{v}, \overbrace{\frac{1}{s^{\widehat{q}_{v+1}}}, ..., \frac{1}{s^{\widehat{q}_r}}}^{r-v}, 0_{p-r,m-r}\right] \quad (2)$$

where $r \geq v \geq k \geq 0$ and $q_1 \geq q_2 \geq ... \geq q_k > 0 = q_{k+1} = \cdots = q_v$, $\widehat{q}_r \geq \widehat{q}_{r-1} \geq \cdots \geq \widehat{q}_{v+1} > 0$ are respectively the orders of the *poles* and the *zeros* of $A(s)$ at $s = \infty$. Finally if $A(s)$ is as in 1 with Smith-McMillan form at $s = \infty$ as in 2 then it turns out [1] that

$$q = q_1 \quad (3)$$

II. Realization theory for polynomial matrices

We start with some known facts and basic results regarding the "realization theory" of *polynomial matrices.* What follows can be seen as an extension to the case of *polynomial matrices* of the results regarding the realization theory of *proper* rational matrices .

Definition 1: Let $A(s) \in \mathbb{R}[s]^{p\times m}, rank_{\mathbf{R}(s)}\ A(s) = r \leq \min(p, m)$. A quadruple of matrices $C_\infty \in \mathbb{R}^{p\times\mu}, A_\infty \in \mathbb{R}^{\mu\times\mu}, B_\infty \in \mathbb{R}^{\mu\times m}, D_\infty \in \mathbb{R}^{p\times m}$, $\mu \in \mathbb{Z}^+$ is called a *generalized state space* (GSS) *realization* of $A(s)$ iff the GSS system denoted by Σ_g and defined by:

$$A_\infty \rho x_\infty(t) = x_\infty(t) - B_\infty u(t) \quad (4)$$

$$y(t) = C_\infty x_\infty(t) + D_\infty u(t) \tag{5}$$

(where ρ is the differential operator $\frac{d}{dt}$in the continuous time case or the forward shift operator: $\rho^i x(t) := x(t+i), i \in \mathbb{Z}^+$ in the discrete time case), or after Laplace or $\mathcal{Z}$ transformation in Rosenbrock 'system matrix' form:

$$\begin{bmatrix} I_\mu - sA_\infty & B_\infty \\ -C_\infty & D_\infty \end{bmatrix} \begin{bmatrix} X_\infty(s) \\ -U(s) \end{bmatrix} = \begin{bmatrix} 0 \\ -Y(s) \end{bmatrix} \tag{6}$$

has transfer function matrix between $Y(s) = \mathcal{L}_-\{y(t)\}$ and $U(s) = \mathcal{L}_-\{u(t)\}$ $(Y(s) = \mathcal{Z}\{y(t)\}$ and $U(s) = \mathcal{Z}\{u(t)\})$ the *polynomial* matrix $A(s)$ i.e. if

$$A(s) = C_\infty (I_\mu - sA_\infty)^{-1} B_\infty + D_\infty \tag{7}$$

The variable $x_\infty(t) : \mathbb{R} \to \mathbb{R}^\mu$ in (4) is called the (fast) *generalized state vector of* Σ_g and the positive integer μ is called the *dimension* of Σ_g.

Remark 2: A GSS realization of a $A(s) \in \mathbb{R}[s]^{p\times m}$ can be obtained always from a state space realization of the *strictly proper* rational matrix $\overline{A}(s) := \left(\frac{1}{s}\right) A\left(\frac{1}{s}\right) \in \mathbb{R}^{p\times m}_{pr}(s)$ [3] because if $C_\infty \in \mathbb{R}^{p\times\mu}$,$A_\infty \in \mathbb{R}^{\mu\times\mu}$,$B_\infty \in \mathbb{R}^{\mu\times m}$ is a *state space* realization of $\overline{A}(s)$, i.e. if

$$\left(\frac{1}{s}\right) A\left(\frac{1}{s}\right) = C_\infty (sI_\mu - A_\infty)^{-1} B_\infty \tag{8}$$

then (8) by the substitution $\frac{1}{s} \mapsto s$ gives (7) with $D_\infty = 0_{p,m}$.

Example 3: Consider the polynomial matrix

$$A(s) = \begin{bmatrix} 1 & -s^3 & s^2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{9}$$

A *minimal* state space realization $(C_\infty, A_\infty, B_\infty, D_\infty)$ of

$$\begin{aligned} \overline{A}(s) \quad : \quad &= \left(\frac{1}{s}\right) A\left(\frac{1}{s}\right) = \begin{bmatrix} \frac{1}{s} & -\frac{1}{s^4} & \frac{1}{s^3} \\ 0 & \frac{1}{s} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \\ &\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s & -1 & 0 \\ 0 & 0 & s \\ 0 & s^2 & 1 \end{bmatrix}^{-1} \\ &= N(s)D(s)^{-1} \end{aligned} \tag{10}$$

where $N(s), D(s)$ are right coprime and $D(s)$ is column proper is obtained via Proposition 1.81 in [1] as

$$C_\infty = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A_\infty = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_\infty = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, D_\infty = 0_{3,3}$$

which is also a GSS realization of $A(s)$,i.e. $A(s) = C_\infty (I_\mu - sA_\infty)^{-1} B_\infty$.■

Let $A(s) = A_0 + A_1 s + \ldots + A_{q_1} s^{q_1} \in \mathbb{R}[s]^{p\times m}$, $A_i \in \mathbb{R}^{p\times m}, i = 0,1,2,\ldots,q_1 \geq 1$, $A_{q_1} \neq 0$ and let $C_\infty \in \mathbb{R}^{p\times\mu}, A_\infty \in \mathbb{R}^{\mu\times\mu}, B_\infty \in \mathbb{R}^{\mu\times m}, D_\infty \in \mathbb{R}^{p\times m}$, $\mu \in \mathbb{Z}^+$ be a GSS realization of $A(s)$. Let also $J_\infty = QA_\infty Q^{-1}$, $Q \in \mathbb{R}^{\mu\times\mu}$, $|Q| \neq 0$, be the Jordan normal form of A_∞, and $\overline{C}_\infty := C_\infty Q^{-1}$, $\overline{B}_\infty := QB_\infty$. From (7) it follows that $(I_\mu - sA_\infty)^{-1} \in \mathbb{R}[s]^{\mu\times\mu}$, so that $I_\mu - sA_\infty$ or equivalently $I_\mu - sJ_\infty$ are $\mathbb{R}[s]$-unimodular matrices and J_∞ has in general the form

$$J_\infty = block\ diag\,[J_{\infty 1}, J_{\infty 2}, ..., J_{\infty\eta}, 0_{\tau,\tau}] \in \mathbb{R}^{\mu\times\mu} \tag{11}$$

where

$$J_{\infty i} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix} \in \mathbb{R}^{(\kappa_i+1)\times(\kappa_i+1)} \tag{12}$$

and $\kappa_1 \geq \kappa_2 \geq \ldots \geq \kappa_\eta$, $\kappa_i \in \mathbb{Z}^+$, $i = 1,2,...,\eta$. From (11)(12) it follows that J_∞ (equivalently A_∞) is a nilpotent matrix with index of nilpotency equal to the size $\kappa_1 + 1$ of its largest Jordan block $J_{\infty 1}$ i.e. $J_\infty^{\kappa_1} \neq 0_{\mu,\mu}$, $J_\infty^{\kappa_1+i} = 0_{\mu,\mu}$, $i = 1,2,..$ and it can be easily verified that

$$(I_\mu - sJ_\infty)^{-1} = I_\mu + sJ_\infty + \ldots + s^{\kappa_1} J_\infty^{\kappa_1} \in \mathbb{R}[s]^{\mu\times\mu} \tag{13}$$

From the fact that

$$A(s) = \overline{C}_\infty (I_\mu - sJ_\infty)^{-1} \overline{B}_\infty + D_\infty \tag{14a}$$

and (13) it follows that $\kappa_1 \geq q_1$ and

$$\begin{aligned} \overline{C}_\infty J_\infty^i \overline{B}_\infty &= A_i, \quad i = 0,1,2,\ldots,q_1 \\ \overline{C}_\infty J_\infty^i \overline{B}_\infty &= 0, \quad i = q_1+1, q_1+2,\ldots \end{aligned} \tag{15}$$

We give now a number of definitions and results regarding the structure of the GSS realization of a polynomial matrix $A(s)$.

Definition 4: The *system poles* at $s = \infty$ of the GSS realization $\Sigma_g = [C_\infty, A_\infty, B_\infty, D_\infty]$ of $A(s)$ are the *zeros* at $s = \infty$ of $I_\mu - sA_\infty$. The *generalized order* f_g of Σ_g is the total number of the *system poles* at $s = \infty$ of Σ_g or equivalently the total number of *zeros* at $s = \infty$ of $I_\mu - sA_\infty$. (where multiplicities and orders of the zeros at $s = \infty$ of $I_\mu - sA_\infty$ are accounted for).

From (11)(12) it can be easily seen that the orders of the zeros at $s = \infty$ of $I_\mu - sJ_\infty$ are the integers $\kappa_i \geq 1$, $i = 1,2,\ldots,\eta$ i.e. that the Smith McMillan form $S^\infty_{I_\mu - sJ_\infty}$ at $s = \infty$ of the matrix pencil $I_\mu - sA_\infty$ [2][1] is given by

$$S^\infty_{I_\mu - sJ_\infty} = block\ diag\left[\ sI_{f_g}, I_\tau, \tfrac{1}{s^{\kappa_\eta}}, \ldots, \tfrac{1}{s^{\kappa_1}}\ \right] \tag{16}$$

Now since $I_\mu - sA_\infty$ is a unimodular polynomial matrix (regular matrix pencil) it has no finite zeros i.e. the number n of the finite zeros of $I_\mu - sA_\infty$ is $n := \deg|I_\mu - sA_\infty| = 0$ and has no finite poles. Now from the known fact that in every square and nonsingular polynomial matrix $A(s)$ the total number of zeros of $A(s)$ in $\mathbb{C}\cup\{\infty\}$ is equal to the total number of poles of $A(s)$ in $\mathbb{C}\cup\{\infty\}$ [1] it simply follows that the *generalized order* f_g of Σ_g is given also by the total number of poles at $s = \infty$ of $I_\mu - sA_\infty$ which by definition is its McMillan degree $\delta_M [I_\mu - sA_\infty]$. Summarizing we have that

$$\begin{aligned} f_g: &= \text{\# of zeros at } s=\infty \text{ of } [I_\mu - sA_\infty] \quad (17) \\ &\overset{(16)}{=} \sum_{i=1}^{\eta} \kappa_i = rank_{\mathbb{R}} J_\infty = rank_{\mathbb{R}} A_\infty \\ &= \text{\# of poles at } s=\infty \text{ of } [I_\mu - sA_\infty] \\ &= :\delta_M [I_\mu - sA_\infty] \end{aligned}$$

A. Irreducibility at infinity

We examine now the concept of *irreducibility at* $s = \infty$ of a GSS realization of a polynomial matrix. This concept is analogous to the concept of *irreducibility in* $\mathbb{C}$ of a *state space* realization of a *proper* rational matrix. To this end we introduce a number of results that are necessary.

Let $A(s) \in \mathbb{R}[s]^{p\times m}$, and let $\overline{C}_\infty \in \mathbb{R}^{p\times\mu}, J_\infty \in \mathbb{R}^{\mu\times\mu}, \overline{B}_\infty \in \mathbb{R}^{\mu\times m}, D_\infty \in \mathbb{R}^{p\times m}$ be a GSS realization of $A(s)$ with J_∞ in Jordan normal form:

$$J_\infty = block\ diag\,[J_{\infty 1}, J_{\infty 2}, ..., J_{\infty\eta}, 0_{\tau,\tau}] \in \mathbb{R}^{\mu\times\mu} \quad (18)$$

$$J_{\infty i} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \in \mathbb{R}^{(\kappa_i+1)\times(\kappa_i+1)} \quad (19)$$

$\kappa_1 \geq \kappa_2 \geq \ldots \geq \kappa_\eta$, $\kappa_i \in \mathbb{Z}^+$ so that $\mu := \sum_{i=1}^{\eta} (\kappa_i + 1) + \tau$,

$$\overline{C}_\infty = \begin{bmatrix} C_{\infty 1} & \cdots & C_{\infty\eta} & C_{\infty\eta+1} \end{bmatrix} \in \mathbb{R}^{p\times\mu} \quad (20)$$

$$C_{\infty i} = \begin{bmatrix} c_{i1} & \cdots & c_{i\kappa_i+1} \end{bmatrix} \in \mathbb{R}^{p\times(\kappa_i+1)}, c_{ij} \in \mathbb{R}^{p\times 1} \quad (21)$$

$$\overline{B}_\infty = \begin{bmatrix} B_{\infty 1} \\ B_{\infty 2} \\ \vdots \\ B_{\infty\eta} \\ B_{\infty\eta+1} \end{bmatrix} \in \mathbb{R}^{\mu\times m} \quad (22)$$

$$B_{\infty i} = \begin{bmatrix} b_{i1}^\top \\ b_{i2}^\top \\ \vdots \\ b_{i\kappa_i}^\top \\ b_{i\kappa_i+1}^\top \end{bmatrix} \in \mathbb{R}^{(\kappa_i+1)\times m}, b_{ij}^\top \in \mathbb{R}^{1\times m} \quad (23)$$

and consider the singular matrix pencils:

$$\begin{bmatrix} I_{\kappa_i+1} - sJ_{\infty i} \\ C_{\infty i} \end{bmatrix} = \begin{bmatrix} I_{\kappa_i+1} \\ C_{\infty i} \end{bmatrix} - s \begin{bmatrix} J_{\infty i} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} I_{\kappa_i+1} - sJ_{\infty i} & B_{\infty i} \end{bmatrix} = \begin{bmatrix} I_{\kappa_i+1} & B_{\infty i} \end{bmatrix} - s \begin{bmatrix} J_{\infty i} & 0 \end{bmatrix}$$

then we can state and prove the following

Proposition 5: (i) the singular matrix pencil

$$\begin{bmatrix} I_{\kappa_i+1} - sJ_{\infty i} \\ C_{\infty i} \end{bmatrix} \quad (24)$$

has no zeros at $s = \infty$ iff

$$c_{i1} \neq 0 \quad (25)$$

or equivalently iff

$$rank_{\mathbb{R}} \begin{bmatrix} J_{\infty i} \\ C_{\infty i} \end{bmatrix} = \kappa_i + 1 \quad (26)$$

(ii) the singular matrix pencil

$$\begin{bmatrix} I_{\kappa_i+1} - sJ_{\infty i} & B_{\infty i} \end{bmatrix} \quad (27)$$

has no zeros at $s = \infty$ iff

$$b_{i\kappa_i+1}^\top \neq 0 \quad (28)$$

or equivalently iff

$$rank_{\mathbb{R}} [J_{\infty i}, B_{\infty i}] = \kappa_i + 1 \quad (29)$$

In view of the above one can easily obtain the following

Corollary 6: Let $C_{\infty i} = \begin{bmatrix} c_{i1} & c_{i2} & \dots & c_{i\kappa_i} & c_{i\kappa_i+1} \end{bmatrix} \in \mathbb{R}^{p\times(\kappa_i+1)}$ such that

$$c_{i1} = c_{i2} = ... = c_{i\sigma_i - 1} = 0, \quad (30)$$

where $c_{i\sigma_i} \neq 0$ and $1 \leq \sigma_i \leq \kappa_i + 1$. Then $\begin{bmatrix} I_{\kappa_i+1} - sJ_{\infty i} \\ C_{\infty i} \end{bmatrix}$ has one zero at $s = \infty$ of order $\sigma_i - 1$. By analogy a similar result holds for the pencil $\begin{bmatrix} I_{\kappa_i+1} - sJ_{\infty i} & B_{\infty i} \end{bmatrix}$.

Due to the Jordan block structure of $J_\infty \in \mathbb{R}^{\mu\times\mu}$ as in 18 and the block structures of $\overline{C}_\infty$ and $\overline{B}_\infty$ as in 20 -23 the above corollary implies also the following

Corollary 7: Let $A(s) \in \mathbb{R}[s]^{p\times m}$, and let $\overline{C}_\infty \in \mathbb{R}^{p\times\mu}, J_\infty \in \mathbb{R}^{\mu\times\mu}, \overline{B}_\infty \in \mathbb{R}^{\mu\times m}, D_\infty \in \mathbb{R}^{p\times m}$, $\mu \in Z^+$ be a GSS realization of $A(s)$ with J_∞ in Jordan normal form as in 18 and 19 then

(i) $\begin{bmatrix} I_\mu - sJ_{\infty i} \\ \overline{C}_\infty \end{bmatrix}$ has no zeros at $s = \infty$ iff

$$rank_{\mathbb{R}} \begin{bmatrix} J_\infty \\ \overline{C}_\infty \end{bmatrix} = \mu$$

(ii) $\begin{bmatrix} I_\mu - sJ_{\infty i} & \overline{B}_\infty \end{bmatrix}$ has no zeros at $s = \infty$ iff

$$rank_{\mathbb{R}} \begin{bmatrix} J_\infty & \overline{B}_\infty \end{bmatrix} = \mu$$

Corollary 8: $C_{\infty i} J_{\infty i} = 0 \Leftrightarrow c_{i1} = c_{i2} = \ldots c_{i\kappa_i} = 0 \overset{Corol.6}{\Rightarrow} \begin{bmatrix} I_{\kappa_i+1} - sJ_{\infty i} \\ C_{\infty i} \end{bmatrix}$ has one zero at $s = \infty$ of order κ_i.

From the above discussion it follows that if for some $i = 1, 2, \ldots, \eta$,the matrix pencil $\begin{bmatrix} I_{\kappa_i+1} - sJ_{\infty i} \\ C_{\infty i} \end{bmatrix}$ has zeros at $s = \infty$ of order κ_i then when $\overline{C}_\infty (I_\mu - sJ_\infty)^{-1}$ is formed, the system poles at $s = \infty$ of the GSS realization $\Sigma_g = [\overline{C}_\infty, J_\infty, \overline{B}_\infty, D_\infty]$ that correspond to the zero at $s = \infty$ of $\begin{bmatrix} I_{\kappa_i+1} - sJ_{\infty i} \\ C_{\infty i} \end{bmatrix}$ of order κ_i "cancel out" and they do not appear as *poles* at $s = \infty$ of the polynomial matrix $A(s) = \overline{C}_\infty (I_\mu - sJ_\infty)^{-1} \overline{B}_\infty$. Thus a GSS realization $\widehat{\Sigma}_g = \left[\widehat{C}_\infty, \widehat{J}_\infty, \widehat{B}_\infty, \widehat{D}_\infty\right]$ of $A(s)$ is obtained which has generalized order $\widehat{f}_g := \kappa_1 + \kappa_3 < f_g$. Similar remarks apply if for some $i \in \{1,2,3\}$ the matrix pencil $[I_{\kappa_i+1} - sJ_{\infty i}, B_{\infty i}]$ has zeros at $s=\infty$.

The analysis in the above example gives rise to the concept of *decoupling zeros a*t $s=\infty$ of a GSS realization $\Sigma_g = [C_\infty, A_\infty, B_\infty, D_\infty]$ of a polynomial matrix $A(s)$.

Definition 9: The *input decoupling zeros (i.d.z.) (output decoupling zeros (o.d.z)) at* $s=\infty$ of a GSS realization $\Sigma_g = [C_\infty, A_\infty, B_\infty, D_\infty]$ of a polynomial matrix $A(s)$ are the zeros at $s=\infty$ of the singular matrix pencil $[I_\mu - sA_\infty, B_\infty]$ $\left(\begin{bmatrix} I_\mu - sA_\infty \\ C_\infty \end{bmatrix}\right)$ .The *input-output decoupling zeros* (*i.o.d.z.)* at $s=\infty$ of Σ_g are the *common* zeros at $s=\infty$ of the singular matrix pencils $[I_\mu - sA_\infty, B_\infty]$ and $\begin{bmatrix} I_\mu - sA_\infty \\ C_\infty \end{bmatrix}$. The *decoupling zeros* (*d.z.*) at $s=\infty$ of Σ_g are the elements of the set: $\{$i.d.z. at $s=\infty$ of $\Sigma_g\} + \{$o.d.z. at $s=\infty$ of $\Sigma_g\} - \{$i.o.d. zeros at $s=\infty$ of $\Sigma_g\}$.

Remark 10: Candidates for *(i.d.z.)* and *(o.d.z) at* $s=\infty$ of a GSS realization of $\Sigma_g = [C_\infty, A_\infty, B_\infty, D_\infty]$ of a polynomial matrix $A(s)$ are the zeros at $s=\infty$ of $I_\mu - sA_\infty$ i.e. the *system poles of* Σ_g at $s=\infty$.

Example 11: Let $p=m=r=3$, and

$$A(s) = \begin{bmatrix} s+1 & s^2 & 0 \\ 0 & s+1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ S^\infty_{A(s)} = \begin{bmatrix} s^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

i.e. $A(s)$ has one pole at $s=\infty$ of order $q_1 = 2$ and no zeros at $s=\infty$. A GSS realization $\Sigma_g = \left(\overline{C}_\infty, J_\infty, \overline{B}_\infty\right)$ of $A(s)$ with $\mu = 7, \eta = 2, \kappa_1 = 2 = q_1, \kappa_2 = 1, \tau = 2$ is given by

$$\overline{C}_\infty = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix},$$

$$J_\infty = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \overline{B}_\infty = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \tag{31}$$

and the generalized order of Σ_g is $f_g = rank_{\mathbf{R}} J_\infty = \kappa_1 + \kappa_2 = 3$. Since

$$C_{\infty 2} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} = C_{\infty 2}\left[I_{\kappa_2} - sJ_{\infty 2}\right]$$

we obtain that

$$\begin{bmatrix} I_{\kappa_2} - sJ_{\infty 2} \\ C_{\infty 2} \end{bmatrix} = \begin{bmatrix} 1 & -s \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} I_{\kappa_2} \\ C_{\infty 2} \end{bmatrix}\left[I_{\kappa_2} - sJ_{\infty 2}\right]$$

So $\begin{bmatrix} I_{\kappa_2} - sJ_{\infty 2} \\ C_{\infty 2} \end{bmatrix}$ has one zero at $s=\infty$ (which is the zero at $s=\infty$ of $I_{\kappa_2} - sJ_{\infty 2}$ and which, from $S^\infty_{I_{\kappa_2} - sJ_{\infty 2}} = diag\left[\ sI_{\kappa_2 - 1}, \frac{1}{s^{\kappa_2}}\ \right] = \begin{bmatrix} s & 0 \\ 0 & \frac{1}{s} \end{bmatrix}$, has order $\kappa_2 = 1$) and thus $\Sigma_g = \left[\overline{C}_\infty, J_\infty, \overline{B}_\infty\right]$ has an o.d.z. at $s=\infty$. Similarly and since

$$B_{\infty 2} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \left[I_{\kappa_2} - sJ_{\infty 2}\right] B_{\infty 2}$$

$\begin{bmatrix} I_{\kappa_2} - sJ_{\infty 2} & B_{\infty 2} \end{bmatrix}$ has one zero at $s=\infty$ of order $\kappa_2 = 1$ and thus $\Sigma_g = \left[\overline{C}_\infty, J_\infty, \overline{B}_\infty\right]$ has an i.d.z. at $s=\infty$. Finally, and according to Definition 11 Σ_g has one i.o.d.z. at $s=\infty$ and one d.z. at $s=\infty$. Since $C_{\infty 2}B_{\infty 2} = 0$ a GSS realization $\widehat{\Sigma}_g = \left[\widehat{C}_\infty, \widehat{J}_\infty, \widehat{B}_\infty\right]$ of $A(s)$ can be obtained as described above. This GSS realization is given by

$$\widehat{C}_\infty = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}, \tag{32}$$

$$\widehat{J}_\infty = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \widehat{B}_\infty = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

The *generalized order* of $\widehat{\Sigma}_g$ is $\widehat{f}_g = rank_{\mathbf{R}} \widehat{J}_\infty = \kappa_1 = 2$. Since the matrix pencil

$$\left[I_\mu - s\widehat{J}_\infty, \widehat{B}_\infty\right] = \begin{bmatrix} 1 & -s & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -s & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 \end{bmatrix}$$

(as a *row proper* polynomial matrix) has no zeros at $s=\infty$ [1], $\widehat{\Sigma}_g$ has no *input decoupling zeros* at $s=\infty$. Similarly $\begin{bmatrix} I_\mu - s\widehat{J}_\infty \\ \widehat{C}_\infty \end{bmatrix}$ has no zeros at $s=\infty$ and so $\widehat{\Sigma}_g$ has no *output decoupling zeros* at $s=\infty$. Finally $\widehat{\Sigma}_g$ has no *decoupling zeros* at $s=\infty$.■

The above considerations give rise to the following

Definition 12: A GSS realization $\widehat{\Sigma}_g = [C_\infty, A_\infty, , B_\infty, , D_\infty]$ of a polynomial matrix $A(s)$ which has no input and no output decoupling zeros at $s=\infty$ is called an *irreducible at* $s=\infty$ *GSS realization of* $A(s)$.

From Corollary 7 we obtain

Proposition 13: A GSS realization $\Sigma_g = \overline{C}_\infty \in \mathbb{R}^{p\times\mu}, J_\infty \in \mathbb{R}^{\mu\times\mu}, \overline{B}_\infty \in \mathbb{R}^{\mu\times m}, D_\infty \in \mathbb{R}^{p\times m}$ of a polynomial matrix $A(s)$ has no input and no output decoupling zeros at $s=\infty$ i.e. it is irreducible at $s=\infty$ or equivalently $\left[I_\mu - sJ_\infty, \overline{B}_\infty\right]$ and $\begin{bmatrix} I_\mu - sJ_\infty \\ \overline{C}_\infty \end{bmatrix}$ have no zeros at $s=\infty$ iff $rank_{\mathbf{R}}\left[J_\infty, \overline{B}_\infty\right] = \mu$ and $rank_{\mathbf{R}}\begin{bmatrix} J_\infty \\ \overline{C}_\infty \end{bmatrix} = \mu$.

The next Proposition says that if the Smith-McMillan form $S^\infty_{A(s)}$ at $s=\infty$ of $A(s)$ is given by 2 and $C_\infty, A_\infty, B_\infty, D_\infty = 0$ is a GSS realization of $A(s)$ which is obtained as in Remark 2, i.e. as a *minimal* state space realization of the strictly proper rational matrix $\overline{A}(s)$:=

$\frac{1}{s}A\left(\frac{1}{s}\right) \in \mathbb{R}_{pr}(s)^{p\times m}$, then this realization is an *irreducible at* $s=\infty$ GSS realization of $A(s)$ and the number η of the Jordan blocks $J_{\infty i}$ in the Jordan form J in 11 satisfy

$$\eta = k$$

while the indices κ_i of the sizes $\kappa_i+1, i=1,2,...,\eta$ of the Jordan blocks $J_{\infty i}$ of J_∞ in 12 are given by

$$\kappa_i = q_i$$

where $q_i > 0,\ i=1,2,...,k$ are the non zero orders of the poles at $s=\infty$ appearing in the Smith McMillan form $S^{\infty}_{A(s)}$ at $s=\infty$ of $A(s)$. This proposition gives us also the necessary tools for the investigation in the sequel of what constitutes a *minimal* GSS realization of a polynomial matrix $A(s)$.

Proposition 14: Let $A(s) \in \mathbb{R}[s]^{p\times m}$ with $rank_{\mathbb{R}(s)}A(s)=r$ and Smith-McMillan form at $s=\infty$

$$S^{\infty}_{A(s)} = diag(\overbrace{s^{q_1},...,s^{q_k},I_{v-k}}^{v},\frac{1}{s^{\widehat{q}_{v+1}}},...,\frac{1}{s^{\widehat{q}_r}},0_{p-r,m-r}) \tag{33}$$

Then (i) the McMillan degree $\delta_M\left(\overline{A}(s)\right)$ of the strictly proper rational matrix: $\overline{A}(s) := \frac{1}{s}A\left(\frac{1}{s}\right) \in \mathbb{R}_{pr}(s)^{p\times m}$ is given by

$$\mu = \delta_M\left(\overline{A}(s)\right) = \sum_{i=1}^{k}(q_i+1)+v-k = \sum_{i=1}^{k}q_i+v \tag{34}$$

Let $C_\infty,\ A_\infty,\ B_\infty$ be a *minimal state space realization* of $\overline{A}(s) \in \mathbb{R}_{pr}(s)^{p\times m}$, i.e. let that

$$\overline{A}(s) := \frac{1}{s}A\left(\frac{1}{s}\right) = C_\infty\left(sI_\mu - A_\infty\right)^{-1}B_\infty \tag{35}$$

holds true while the following equivalent conditions are satisfied

$$rank_{\mathbf{R}}\left[B_\infty, J_\infty B_\infty, \ldots A_\infty^{\mu-1}B_\infty\right] = rank_{\mathbf{R}}\begin{bmatrix} C_\infty \\ C_\infty A_\infty \\ \vdots \\ C_\infty A_\infty^{\mu-1}\end{bmatrix} = \mu \tag{36}$$

or equivalently

$$rank_{\mathbf{R}}\left[A_\infty, B_\infty\right] = rank_{\mathbf{R}}\begin{bmatrix} A_\infty \\ C_\infty\end{bmatrix} = \mu \tag{37}$$

and let $J_\infty := QA_\infty Q^{-1},\ |Q| \neq 0$ be the Jordan normal form of A_∞.

Then (ii)

$$J_\infty = block\ diag\left[J_{\infty 1}, J_{\infty 2}, ..., J_{\infty k}, 0_{v-k,v-k}\right] \in \mathbb{R}^{\mu\times\mu} \tag{38}$$

where

$$J_{\infty i} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix} \in \mathbb{R}^{(q_i+1)\times(q_i+1)}\ i=1,\ldots,k \tag{39}$$

and (iii) $C_\infty,\ A_\infty,\ B_\infty$ (or equivalently $\overline{C}_\infty := C_\infty Q^{-1}$, $J_\infty = QA_\infty Q^{-1}$, $\overline{B}_\infty := QB_\infty$) constitute *irreducible at* $s=\infty$ *GSS realizations* of $A(s)$.

Given a GSS realization $\Sigma_g = [C_\infty, A_\infty, B_\infty, D_\infty]$ of a polynomial matrix $A(s) \in \mathbb{R}[s]^{p\times m}$ the above considerations give rise to the next corollary which gives a relation between (i) the *set of zeros at* $s=\infty$ of the matrix pencil $I_\mu - sA_\infty$,(ii) the *set of poles at* $s=\infty$ of $A(s)$, and (iii) the *set of decoupling zeros* at $s=\infty$ of Σ_g.

Corollary 15:

$$\{\text{zeros at } s=\infty \text{ of } [I_\mu - sA_\infty]\} \equiv$$

$$\{\text{poles at } s=\infty \text{ of } A(s)\} \cup \{\text{dec. zeros at } s=\infty \text{ of } \Sigma_g\ \} \tag{40}$$

Remark 16: Using (17), (40) can be written as

$$\begin{aligned} f_g &= \delta_M\left[I_\mu - sA_\infty\right] = && (41)\\ &= \delta_M\left(A(s)\right) + \#\text{ dec. zeros at } s=\infty \text{ of } \Sigma_g \end{aligned}$$

Example 17: Consider the polynomial matrix $A(s)$ in example 11 and its GSS realization $\Sigma_g = \left(\overline{C}_\infty, J_\infty, \overline{B}_\infty\right)$ given by (31). Σ_g has one decoupling zeros at $s=\infty$ of order $\kappa_2 = 1$ and (40) can be written as $f_g = \kappa_1 + \kappa_2 = q_1 + \kappa_2 = 2+1 = 3$

Eqn. (40) gives rise to the inequality

$$\begin{aligned} f_g &:= \#\text{ of zeros at } s=\infty \text{ of } [I_\mu - sA_\infty] && (42)\\ &\geq \#\text{ of poles at } s=\infty \text{ of } A(s) =: \delta_M\left(A(s)\right) \end{aligned}$$

If a GSS realization $\widehat{\Sigma}_g = [C_\infty, A_\infty,, B_\infty,, D_\infty]$ of a polynomial matrix $A(s)$ is irreducible at $s=\infty$, then its generalized order takes its least value:

$$\widehat{f}_g = \delta_M\left[I_\mu - sA_\infty\right] = rank_{\mathbf{R}}A_\infty = \delta_M\left(A(s)\right) \tag{43}$$

which by definition is the McMillan degree $\delta_M\left(A(s)\right)$ of $A(s)$. In such a case the irreducible at $s=\infty$ GSS realization $\widehat{\Sigma}_g = [C_\infty, A_\infty,, B_\infty,, D_\infty]$ of $A(s)$ has the *least generalized order* $\widehat{f}_g$ among the generalized orders of all GSS realizations which give rise to $A(s)$. As indicated by (43) the *least* generalized order $\widehat{f}_g$ of $\widehat{\Sigma}_g$ can then be determined directly from the McMillan degree of the polynomial matrix $A(s)$.

Definition 12 together with the above discussion and (42) give rise to

Theorem 18: A GSS realization $\widehat{\Sigma}_g = [C_\infty, A_\infty,, B_\infty,, D_\infty]$ of a polynomial matrix $A(s)$ with Smith-McMillan form at $s=\infty$ as in (33) is *irreducible at* $s=\infty$ iff

$$\begin{aligned} \widehat{f}_g &:= \delta_M\left[I_\mu - sA_\infty\right] && (44)\\ &= rank_{\mathbf{R}}A_\infty = \sum_{i=1}^{k}q_1 = \delta_M\left(A(s)\right) \end{aligned}$$

Example 19: Consider the polynomial matrix $A(s)$ in example 11 and its irreducible at $s=\infty$ GSS realization $\widehat{\Sigma}_g = \left[\widehat{C}_\infty, \widehat{J}_\infty, \widehat{B}_\infty\right]$ in (32) and notice that

$$\widehat{f}_g = q_1 = 2 = \delta_M\left(A(s)\right)$$

The next theorem is the analogue of Theorem 2.56 in [1] for the case of an irreducible at $s=\infty$ GSS realizations of a polynomial matrix $A(s)$.

Theorem 20: Let $A(s) \in \mathbb{R}[s]^{p\times m}$ with $\text{rank}_{\mathbb{R}(s)}A(s) = r$ and Smith-McMillan form at $s=\infty$ as in (33).

Let $\widehat{\Sigma}_g = [C_\infty, A_\infty, B_\infty, D_\infty]$ be an *irreducible at* $s=\infty$ GSS realization of $A(s)$.Then the *zero structure* at $s=\infty$ of $I_\mu - sA_\infty$ is isomorphic to the *pole structure* at $s=\infty$ of $A(s)$ i.e. if $S^\infty_{A(s)}$ is given by (33) then the Smith-McMillan form at $s=\infty$ of $I_\mu - sA_\infty$ is given by

$$S^\infty_{[I_\mu - sA_\infty]} = diag\left[sI_{\widehat{f}_g}, I_{v-k}, \frac{1}{s^{q_k}}, \cdots, \frac{1}{s^{q_1}}\right] \tag{45}$$

where $\widehat{f}_g = rank_{\mathbf{R}} A_\infty = \sum_{i=1}^k q_i = \mu - v$.

Example 21: Consider the polynomial matrix $A(s)$ in example 11 and its irreducible at $s=\infty$ GSS realization $\widehat{\Sigma}_g = \left[\widehat{C}_\infty, \widehat{J}_\infty, \widehat{B}_\infty\right]$ in (32). Notice that $S^\infty_{A(s)} = diag\left[s^2, 1, 1\right]$ i.e. $q_1 = 2 = \widehat{f}_g$, $q_2 = q_3 = 0$, $k=1$, $v=3$ and $S^\infty_{I_\mu - s\widehat{J}_\infty} = diag\left[s, s, 1, 1, \frac{1}{s^2}\right]$. So that the pole at $s=\infty$ of $A(s)$,denoted by $s^{q_1} = s^2$ in $S^\infty_{A(s)}$ appears as a zero at $s=\infty$ of $I_\mu - s\widehat{J}_\infty$, denoted by $\frac{1}{s^{q_1}} = \frac{1}{s^2}$ in $S^\infty_{\left[I_\mu - s\widehat{J}_\infty\right]}$.

Example 22: Let

$$J_\infty = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, B_\infty = C_\infty = I_3, D_\infty = 0_{3,3} \tag{46}$$

Obviously $[I_3, J_\infty, I_3, 0_{3,3}]$ is an *irreducible at* $s=\infty$ *GSS realization* of

$$A(s) := [I_3 - sJ_\infty]^{-1} = \begin{bmatrix} 1 & -s & 0 \\ 0 & 1 & -s \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & s & s^2 \\ 0 & 1 & s \\ 0 & 0 & 1 \end{bmatrix}$$

Now

$$S^\infty_{[I_3 - sJ_\infty]} = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & \frac{1}{s^2} \end{bmatrix}, \quad S^\infty_{A(s)} = \begin{bmatrix} s^2 & 0 & 0 \\ 0 & \frac{1}{s} & 0 \\ 0 & 0 & \frac{1}{s} \end{bmatrix} \tag{47}$$

so that the *zero structure* at $s=\infty$ of $I_3 - sJ_\infty$ is isomorphic to the *pole structure* at $s=\infty$ of $A(s)$.

B. Minimal GSS realizations of a polynomial matrix

We discuss now the concept of a *minimal* GSS realization of a *polynomial* matrix. Although the concept of *irreducibility* $(in\ \mathbb{C})$ and of *minimality* of a *state space* realization of a *proper* rational matrix coincide i.e. *irreducibility* $(in\ \mathbb{C})$ of a state space realization of a proper rational matrix implies and is implied by the *minimality* of the dimension of the state space realization, this is not in general true for the analogous concepts of *irreducibility at* $s=\infty$ and of *minimality* of a GSS realization of a *polynomial* matrix. In the following we firstly define what constitutes a *minimal* GSS *realization* of a polynomial matrix and then illustrate the above points by showing that *minimality* of a GSS realization of a *polynomial* matrix implies its *irreducibility at* $s=\infty$ (Theorem 24) but that the reverse is not true in general. Through these results we give the necessary tools for obtaining (i) a minimal GSS realization of a *polynomial matrix* $A(s) \in \mathbb{R}[s]^{p\times m}$ and (ii) the least value of the dimension $\overline{\mu}$ of the generalized state vector $x_\infty(t) \in \mathbb{R}^{\overline{\mu}}$ appearing in a *minimal* GSS realization of $A(s)$. We start with

Definition 23: A GSS realization $\widetilde{\Sigma}_g = [\widetilde{C}_\infty \in \mathbb{R}^{p\times\widetilde{\mu}}, \widetilde{J}_\infty \in \mathbb{R}^{\widetilde{\mu}\times\widetilde{\mu}}, \widetilde{B}_\infty \in \mathbb{R}^{\widetilde{\mu}\times m}, \widetilde{D}_\infty \in \mathbb{R}^{p\times m}]$of a polynomial matrix $A(s) \in \mathbb{R}[s]^{p\times m}$ is called *minimal* if it has the least number of generalized states or equivalently if its *dimension* $\widetilde{\mu}$ is minimal i.e. $\widetilde{\mu} \leq \mu$ for every dimension μ of all other GSS realizations $\Sigma_g = [C_\infty \in \mathbb{R}^{p\times\mu}, A_\infty \in \mathbb{R}^{\mu\times\mu}, B_\infty \in \mathbb{R}^{\mu\times m}, D_\infty \in \mathbb{R}^{p\times m}]$ of $A(s)$. The dimension $\widetilde{\mu}$ of a minimal GSS realization of a polynomial matrix $A(s)$ is called the *least dimension* of $A(s)$.

The next Theorem gives a necessary and sufficient condition for a GSS realization of a polynomial matrix to be *minimal.*

Theorem 24: [7]Let $A(s) \in \mathbb{R}[s]^{p\times m}$ with $\text{rank}_{\mathbb{R}(s)}A(s) = r$ and Smith-McMillan form at $s=\infty$

$$S^\infty_{A(s)} = diag(\overbrace{s^{q_1}, ..., s^{q_k}, I_{v-k}}^{v}, \frac{1}{s^{\widehat{q}_{v+1}}}, ..., \frac{1}{s^{\widehat{q}_r}}, 0_{p-r,m-r})$$

Let $\widetilde{\Sigma}_g = \left[\widetilde{C}_\infty, \widetilde{J}_\infty, \widetilde{B}_\infty, \widetilde{D}_\infty\right]$ be a GSS realization of $A(s)$ with $\widetilde{J}_\infty$ in Jordan normal form. Then $\widetilde{\Sigma}_g = \left[\widetilde{C}_\infty, \widetilde{J}_\infty, \widetilde{B}_\infty, \widetilde{D}_\infty\right]$ is a minimal GSS realization of $A(s)$ iff

$$\widetilde{\mu} = \delta_M\left[\overline{A}(s)\right] - (v-k) = \sum_{i=1}^k (q_i + 1) = k + \sum_{i=1}^k q_i \tag{48}$$

where $\overline{A}(s) := \frac{1}{s}A\left(\frac{1}{s}\right) \in \mathbb{R}_{pr}(s)^{p\times m}$.

Remark 25: Two important conclusions that follow from Theorem 25 are:

(i) a necessary condition for a GSS realization $\widetilde{\Sigma}_g = \left[\widetilde{C}_\infty \in \mathbb{R}^{p\times\widetilde{\mu}}, \widetilde{J}_\infty \in \mathbb{R}^{\widetilde{\mu}\times\widetilde{\mu}}, \widetilde{B}_\infty \in \mathbb{R}^{\widetilde{\mu}\times m}, \widetilde{D}_\infty \in \mathbb{R}^{p\times m}\right]$ of a polynomial matrix $A(s)$ to be *minimal* is that the realization must be *irreducible* at $s=\infty$ and that the realization has no non-dynamic variables.

(ii) the least dimension $\widetilde{\mu}$ that a GSS realization of $A(s)$ may have is the number $\widetilde{\mu} = \sum_{i=1}^k (q_i + 1)$,where $q_i > 0$, $i = 1, 2, \ldots, k$ are the nonzero orders of the poles at $s=\infty$ of $A(s)$.□

Remark 26: Notice that if $v=k$, i.e. if there are not "non-dynamic variables" (equivalently if there is no I_{v-k} block in (33)) then $\widetilde{\mu} \equiv \mu$ and a minimal state space realization $C_\infty, J_\infty, B_\infty$ of $\overline{A}(s) \in \mathbb{R}_{pr}(s)^{p\times m}$ coincides with a *minimal* GSS realization $\widetilde{C}_\infty, \widetilde{J}_\infty, \widetilde{B}_\infty, \widetilde{D}_\infty$ of $A(s) \in \mathbb{R}[s]^{p\times m}$.

Based on the above results we can now state

Corollary 27: A GSS realization of a polynomial matrix $A(s)$ is a *minimal* GSS realization if it is an *irreducible at* $s=\infty$ GSS realization and *has no non-dynamic variables.*

Example 28: Consider examples 11 and 21. $S^0_{A(s)} = diag\left[\frac{1}{s^3}, \frac{1}{s}, \frac{1}{s}\right]$, $\mu = \delta_M\left[\overline{A}(s)\right] = \sum_{i=1}^k (q_i + 1) + v -$

$k=2+1+3-1=5$. The least dimension is $\widetilde{\mu}=\sum_{i=1}^{k}(q_i+1)=2+1=3$. A non-minimal but irreducible at $s=\infty$ GSS realization of $A(s)$ is given by (32) while a *minimal* GSS realization of $A(s)$ is

$$\widetilde{C}_\infty = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \widetilde{J}_\infty = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\widetilde{B}_\infty = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \widetilde{D}_\infty = \widehat{C}_\infty \widehat{B}_\infty = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

For conventional state space realizations of a *proper* rational matrix $A(s)$ the least possible dimension of a state space realization is the *least order* $\nu(A(s))$ of $A(s)$ [5].

If $A(s)$ is a $p\times m$ column (row) proper polynomial matrix with $p\geq m$ $(p\leq m)$ then an *irreducible at* $s=\infty$ GSS realization of $A(s)$ as well as the Smith-McMillan form at $s=\infty$ of $A(s)$ can be obtained by inspection. This is stated in the following

Proposition 29: Let $A(s)\in\mathbf{R}[s]^{p\times m}, p\geq m$, be column proper and let $v_j=\deg a_j(s)$, where $a_j(s)=a_{j0}+a_{j1}s+\cdots+a_{jv_j}s^{v_j}\in\mathbf{R}[s]^{p\times 1}, j\in\mathbf{m}$ are the m columns of $A(s)$. Then an *irreducible at* $s=\infty$ GSS *realization* $C_\infty, J_\infty, B_\infty$ of $A(s)$ can be obtained by inspection and is given by

$$C_\infty=[a_{1v_1},\cdots,a_{10}\mid a_{2v_2},\cdots,a_{20}\mid\cdots\mid a_{mv_m},\cdots,a_{m0}] \tag{49}$$

$$J_\infty = block\ diag\,[J_{\infty 1}, J_{\infty 2},\cdots,J_{\infty m}]\in\mathbf{R}^{\mu\times\mu} \tag{50}$$

$$J_{\infty j}=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}\in\mathbf{R}^{(v_i+1)\times(v_i+1)} \tag{51}$$

$$B_\infty = blok\ diag\,[b_1,b_2,\ldots,b_m]\in\mathbf{R}^{\mu\times m} \tag{52}$$

$$b_j=[0,0,\ldots,0,1]^\top\in\mathbf{R}^{(v_j+1)\times 1}\qquad j\in\mathbf{m} \tag{53}$$

$$\mu=\sum_{j=1}^{m}(v_j+1) \tag{54}$$

Remark 30: A dual result can be obtained for every $p\times m$ $(p\leq m)$ *row proper* polynomial matrix $A(s)$ by transposing $A(s)$ and applying Remark 3.72. Comparing the dimensions of the Jordan blocks $J_{\infty i}$ in (39) of Proposition 3.64 and those of the Jordan blocks in (51) of Proposition 3.70 we can easily conclude that if $A(s)$ is column (row) proper then its column (row) degrees $v_j, j\in\mathbf{m}$ $(r_i,\ i\in\mathbf{p})$ are equal to the orders of its *poles* at $s=\infty$, i.e. that $v_j\equiv q_j$, $j\in\mathbf{m}$ $(r_i\equiv q_i,\ i\in\mathbf{p})$ and that $A(s)$ has no zeros at $s=\infty$.

Theorem 31: Let $A(s)\in\mathbb{R}[s]^{p\times m}$ with Smith-McMillan form at $s=\infty$ given by

$$S_{A(s)}^{\infty}=diag(\overbrace{s^{q_1},...,s^{q_k},I_{v-k}}^{v},\frac{1}{s^{\widehat{q}_{v+1}}},...,\frac{1}{s^{\widehat{q}_r}},0_{p-r,m-r}) \tag{55}$$

If the GSS realization C_∞,A_∞,B_∞,D_∞ of $A(s)$ is *irreducible at* $s=\infty$ then the *zero structure* in $\mathbb{C}$ (at $s=\infty$) of $A(s)$ is isomorphic to the *zero structure* in $\mathbb{C}$ (at $s=\infty$) of the GSS Rosenbrock "system matrix" (singular matrix pencil):

$$P(s):=\begin{bmatrix} I_\mu - sA_\infty & B_\infty \\ -C_\infty & D_\infty \end{bmatrix}$$
$$= s\begin{bmatrix} -A_\infty & 0_{\mu,m} \\ 0_{p,\mu} & 0_{p,m} \end{bmatrix}+\begin{bmatrix} I_\mu & B_\infty \\ -C_\infty & D_\infty \end{bmatrix} \tag{56}$$

and if $S_{A(s)}^{\infty}$ is given by 55 then the Smith-McMillan form at $s=\infty$ of $P(s)$ is given by

$$S_{P(s)}^{\infty}=diag\left[sI_{\mu-\nu},I_{v-k},\frac{1}{s^{\widehat{q}_{k+1}}},\ldots,\frac{1}{s^{\widehat{q}_r}},0\right]$$

III. Conclusions

In this note we have investigated the mechanism of cancelations of zeros at $s=\infty$, in the proccess of transforming a polynomial transfer function to its equivalent "pure" generalized state space realization. The concepts of irreducibility and minimality of the realization have been discussed and the role of of *dynamic* and *non dynamic* variables appearing in generalized state space realizations has been examined. Finally, the isomorphism between zeros at infinity of the "infinite pole pencil" and the Rosenbrock system matrix of an irreducible at infinity generalized state space realization and the *pole and zero structure at infinity* of such a polynomial matrix has been presented.

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