On a first order hold discretization for singular systems

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***Abstract*—This note proposes a new discretization method. The proposed sampled system is described in terms of the Markov parameters of the system and therefore the proposed method is easily implemented. The methodology we use is a first-order hold discretization for the input and first-order approximation of its derivatives.**

I. Introduction

Consider the linear time-invariant state space system of the form

$$\dot{x}_c(t) = Ax_c(t) + Bu(t) \tag{1}$$

with solution given by

$$x_c(t) = e^{At}x_c(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau \tag{2}$$

By assuming that the input $u(t)$ is constant in the interval $[kT, kT+T)$ i.e.

$$u(t) = u(kT) \qquad \forall t \in [kT, kT+T)$$

a *zero-order hold* discretized model of (1) is given [4] by :

$$x_d((k+1)T) = \tilde{A}x_d(kT) + \tilde{B}u(kT) \tag{3}$$

where

$$\tilde{A} = e^{AT},\ \tilde{B} = \left[\int_0^T e^{Aw}dw\right]B$$

The connection between the transfer function of (2) and the ones in (3) is given by

$$\left(zI_n - \tilde{A}\right)^{-1}\tilde{B} = \left(1 - z^{-1}\right)Z\left\{L^{-1}\left\{\frac{H_{spr}(s)}{s}\right\}\right\} \tag{4}$$

$$H_{spr}(s) = (sI_n - A)^{-1}B$$

where $L^{-1}[\cdot]$ denotes the inverse Laplace transform and $Z[\cdot]$ denotes the Z-transform. In case the input $u(t)$ is approximated by the first order polynomial

$$u(t) = u(kT) + \frac{u(kT) - u((k-1)T)}{T}(\tau - kT)$$

$\forall t \in [kT, kT+T)$ we get the causal *first-order hold* discretized model of (8) as given by [2] :

$$x((k+1)T) = \hat{A}x(kT) + \hat{B}_0u(kT) + \hat{B}_{-1}u((k-1)T) \tag{5}$$

where

$$\hat{A} = e^{AT},\ \hat{B}_0 = \left[\int_0^T e^{Aw}\left(2 - \frac{w}{T}\right)dw\right]B,$$

$$\hat{B}_{-1} = \left[\int_0^T e^{Aw}\left(\frac{w}{T} - 1\right)dw\right]B$$

The connection between the transfer function of (2) and the ones in (5) is given by

$$\left(zI_n - \tilde{A}\right)^{-1}\left(\hat{B}_0 + \hat{B}_{-1}z^{-1}\right) = \tag{6}$$

$$= \left(1 - z^{-1}\right)^2 Z\left\{L^{-1}\left[\frac{(Ts+1)H(s)}{Ts^2}\right]\right\}$$

Alternatively, we can approximate the input $u(t)$ by the first order polynomial

$$u(t) = u(kT) + \frac{u((k+1)T) - u(kT)}{T}(\tau - kT)$$

$\forall t \in [kT, kT+T)$ and get the *first-order hold* discretized model

$$x((k+1)T) = \hat{A}x(kT) + \hat{B}_0u(kT) + \hat{B}_1u((k+1)T) \tag{7}$$

where

$$\hat{A} = e^{AT},\ \hat{B}_0 = \left[\int_0^T e^{Aw}\frac{w}{T}dw\right]B,$$

$$\hat{B}_1 = \left[\int_0^T e^{Aw}(1 - \frac{w}{T})dw\right]B$$

The above system is called *triangle-hold* equivalent model [9] in order to distinguish it from the *first-order hold* model. Its transfer function is given by

$$\left(zI_n - \tilde{A}\right)^{-1}\left(\hat{B}_0 + \hat{B}_1z\right) = \frac{(z-1)^2}{Tz}Z\left\{L^{-1}\left[\frac{H(s)}{s^2}\right]\right\}$$

A more general class of systems, are the linear time-invariant non-homogeneous singular systems of the form

$$E\dot{x}(t) = Ax(t) + Bu(t) \tag{8}$$

where $E, A \in R^{n\times n}, B \in R^{m\times n}$,$C \in R^{p\times n}$ and $D \in R^{p\times m}$ and E is not necessary nonregular. (8) is assumed to be regular

$\det[sE-A] \neq 0$. Systems of the above form are usually called singular systems, descriptor systems, generalized state space systems, semistate systems etc. Descriptor systems appear in the modelling of many physical phenomena, such as engineering systems, social economic systems, network analysis, biological systems, etc. An extended reference on descriptor systems may be found in [1].

In case where E is singular, we may use the forward or backward Euler method, or even the Gears method [6] in order to get a discretized singular model of (8). In this work, based a) on a recent work of [7] concerning the solution of singular systems in terms of the Laurent expansion terms of $(sE-A)^{-1}$, b) on similar techniques applied for the derivation of a zero-order models (either for state-space systems [4] or for singular systems [3]), and c) on some remarks concerning the discretization of singular systems by using the Euler-approximation [8], we propose a discretization method of the singular system (8). More specifically, after some preliminary results concerning the solution and the Markov parameters of a singular system, we present in the third section a non-causal triangular-hold (and first-order hold) discretization method for singular systems. The methodology that we are using is based on : a) a first-order hold discretization for the input $u(t)$ and b) a forward Euler approximation for its derivatives. The proposed models are described in terms of the Markov parameters of the system.

II. Preliminary results

Consider the singular system described by (8). Its resolvent matrix can be expressed in a power series expansion of s as follows

$$\Phi(s) = (sE-A)^{-1} = \sum_{k=-\mu}^{\infty} \Phi_k(E,A)s^{-k-1} \tag{9}$$

where μ is the index of nilpotency of the pencil $sE-A$. The matrices $\Phi_i(E,A)$ are uniquely defined by the relations [5] :

$$E\Phi_k = A\Phi_{k-1}, k=-\mu,...,-2,-1 \tag{10}$$

$$E\Phi_0 - A\Phi_{-1} = I_n \tag{11}$$

$$\Phi_k = (\Phi_0 A)^k \Phi_0 = \Phi_{k-1}A\Phi_0 \ (k=1,2,...) \tag{12}$$

$$\Phi_{-k} = -\Phi_{-k+1}E\Phi_{-1} = (-\Phi_{-1}E)^{k-1}\Phi_{-1} \ (k=2,3...,\mu)$$

with $\Phi_{-\mu-1}=0$, or equivalently by the relations

$$\Phi_k E = \Phi_{k-1}A, k=-\mu,...,-2,-1 \tag{13}$$

$$\Phi_0 E - \Phi_{-1}A = I_n \tag{14}$$

$$\Phi_k = \Phi_0 (A\Phi_0)^k = \Phi_0 A\Phi_{k-1} \ (k=1,2,...) \tag{15}$$

$$\Phi_{-k} = -\Phi_{-1}E\Phi_{-k+1} = \Phi_{-1}(-E\Phi_{-1})^{k-1} \ (k=2,3...,\mu) \tag{16}$$

Based on the above relations, the following properties can be derived

$$\Phi_i E\Phi_j = \Phi_j E\Phi_i \ (\forall i,j) \tag{17}$$

$$\Phi_i E\Phi_j = \left\{ \begin{array}{ll} -\Phi_{i+j} & i<0, j<0 \\ \Phi_{i+j} & i\geq 0, j\geq 0 \\ 0 & ij\leq 0 \text{ and } \\ & |i|+|j|\neq 0 \end{array} \right\} \tag{18}$$

$$\Phi_i A\Phi_j = \left\{ \begin{array}{ll} -\Phi_{i+j+1} & i<0, j<0 \\ \Phi_{i+j+1} & i\geq 0, j\geq 0 \\ 0 & ij\leq 0 \text{ and } \\ & |i|+|j|\neq 0 \end{array} \right\} \tag{19}$$

where $|\cdot|$ is the absolute value of the argument matrix.

The solution of (8) in terms of the resolvent matrix of the singular systems is given by [7] :

$$\begin{aligned} x(t) &= e^{\Phi_0 At}\Phi_0 Ex(0-)+ \\ &+ \int_0^t e^{\Phi_0 A(t-\tau)}\Phi_0 Bu(\tau)d\tau + \sum_{i=0}^{\mu-1}\Phi_{-i-1}Bu^{(i)}(t) \\ &+ \sum_{i=0}^{\mu-1}\Phi_{-i-1}\left(\delta^{(i)}(t)Ex(0-) + \sum_{j=0}^{i-1}\delta^{(i-j-1)}Bu^{(j)}(0-)\right) \end{aligned} \tag{20}$$

III. State space first-order hold discretization of non-homogeneous singular systems

A *zero-order hold* state-space discretization of (8) is given by [3] :

$$x((k+1)T) = \tilde{A}x(kT) + \tilde{B}(\sigma)u(kT) \tag{21}$$

$$x(0) = \Phi_0 Ex(0-) + \sum_{i=0}^{\mu-1}(-\Phi_{-1}E)^i\Phi_{-1}Bu^{(i)}(0+)$$

where

$$\tilde{A} = e^{\Phi_0 AT} \quad ; \quad \tilde{B}(\sigma) = \sum_{i=0}^{\mu}\tilde{B}_i\sigma^i \tag{22}$$

$$\tilde{B}_0 = \left[\int_0^T e^{\Phi_0 Aw}dw\right]\Phi_0 B + \sum_{i=1}^{\mu}(-1)^i\Phi_{-i}BT^{1-i}$$

$$\tilde{B}_\ell = \sum_{j=\ell}^{\mu}(-1)^{j-\ell}\Phi_{-j}BT^{1-j}\begin{pmatrix} j \\ j-\ell \end{pmatrix}$$

where $\ell = 1,2,..,\mu$ and $\sigma^i u(kT) = u((k+i)T)$. Note that according to [3]

$$\begin{aligned} &\left(zI-\tilde{A}\right)^{-1}\tilde{B}(z) = \\ &= \left(1-z^{-1}\right)Z\left\{L^{-1}\left\{\frac{H_{spr}(s)}{s}\right\}\right\} + H_{pol}\left(\frac{z-1}{T}\right) \end{aligned} \tag{23}$$

where $H_{spr}(s)$ (resp. $H_{pol}(s)$) denotes the strictly proper part (resp. polynomial part) of the transfer function matrix $H(s) = (sE-A)^{-1}B$. In the next theorem we propose a non-causal *triangular-order hold* state-space discretization of (8).

Theorem 1: Using a *triangular-order hold* approximation of the input $u(t)$ and its derivatives, the continuous time nonhomogeneous singular system (8) is discretized to yield the state space system

$$x((k+1)T) = \tilde{A}x(kT) + \tilde{B}(\sigma)\, u(kT) \quad (24)$$

$$x(0) = \Phi_0 Ex(0-) + \sum_{i=0}^{\mu-1} (-\Phi_{-1}E)^i \Phi_{-1} Bu^{(i)}(0+) \quad (25)$$

where

$$\tilde{A} = e^{\Phi_0 AT} \quad (26)$$

$$\tilde{B}(\sigma) = \sum_{i=0}^{\mu} \tilde{B}_i \sigma^i \quad \text{with} \quad \sigma^i u(kT) = u((k+i)T) \quad (27)$$

$$\tilde{B}_0 = \left[\int_0^T e^{\Phi_0 Aw} \frac{w}{T} dw\right] \Phi_0 B + \left(\sum_{i=1}^{\mu} (-1)^i \Phi_{-i} BT^{1-i}\right) \quad (28)$$

$$\tilde{B}_1 = \left[\int_0^T e^{\Phi_0 Aw} (1 - \frac{w}{T}) dw\right] \Phi_0 B + \sum_{i=1}^{\mu} (-1)^{i-1} i \Phi_{-i} BT^{1-i} \quad (29)$$

$$\tilde{B}_\ell = \sum_{j=\ell}^{\mu} (-1)^{j-\ell} \Phi_{-j} BT^{1-j} \begin{pmatrix} j \\ j-\ell \end{pmatrix} \quad \ell = 2, 3, .., \mu \quad (30)$$

Proof: At the sample times kT, $k = 0, 1, ...$ (where $0 \equiv 0+$) the state of (8) is given by

$$x(kT) = e^{\Phi_0 AkT} \Phi_0 Ex(0-) + \int_0^{kT} e^{\Phi_0 A(kT-\tau)} \Phi_0 Bu(\tau) d\tau + + \sum_{i=0}^{\mu-1} (-\Phi_{-1}E)^i \Phi_{-1} Bu^{(i)}(kT) \quad \text{for } k = 1, 2, ... \quad (31)$$

and

$$x(0) = \Phi_0 Ex(0-) + \sum_{i=0}^{\mu-1} (-\Phi_{-1}E)^i \Phi_{-1} Bu^{(i)}(0+) \text{ for } k = 0$$

The state at the $(k+1)$th step can be expressed in terms of the state at the kth step as follows

$$\begin{aligned} x((k+1)T) &= e^{\Phi_0 A(kT+T)} \Phi_0 Ex(0-) + \\ &+ \int_0^{kT+T} e^{\Phi_0 A(kT+T-\tau)} \Phi_0 Bu(\tau) d\tau + \sum_{i=0}^{\mu-1} \Phi_{-i-1} Bu^{(i)}((k+1)T) \\ &= e^{\Phi_0 AT} e^{\Phi_0 AkT} \Phi_0 Ex(0-) + \\ &+ e^{\Phi_0 AT} \int_0^{kT} e^{\Phi_0 A(kT-\tau)} \Phi_0 Bu(\tau) d\tau + \\ &+ e^{\Phi_0 AT} \int_{kT}^{(k+1)T} e^{\Phi_0 A(kT-\tau)} \Phi_0 Bu(\tau) d\tau + \\ &+ \sum_{i=0}^{\mu-1} \Phi_{-i-1} Bu^{(i)}((k+1)T) = \\ &= e^{\Phi_0 AT} \left(e^{\Phi_0 AkT} \Phi_0 Ex(0-) + \int_0^{kT} e^{\Phi_0 A(kT-\tau)} \Phi_0 Bu(\tau) d\tau\right) + \\ &+ \underbrace{\int_{kT}^{(k+1)T} e^{\Phi_0 A(kT+T-\tau)} \Phi_0 Bu(\tau) d\tau}_{I_0} + \\ &+ \sum_{i=0}^{\mu-1} \Phi_{-i-1} Bu^{(i)}((k+1)T) \end{aligned} \quad (32)$$

In order to obtain a *triangular-order hold* discretization we assume that the input $u(t)$ is approximated in the interval $[kT, kT+T)$ by a first order polynomial of the form

$$u(\tau) = u(kT) + \frac{u((k+1)T) - u(kT)}{T} (\tau - kT)$$

$\forall \tau \in [kT, kT+T)$ and thus

$$\begin{aligned} I_0 &= \int_{kT}^{(k+1)T} e^{\Phi_o A(kT+T-\tau)} \Phi_o Bu(\tau) d\tau = \\ &= \int_{kT}^{(k+1)T} e^{\Phi_o A(kT+T-\tau)} \Phi_o Bu(kT) d\tau + \\ &+ \int_{kT}^{(k+1)T} e^{\Phi_o A(kT+T-\tau)} \Phi_o B \times \\ &\times \left(\frac{u((k+1)T) - u(kT)}{T}\right) (\tau - kT) d\tau = \\ &= \underbrace{\left[\int_{kT}^{(k+1)T} e^{\Phi_o A(kT+T-\tau)} d\tau\right]}_{I_1} \Phi_o Bu(kT) + \\ &+ \underbrace{\left[\int_{kT}^{(k+1)T} e^{\Phi_o A(kT+T-\tau)} (\tau - kT) d\tau\right]}_{I_2} \times \\ &\times \Phi_o B \left(\frac{u((k+1)T) - u(kT)}{T}\right) \end{aligned} \quad (33)$$

Let $w = kT + T - \tau$. Then $dw = -d\tau$ and

$$I_1 = \int_{kT}^{(k+1)T} e^{\Phi_o A(kT+T-\tau)} d\tau = = \int_T^0 e^{\Phi_o Aw} (-dw) = \int_0^T e^{\Phi_o Aw} dw \quad (34)$$

$$I_2 = \int_{kT}^{(k+1)T} e^{\Phi_o A(kT+T-\tau)} (\tau - kT) d\tau = = \int_T^0 e^{\Phi_o Aw} (T-w)(-dw) = = \int_0^T e^{\Phi_o Aw} (T-w) dw \quad (35)$$

By substituting (34) and (35) into (33) we get

$$\begin{aligned} I_0 &= \left[\int_0^T e^{\Phi_o Aw} dw\right] \Phi_0 Bu(kT) + \\ &+ \left[\int_0^T e^{\Phi_o Aw} (T-w) dw\right] \times \\ &\times \Phi_o B \left[\frac{u((k+1)T) - u(kT)}{T}\right] = \\ &= \left[\int_0^T e^{\Phi_o Aw} \left(1 - \frac{T-w}{T}\right) dw\right] \Phi_0 Bu(kT) + \\ &+ \left[\int_0^T e^{\Phi_o Aw} (\frac{T-w}{T}) dw\right] \Phi_0 Bu((k+1)T) = \\ &= \underbrace{\left[\int_0^T e^{\Phi_0 Aw} \frac{w}{T} dw\right] \Phi_0 B}_{\hat{B}_0} u(kT) + \\ &+ \underbrace{\left[\int_0^T e^{\Phi_0 Aw} (1 - \frac{w}{T}) dw\right] \Phi_0 B}_{\hat{B}_1} u((k+1)T) = \end{aligned}$$

Substituting $I_0 = \hat{B}_0 u(kT) + \hat{B}_1 u((k+1)T)$ in equation (32) we have that

$$\begin{aligned} x((k+1)T) &= e^{\Phi_0 AT} \begin{pmatrix} e^{\Phi_0 AkT} \Phi_0 Ex(0-) + \\ + \int_0^{kT} e^{\Phi_0 A(kT-\tau)} \Phi_0 Bu(\tau) d\tau \end{pmatrix} + \\ &+ \hat{B}_0 u(kT) + \hat{B}_1 u((k+1)T) + \\ &+ \sum_{i=0}^{\mu-1} \Phi_{-i-1} Bu^{(i)}((k+1)T) = \\ &= e^{\Phi_0 AT} \underbrace{\begin{pmatrix} e^{\Phi_0 AkT} \Phi_0 Ex(0-) + \\ + \int_0^{kT} e^{\Phi_0 A(kT-\tau)} \Phi_0 Bu(\tau) d\tau + \\ + \sum_{i=0}^{\mu-1} \Phi_{-i-1} Bu^{(i)}(kT) \end{pmatrix}}_{x(kT)} \\ &+ \hat{B}_0 u(kT) + \hat{B}_1 u((k+1)T) + \\ &+ \sum_{i=0}^{\mu-1} \Phi_{-i-1} Bu^{(i)}((k+1)T) - e^{\Phi_0 AT} \sum_{i=0}^{\mu-1} \Phi_{-i-1} Bu^{(i)}(kT) \\ &= e^{\Phi_0 AT} x(kT) + \hat{B}_0 u(kT) + \hat{B}_1 u((k+1)T) + \\ &+ \sum_{i=0}^{\mu-1} \Phi_{-i-1} Bu^{(i)}((k+1)T) - e^{\Phi_0 AT} \sum_{i=0}^{\mu-1} \Phi_{-i-1} Bu^{(i)}(kT) \end{aligned} \quad (36)$$

Since

$$\begin{aligned} & e^{\Phi_0 AT} \sum_{i=0}^{\mu-1} \Phi_{-i-1} Bu^{(i)}(kT) = \\ &= \sum_{i=0}^{\mu-1} \left(I + \frac{\Phi_0 AT}{1!} + \frac{(\Phi_0 AT)(\Phi_0 AT)}{2!} + ...\right) \Phi_{-i-1} Bu^{(i)}(kT) \\ &\overset{(19)}{=} \sum_{i=0}^{\mu-1} \Phi_{-i-1} Bu^{(i)}(kT) \end{aligned}$$

we rewrite (36) as

$$\begin{aligned} x((k+1)T) &= e^{\Phi_0 AT} x(kT) + \hat{B}_0 u(kT) + \hat{B}_1 u((k+1)T) + \\ &+ \sum_{i=0}^{\mu-1} \Phi_{-i-1} Bu^{(i)}((k+1)T) - \sum_{i=0}^{\mu-1} \Phi_{-i-1} Bu^{(i)}(kT) = \\ &= e^{\Phi_0 AT} x(kT) + \hat{B}_0 u(kT) + \hat{B}_1 u((k+1)T) + \\ &+ \sum_{i=0}^{\mu-1} \Phi_{-i-1} BT \frac{(u^{(i)}((k+1)T) - u^{(i)}(kT))}{T} \end{aligned}$$

Now by using forward Euler approximation for the derivatives of u i.e.

$$u^{(i+1)}(kT) \simeq \frac{\left(u^{(i)}((k+1)T) - u^{(i)}(kT)\right)}{T}$$

we obtain that

$$\begin{aligned} x((k+1)T) &= e^{\Phi_0 AT} x(kT) + \hat{B}_0 u(kT) + \hat{B}_1 u((k+1)T) + \\ &+ \sum_{i=0}^{\mu-1} \Phi_{-i-1} BTu^{(i+1)}(kT) = \\ &= e^{\Phi_0 AT} x(kT) + \hat{B}_0 u(kT) + \hat{B}_1 u((k+1)T) + \\ &+ \sum_{i=1}^{\mu} \Phi_{-i} BTu^{(i)}(kT) \end{aligned} \quad (37)$$

The use of the approximation

$$u^{(i)}(kT) = \frac{1}{T^i} \sum_{\ell=0}^{i} (-1)^\ell \begin{pmatrix} i \\ \ell \end{pmatrix} u((k+i-\ell)T)$$

in equation (37) yields

$$\begin{aligned} x((k+1)T) &= e^{\Phi_0 AT} x(kT) + \hat{B}_0 u(kT) + \hat{B}_1 u((k+1)T) + \\ &+ \sum_{i=1}^{\mu} \Phi_{-i} BT \left[\frac{1}{T^i} \sum_{\ell=0}^{i} (-1)^\ell \begin{pmatrix} i \\ \ell \end{pmatrix} u((k+i-\ell)T)\right] \end{aligned}$$

or in more compact form we get the system in (24) that completes the proof. ∎

Note that in the state-space case where $E = I_n, \Phi_0 = I_n, \Phi_{-i} = 0, i > 0$, the proposed model (24) coincides with (7). The connection between the transfer function matrix of the continuous time system $H(s) = H_{spr}(s) + H_{pol}(s)$ and the ones of the discrete time system $G(z) = \left(zI_3 - \tilde{A}\right)^{-1} \tilde{B}(z)$ is given by

$$\left(zI_3 - \tilde{A}\right)^{-1} \tilde{B}(z) =$$
$$= \frac{(z-1)^2}{Tz} Z\left\{L^{-1}\left[\frac{H_{spr}(s)}{s^2}\right]\right\} + H_{pol}\left(\frac{z-1}{T}\right)$$

The following Theorem establish the connection between the zeros of the matrix $sI_n - \Phi_0 A$ and the zeros of the pencil $sE - A$ and thus establish the connection between stability of the continuous time system (8) and its triangular-order hold discretization (24).

Theorem 2: [3] Let λ_i, μ_i and ν_i be the zeros of the matrix pencils $sI_n - \Phi_0 A, sE - A$ and $\left[\begin{array}{cc} sI_n & \Phi_0 \end{array}\right]$ i.e. $rank_R[\lambda_i I_n - \Phi_0 A] \leq n$, $rank_R[\mu_i E - A] \leq n$ and $rank_R\left[\begin{array}{cc} \nu_i I_n & \Phi_0 \end{array}\right] \leq n$. Then

$$\{\lambda_i\} = \{\mu_i\} + \{\nu_i\}$$

where $\{\cdot\}$ denotes the set of the specific zeros.

According to the above Theorem, it is easily shown that if the system (8) is marginally stable then the sampled system (24) is marginally stable (or stable if Φ_0 is nonsingular).

We can also introduce the following variables

$$x_i(kT) = u((k+i-1)T), \; i = 1,..,\mu$$
$$v(kT) = u((k+\mu)T)$$

in order to get the equivalent discrete state-space model

$$\tilde{x}((k+1)T) = \hat{A}\tilde{x}(kT) + \hat{B}v(kT)$$
$$y(kT) = \hat{C}\tilde{x}(kT)$$

where $\tilde{x}(kT) = \left[\begin{array}{cccc} x(kT)^T & x_1(kT)^T & \cdots & x_\mu(kT)^T \end{array}\right]^T$ and

$$\hat{A} = \left[\begin{array}{ccccc} \tilde{A} & \tilde{B}_0 & \tilde{B}_1 & \cdots & \tilde{B}_{\mu-1} \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ 0 & 0 & 0 & \cdots & 0 \end{array}\right]; \hat{B} = \left[\begin{array}{c} \tilde{B}_\mu \\ 0 \\ \vdots \\ I \end{array}\right]$$
$$\hat{C} = \left[\begin{array}{cccc} I & 0 & \cdots & 0 \end{array}\right]$$

Remark 3: In the case where : a) we assume that the input $u(t)$ is approximated in the interval $[kT, kT+T)$ by a first order polynomial (backward Euler approximation for the first derivative of u) of the form

$$u(\tau) = u(kT) + \frac{u(kT) - u((k-1)T)}{T}(\tau - kT)$$

$\forall \tau \in [kT, kT+T)$ and b) by using backward Euler approximation for all the derivatives of u i.e.

$$u^{(i+1)}(kT) \simeq \frac{\left(u^{(i)}(kT) - u^{(i)}((k-1)T)\right)}{T}$$

or equivalently

$$u^{(i)}((k+1)T) = \frac{1}{T^i}\sum_{\ell=0}^{i}(-1)^\ell \binom{i}{\ell} u((k+1-\ell)T)$$

we get, using similar lines with the above proof, a *causal first-order hold* discretization of the form

$$x((k+1)T) = \tilde{A}x(kT) + \tilde{B}(\sigma)u(kT) \qquad (38)$$
$$x(0) = \Phi_0 Ex(0-) + \sum_{i=0}^{\mu-1}(-\Phi_{-1}E)^i \Phi_{-1}Bu^{(i)}(0+)$$

where $\tilde{A} = e^{\Phi_0 AT}$ and

$$\tilde{B}(\sigma) = \sum_{i=-1}^{\mu-1}\tilde{B}_i\sigma^i \quad \text{with} \quad \sigma^i u(kT) := u((k-i)T)$$
$$\tilde{B}_{-1} = \sum_{i=1}^{\mu}\Phi_{-i}BT^{1-i}$$
$$\tilde{B}_0 = \left[\int_0^T e^{\Phi_0 Aw}(2 - \frac{w}{T})dw\right]\Phi_0 B +$$
$$+ \sum_{i=1}^{\mu}(-1)^1\Phi_{-i}BT^{1-i}\left(\begin{array}{c} i \\ i-1 \end{array}\right)$$
$$\tilde{B}_1 = \left[\int_0^T e^{\Phi_0 Aw}(\frac{w}{T} - 1)dw\right]\Phi_0 B +$$
$$+ \sum_{i=2}^{\mu}(-1)^2\Phi_{-i}BT^{1-i}\left(\begin{array}{c} i \\ i-2 \end{array}\right)$$
$$\tilde{B}_\ell = \sum_{j=\ell+1}^{\mu}(-1)^{\ell+1}\Phi_{-j}BT^{1-j}\left(\begin{array}{c} j \\ j-(\ell+1) \end{array}\right)$$

where $\ell = 2, 3, .., \mu - 1$. Note that in the state-space case where $E = I_n, \Phi_0 = I_n, \Phi_{-i} = 0, i > 0$, the proposed model (38) coincides with (5). The connection between the transfer function matrix of the continuous time system $H(s) = H_{spr}(s) + H_{pol}(s)$ and the ones of the first order-hold discrete time system $G(z) = \left(zI_3 - \tilde{A}\right)^{-1}\tilde{B}(z)$ is given by

$$\left(zI - \tilde{A}\right)^{-1}\tilde{B}(z) =$$
$$= \left(1 - z^{-1}\right)^2 Z\left\{\frac{(Ts+1)H_{spr}(s)}{Ts^2}\right\} + H_{pol}\left(\frac{1 - z^{-1}}{T}\right)$$

Similarly, with the remark given above we can denote the following variables

$$x_i(kT) = u((k - \mu + i)T), \; i = 1, ..., \mu$$
$$v(kT) = u((k+1)T)$$

in order to get the equivalent discrete state-space model

$$\tilde{x}((k+1)T) = \hat{A}\tilde{x}(kT) + \hat{B}v(kT)$$
$$y(kT) = \hat{C}\tilde{x}(kT)$$

where $\tilde{x}(kT) = \left[\begin{array}{cccc} x(kT)^T & x_1(kT)^T & \cdots & x_\mu(kT)^T \end{array}\right]^T$

and

$$\hat{A}=\begin{bmatrix} \tilde{A} & \tilde{B}_{\mu-1} & \tilde{B}_{\mu-2} & \cdots & \tilde{B}_0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix};\hat{B}=\begin{bmatrix} \tilde{B}_{-1} \\ 0 \\ 0 \\ \vdots \\ I \end{bmatrix}$$

$$\hat{C}=\begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix}$$

Example 4: Let us consider the singular system in [3]

$$\underbrace{\begin{bmatrix} -\rho+38 & 12\rho+54 & 37\rho+47 \\ 2\rho-3 & 6\rho+11 & 13\rho+32 \\ -\rho+3 & 2\rho+9 & 8\rho+13 \end{bmatrix}}_{\rho E-A}\underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}}_{x(t)}=\underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B}u(t)$$

$$x(0-)=\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$$

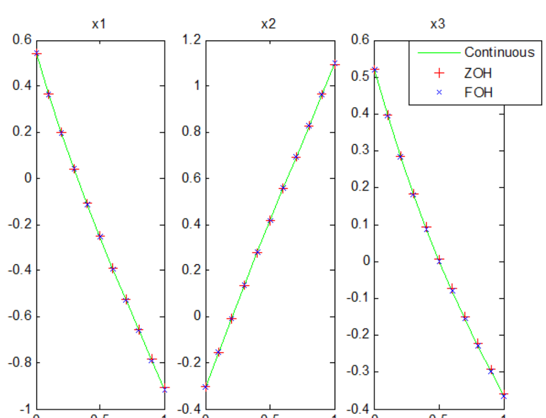
where $\rho x(t)=dx(t)/dt$. Let us consider that the input $u(t)=t$ and the sampling period $T=0.1$. Then the triangular-order hold discrete time system is the following:

$$x((k+1)T)=$$
$$=\begin{bmatrix} 0.9247 & -0.10040 & -0.12549 \\ 0.025099 & 1.0335 & 0.041831 \\ -0.083663 & -0.11155 & 0.86056 \end{bmatrix}x(kT)+$$
$$+\begin{bmatrix} 2.2736 \\ -2.9245 \\ 0.95683 \end{bmatrix}u(kT)+\begin{bmatrix} 0.25192 \\ -0.41731 \\ 0.18269 \end{bmatrix}u((k+1)T)+$$
$$+\begin{bmatrix} 1.2692 \\ -1.6731 \\ 0.57692 \end{bmatrix}u((k+2)T)$$

under the initial conditions

$$x(0)=\Phi_0 Ex(0-)+\Phi_{-1}Bu(0+)-$$
$$-\Phi_{-1}E\Phi_{-1}Bu^{(1)}(0+)=\begin{bmatrix} \frac{141}{260} & -\frac{159}{520} & \frac{27}{52} \end{bmatrix}^T$$

The following graph show the plots of the continous time functions $x_i(t), i=1,2,3$ as well as of their respective approximations that becomes under the zero-order hold and the triangular-order hold discretization models described above.



The following table gives the norm of the difference between the smooth solution and the zero-order hold approximation, as well as, between the smooth solution and the triangular-order hold approximation.

Norm	$\|\cdot\|_1$	$\|\cdot\|_2$	$\|\cdot\|_\infty$
$x_1^{con}-x_1^{zoh}$	0.7290	0.0736	0.0076
$x_1^{con}-x_1^{foh}$	0.0032	3.2469×10^{-4}	3.3846×10^{-5}
$x_2^{con}-x_2^{zoh}$	0.2463	0.0249	0.0026
$x_2^{con}-x_2^{foh}$	0.0044	4.4137×10^{-4}	4.4615×10^{-5}
$x_3^{con}-x_3^{zoh}$	0.8077	0.0816	0.0085
$x_3^{con}-x_3^{foh}$	0.0015	1.5011×10^{-4}	3.0769×10^{-5}

IV. Conclusions

A triangular-order hold and a first-order hold discretization method of a singular system have been considered in this work. The first method is using forward Euler approximations of the derivatives of the input $u(t)$ whereas the second one is using backward Euler approximations. Both the proposed sampled systems are described in terms of the Markov parameters of the system and therefore are easily implemented. Certain questions regarding the stability of the sampled systems have been investigated. The whole theory has been illustrated via an example.

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