CONSTRUCTION OF ALGEBRAIC-DIFFERENTIAL EQUATIONS WITH GIVEN SMOOTH-IMPULSIVE BEHAVIOR*

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Abstract. The paper proposes a new method for the construction of a class of systems of algebraic and differential equations of the form $A(\rho)\beta(t) = 0, \rho := dt$ where $A(\rho) = A_0 + A_1\rho + \cdots + A_q\rho^q \in \mathbb{R}[\rho]^{r\times r}$ and $\det[A(\rho)] \neq 0$ ($A(\rho)$ is regular) with given smooth and impulsive behavior.

Key words. behavior, polynomial matrix, jordan pairs, exact modelling

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1. Introduction.

Consider a system of algebraic and differential equations

$$A(\rho)\beta(t) = 0, \rho := d/dt \tag{1.1}$$

where

$$A(s) = A_q s^q + \cdots + A_1 s + A_0 \tag{1.2}$$

with $A_i \in \mathbb{R}^{r\times r}, A_q \neq 0$ and $\det A(s) \neq 0$ ($A(\rho)$ is regular). Depending of the finite and infinite zero structure of $A(\rho)$ the solution space of this system consists both of smooth and impulsive solutions [2], [7], [8].

According to [9], the crucial role in the modelling of a system plays the behavior of the system and not the algebraic-differential equations that describe the system. More particularly, whereas to a system of algebraic-differential equations corresponds one specific solution space, to a specific solution space corresponds more than one models in form of system of algebraic-differential equations. In [1], [9] and [10] a recursive method is proposed, known as Most Powerful Unfalsified Model (MPUM), which constructs the system of algebraic and differential equations $A(\rho)\beta(t) = 0$ if we know the *smooth* behavior of the system. A different approach to the same problem is given in [2]. Therefore, we observe an attempt to the solution of the **inverse** problem : Given the solution space B of a system (a physical process), find the system of algebraic and differential equations that have B as its solution space. Since, the inverse problem has already been investigated for the smooth behavior case, our main aim in this work is to solve the inverse problem for the case where we know both the smooth and the impulsive behavior of a system. The constructed models are not unique, but are connected by the full unimodular equivalence that preserves both the finite and infinite zero structure of the polynomial matrices that describes the systems [4], [5].

In section 2, we present the connection between the finite and infinite Jordan pairs of a regular polynomial matrix $A(\rho)$, and thus its finite and infinite zero structure, with the smooth and impulsive behavior of the system of algebraic and differential equations $A(\rho)\beta(t) = 0$. In section 3, we study the inverse problem of finding a system $A(\rho)\beta(t) = 0$ with prescribed *smooth* and *impulsive* solution space B .

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2. Finite-Infinite Jordan pairs and its connections with the behavior of a system. The characterization of a *finite Jordan pair* of a polynomial matrix corresponding to a zero s_0 of its determinant is given below.

THEOREM 2.1. *([2], pp.184) Let $(C_{s_0} \in \mathbb{R}^{r\times n}, J_{s_0} \in \mathbb{R}^{n\times n})$ be a matrix pair where J_{s_0} is a Jordan matrix with unique eigenvalue s_0 i.e.*

$$J_{s_0} := \begin{pmatrix} s_0 & 1 & \cdots & 0 \\ 0 & s_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s_0 \end{pmatrix} \in \mathbb{R}^{n\times n}$$

Then the following conditions are necessary and sufficient in order that (C_{s_0}, J_{s_0}) be a Jordan pair of $A(s)$ (given in (1.2)) corresponding to s_0 :

(a) $\det A(s)$ *has a zero at s_0 of multiplicity n,*

(b) rank $\begin{pmatrix} C_{s_0} \\ C_{s_0}J_{s_0} \\ \vdots \\ C_{s_0}J_{s_0}^{n-1} \end{pmatrix} = n$

(c) $A_q C_{s_0} J_{s_0}^q + \cdots + A_1 C_{s_0} J_{s_0} + A_0 C_{s_0} = 0$ ■

Taking a Jordan pair $(C_{s_i} \in \mathbb{R}^{r\times n_i}, J_{s_i} \in \mathbb{R}^{n_i\times n_i})$ for every zero $s_i, i = 1, 2, .., k$ of $A(s)$ we define a *finite Jordan pair* $(C \in \mathbb{R}^{r\times n}, J \in \mathbb{R}^{n\times n})$ of $A(s)$ as

$$C = \begin{pmatrix} C_{s_1} & C_{s_2} & \cdots & C_{s_k} \end{pmatrix} \;\; ; \;\; J = blockdiag \begin{pmatrix} J_{s_1} & J_{s_2} & \cdots & J_{s_k} \end{pmatrix}$$

where $n = n_1 + n_2 + \cdots n_k$ is the total number of finite zeros of $A(s)$ (order accounted for). The column span of $\Psi = Ce^{Jt}$ defines a solution space $B^{\mathbb{C}}$ of (1.1) with $\dim B^{\mathbb{C}} = n$. $B^{\mathbb{C}}$ is known as the smooth solution space of (1.1) defined by [2] and [7]. Based on the definition of a finite Jordan pair we can now give the definition of an *infinite Jordan pair.*

THEOREM 2.2. *([2],pp.185, [6]) Let $(C_\infty \in \mathbb{R}^{r\times \mu}, J_\infty \in \mathbb{R}^{\mu\times\mu})$ be a matrix pair where J_∞ is a Jordan matrix with unique eigenvalue $s_0 = 0$. Then, the following conditions are necessary and sufficient in order that (C_∞, J_∞) be an infinite Jordan pair of $A(s) = A_q s^q + \cdots + A_1 s + A_0$:*

(a) $\det \tilde{A}(s)$ *has a zero at $s_0 = 0$ of multiplicity μ, where* $\tilde{A}(s) = s^q A\left(\frac{1}{s}\right) = A_0 s^q + A_1 s^{q-1} + \cdots + A_q$,

(b) rank $\begin{pmatrix} C_\infty \\ C_\infty J_\infty \\ \vdots \\ C_\infty J_\infty^{\mu-1} \end{pmatrix} = \mu$

(c) $A_0 C_\infty J_\infty^q + \cdots + A_{q-1} C_\infty J_\infty + A_q C_\infty = 0$ ■

Taking a finite Jordan pair $(C_{\infty i} \in \mathbb{R}^{r\times\mu_i}, J_{\infty i} \in \mathbb{R}^{\mu_i\times\mu_i})$ for different algebraic multiplicities of the eigenvalue $s_i = 0, i = 1, 2, .., r$ of $\tilde{A}(s)$ we define an *infinite Jordan pair* $(C_\infty \in \mathbb{R}^{r\times\mu}, J_\infty \in \mathbb{R}^{\mu\times\mu})$ of $A(s)$ as

$$C_\infty = \begin{pmatrix} C_{\infty 1} & C_{\infty 2} & \cdots & C_{\infty r} \end{pmatrix} \in \mathbb{R}^{r\times\mu}$$
$$J_\infty = diag \begin{pmatrix} J_{\infty 1} & J_{\infty 2} & \cdots & J_{\infty r} \end{pmatrix} \in \mathbb{R}^{\mu\times\mu}$$

where $\mu = \sum_{i=1}^{r} \mu_i$ is the total number of infinite elementary divisors of $A(s)$ (order accounted for). The *infinite elementary divisors* (*IED*) of $A(s)$ are defined as the finite elementary divisors (FED) s^{μ_j} of its dual $\tilde{A}(s)$ at $s=0$. It is easily seen from the above that (C_∞, J_∞) is an infinite Jordan pair of $A(s)$ iff it is a finite Jordan pair of its dual polynomial matrix $\tilde{A}(s)$ corresponding to its zero (eigenvalue) at $s=0$. The IEDs of $A(s)$ are given by [3], [7] :

$$s^{\mu_j}, j=1,2,3,..,r$$
$$\mu_j = q - q_j, j=1,2,...,k \text{ with } q_1 = q \geq q_2 \geq \cdots \geq q_k$$
$$\mu_j = q, j=k+1,k+2,..,v$$
$$\mu_j = q + \hat{q}_j, j=v+1,\tau+2,...,r \text{ with } \hat{q}_{v+1} \leq \hat{q}_{v+2} \leq \cdots \leq \hat{q}_r$$

where q_i $(\hat{q}_i)$ are the orders of the *poles* (*zeros*) of $A(s)$ at $s=\infty$. The first kind of IEDs, i.e. the ones with degrees $\mu_j = q - q_j > 0, j=2,3,..,k$ we call *infinite pole IEDs*, since are connected with the orders of the poles $q_j, j=1,2,...,k$ of $A(s)$ at $s=\infty$. The second kind of IEDs, i.e. the ones with degrees $\mu_j = q+\hat{q}_j > 0, j = v+1, v+2,...,r$ we call *infinite zero IEDs*, since are connected with the orders of the zeros $\hat{q}_j, j=v+1,v+2,...,r$ of $A(s)$ at $s=\infty$.

Whereas the finite Jordan pairs of a polynomial matrix are connected with the smooth behavior of the Auto-Regressive (AR) representation (1.1), the infinite Jordan pairs are connected with its impulsive solution space. More specifically the column span of $\Psi_\infty = \sum_{i=0}^{\hat{q}_r - 1} \tilde{C}_{\infty,z} \tilde{J}^i_{\infty,z} \delta^{(i)}(t)$ where

$$\tilde{C}_{\infty,z} := \left[\tilde{C}_{\infty,v+1}, \tilde{C}_{\infty,v+2}, ..., \tilde{C}_{\infty,r}\right] \in \mathbb{R}^{r\times\mu_z}$$
$$\tilde{J}_{\infty,z} := blockdiag\left[\tilde{J}_{\infty,v+1}, \tilde{J}_{\infty,v+2}, ..., \tilde{J}_{\infty,r}\right] \in \mathbb{R}^{\mu_z\times\mu_z}$$

and $\tilde{C}_{\infty,l}$ $(\tilde{J}_{\infty,l})$, with $l=v+1,v+2,...,r$, consists of the first $\hat{q}_l$ columns (the first $\hat{q}_l \times \hat{q}_l$ block) of the matrix $C_{\infty,l}$ $(J_{\infty,l})$, defines a solution space B^∞ of (1.1) with $\dim B^\infty = \mu_z = \sum_{i=v+1}^r \hat{q}_i$. B^∞ is known as the impulsive solution space of (1.1) defined by [7]. The whole solution space $B = B^{\mathbb{C}} + B^\infty$ of (1.1) has dimension

$$\dim B = \dim B^{\mathbb{C}} + \dim B^\infty = n + \mu_z = \deg\det A(s) + \sum_{i=v+1}^{r} \hat{q}_i =$$
$$= \deg\det A(s) + \sum_{i=1}^{k} q_i = \delta_M(A(s))$$

where $\delta_M(\cdot)$ denotes the McMillan degree.

The finite (resp. infinite) Jordan pairs of a regular polynomial matrix $A(s)$, and thus the smooth (resp. impulsive) solution space of an AR-representation, can be easily constructed by techniques given in [2] and [7].

3. Construction of a System of Algebraic-Differential Equations with Given Smooth-Impulsive Solution Space. In the previous section we have shown how to construct the smooth-impulsive behavior of a given homogeneous system of algebraic-differential equations (1.1) by using the finite and infinite Jordan pairs corresponding to the finite and infinite zeros of the polynomial matrix $A(s)$. In this section, we study the **inverse problem** : *Given a specific smooth and impulsive behavior $B \oplus B^\infty$, is it possible to construct a polynomial matrix $A(\rho)$ and therefore a homogeneous system of algebraic-differential equations $A(\rho)\beta(t)=0$ with this prescribed behavior ?* We start by the definition of a *decomposable pair* given in [2].

Definition 3.1. *([2], pp.188) A matrix pair (X,T) is called admissible of order p if $X \in \mathbb{R}^{r\times p}$ and $T \in \mathbb{R}^{p\times p}$. An admissible pair (X,T) of order rq is called a decomposable pair (of degree q) if the following properties are satisfied :*

(a)

$$X = \begin{pmatrix} X_1 & X_2 \end{pmatrix} \quad ; \; T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \tag{3.1}$$

where $(X_1 \in \mathbb{R}^{r\times n}, T_1 \in \mathbb{R}^{n\times n})$ for some n, $0 \le n \le rq$ (so that $(X_2 \in \mathbb{R}^{r\times(rq-n)}$, $T \in \mathbb{R}^{(rq-n)\times(rq-n)}))$.

(b) *the matrix S_{q-1} is nonsingular i.e.*

$$S_{q-1} = \begin{pmatrix} X_1 & X_2T_2^{q-1} \\ X_1T_1 & X_2T_2^{q-2} \\ \vdots & \vdots \\ X_1T_1^{q-1} & X_2 \end{pmatrix} \tag{3.2}$$

A decomposable pair (X,T) will be called decomposable pair (X,T) of the regular polynomial matrix $A(s)$ defined in (1.2) if in addition to (a) and (b) the following conditions holds

(c) $\sum_{i=0}^{q} A_iX_1T_1^i = 0$, $\sum_{i=0}^{q} A_iX_2T_2^{q-i} = 0$. ∎

A decomposable pair appears to include the full spectral information for a polynomial matrix (both finite and infinite). The existence of a decomposable pair for every nonregular polynomial matrix follows from the following Theorem.

Theorem 3.2. *([2], pp.189) Let $A(s)$ defined in (1.2) and (C,J) and (C_∞, J_∞) be its finite and infinite Jordan pairs, respectively. Then*

$$X = \begin{pmatrix} C & C_\infty \end{pmatrix} \quad ; \; T = \begin{pmatrix} J & 0 \\ 0 & J_\infty \end{pmatrix} \tag{3.3}$$

is a decomposable pair of $A(s)$. ∎

An interesting question is, if it is always possible to construct a regular polynomial matrix $A(s)$ that corresponds to a given decomposable pair (X,T) ? The answer is *yes* and is given by the following Theorem.

Theorem 3.3. *([2], pp.197) Let (X,T) be the decomposable pair of degree q proposed in (3.1) and let S_{q-2} defined in (3.6). Then for every $r \times rq$ matrix V such that the matrix $\begin{pmatrix} S_{q-2}^T & V^T \end{pmatrix}^T$ is nonsingular, the polynomial matrix*

$$A(s) = V(I-P)\begin{pmatrix} sI - T_1 & 0 \\ 0 & sT_2 - I \end{pmatrix} \times \left(U_0 + U_1 s + \cdots + U_{q-1}s^{q-1}\right) \tag{3.4}$$

where

$$P = \begin{pmatrix} I & 0 \\ 0 & T_2 \end{pmatrix} S_{q-1}^{-1} \begin{pmatrix} I \\ 0 \end{pmatrix} S_{q-2} \tag{3.5}$$

$$S_{q-2} = \begin{pmatrix} X_1 & X_2T_2^{q-2} \\ X_1T_1 & X_2T_2^{q-3} \\ \vdots & \vdots \\ X_1T_1^{q-2} & X_2 \end{pmatrix} \tag{3.6}$$

and

$$\left(\begin{array}{cccc} U_0 & U_1 & \cdots & U_{q-1} \end{array}\right) := S_{q-1}^{-1} \tag{3.7}$$

has (X,T) *as its decomposable pair.* ■

Therefore given a decomposable pair (X,T) of degree q we can always compose a polynomial matrix $A(s)$ of degree q. Therefore, if we construct from a specific smooth-impulsive solution vector space a decomposable pair, then we are able to build a polynomial matrix $A(s)$ with the specific decomposable pair or otherwise a homogeneous system of algebraic-differential equations $A(\rho)\beta(t) = 0$ with this prescribed behavior. Thus we can easily conclude to the following algorithm :

Algorithm. Construction of a system of algebraic-differential equations with given behavior.

Input : a) the smooth vector space spanned by the vectors $B^C =< \beta_i(t) = \left(\sum_{k=0}^{\sigma_i-1} \beta_{i,k} t^k\right) e^{s_i t} >$ where each $\beta_{i,k}$ is a vector in $\mathbb{C}^r, 0 \leq k \leq \sigma_i - 1, 1 \leq i \leq \ell$, and

b) the impulsive solution vector space spanned by the vectors $B^\infty =< \beta_i(t) = \sum_{k=0}^{\hat{q}_i-1} x_{i,k} \delta^{(\hat{q}_i-1-k)}(t) >$ where $x_{i,k} \in \mathbb{C}^r, 0 \leq k \leq \hat{q}_i - 1, 1 \leq i \leq \ell$.

Step 1. Define a finite Jordan pair of the form

$$C_i = \left(\begin{array}{cccc} \beta_{i,0} & \cdots & (\sigma_i-2)!\beta_{i,\sigma_i-2} & (\sigma_i-1)!\beta_{i,\sigma_i-1} \end{array}\right) \; ; \; J_i = \left(\begin{array}{cccc} s_i & 1 & \cdots & 0 \\ 0 & s_i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s_i \end{array}\right)$$

where $C_i \in \mathbb{R}^{r\times\sigma_i}, J_i \in \mathbb{R}^{\sigma_i\times\sigma_i}, i = 1,2,...,\ell$ and let

$$C = \left(\begin{array}{cccc} C_1 & C_2 & \cdots & C_\ell \end{array}\right) \in \mathbb{R}^{r\times n} \quad ; \quad J = blockdiag\left[J_1, J_2, ..., J_\ell\right] \in \mathbb{R}^{n\times n}$$

with J_i be the Jordan block of order σ_i with eigenvalue s_i and $n = \sum_{j=1}^{\ell} \sigma_i$.

Step 2. Define an infinite Jordan pair of the form

$$C_{\infty,i} = \left(\begin{array}{cccc} x_{i,0} & x_{i,1} & \cdots & x_{i,q+\hat{q}_i-1} \end{array}\right) \in \mathbb{R}^{r\times(q+\hat{q}_i)} \; ; \; J_{\infty,i} = \left(\begin{array}{ccccc} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{array}\right) \in \mathbb{R}^{(q+\hat{q}_i)\times(q+\hat{q}_i)}$$

where the vectors $x_{i,\hat{q}_i}, x_{i,\hat{q}_i+1}, ..., x_{i,q+\hat{q}_i-1}$ are unknowns and let

$$C_\infty = \left(\begin{array}{cccc} C_{\infty,1} & C_{\infty,2} & \cdots & C_{\infty,\ell} \end{array}\right) \in \mathbb{R}^{r\times\mu}$$
$$J_\infty = blockdiag\left[J_{\infty,1}, J_{\infty,2}, ..., J_{\infty,\ell}\right] \in \mathbb{R}^{\mu\times\mu}$$

with $J_{\infty,i}$ be the Jordan block of order $\mu_i = q+\hat{q}_i, i = v+1, v+2, ..., r$ with eigenvalue 0 and $\mu = \sum_{i=1}^{\ell} \mu_i$.

Step 3. Find the first q (starting from $q = 1$ and increasing by step 1) such that

$$S_{q-1} = \left(\begin{array}{cc} C & C_\infty J_\infty^{q-1} \\ CJ & C_\infty J_\infty^{q-2} \\ \vdots & \vdots \\ CJ^{q-1} & C_\infty \end{array}\right) \in \mathbb{R}^{rq\times(n+\mu)}$$

is invertible.

Step 4. Find V such that $\begin{pmatrix} S_{q-2}^T & V^T \end{pmatrix}^T$ is invertible, where S_{q-2} has been defined in (3.6) with (C, J) and (C_∞, J_∞) in place of (X_1, T_1) and (X_2, T_2) respectively.

Step 5. Find P and U_i, $i = 0, 1, ..., q-1$ using (3.5) and (3.7) respectively.

Step 6. Define $A(s)$ as in (3.4).

Output. B^C and B^∞ belongs to the solution space of $A(\rho)\beta(t) = 0$. ■

Note that any *fully unimodular equivalent* (see [4], [5] for definition) polynomial matrix of $A(s)$ is also a solution to our problem.

EXAMPLE 1. *Suppose that we want to find an AR-representation $A(\rho)\beta(t) = 0$ with the following smooth and impulsive solutions*

$$\beta_1(t) = \underbrace{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}_{\beta_{1,1}} e^{2t} + \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\beta_{1,0}} te^{2t} \quad ; \quad \beta_2(t) = \underbrace{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{x_{1,1}} \delta(t) + \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{x_{1,0}} \delta^{(1)}(t)$$

Step 1. *Define the finite Jordan pair*

$$C_f = \begin{pmatrix} \beta_{1,0} & \beta_{1,1} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}; J_f = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

Step 2. *Define the infinite Jordan pair*

$$C_\infty = \begin{pmatrix} 1 & 1 & x_1 & x_3 & x_5 & x_7 \\ 0 & -1 & x_2 & x_4 & x_6 & x_8 \end{pmatrix} ; \; J_\infty = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Step 3. *Since $\begin{pmatrix} C & C_\infty \end{pmatrix} \in \mathbb{R}^{2\times 8}$ then in order the matrix $S_{q-1} \in \mathbb{R}^{2q\times 8}$ to be invertible we must have $q = 4$. Let $S_{q-1} = S_{4-1} = S_3$ defined as*

$$S_3 = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 1 & 1 & x_1 \\ 1 & 1 & 0 & 0 & 0 & 0 & -1 & x_2 \\ 2 & -1 & 0 & 0 & 1 & 1 & x_1 & x_3 \\ 2 & 3 & 0 & 0 & 0 & -1 & x_2 & x_4 \\ 4 & 0 & 0 & 1 & 1 & x_1 & x_3 & x_5 \\ 4 & 8 & 0 & 0 & -1 & x_2 & x_4 & x_6 \\ 8 & 4 & 1 & 1 & x_1 & x_3 & x_5 & x_7 \\ 8 & 20 & 0 & -1 & x_2 & x_4 & x_6 & x_8 \end{pmatrix}$$

Let for example $x_3 = 1, x_2 = 0, x_1 = 0, x_4 = 0, x_5 = 0, x_6 = 0, x_7 = 0, x_8 = 0$. Then $\det S_3 = 65 \neq 0$.

Step 4. *Let*

$$S_2 = \begin{pmatrix} 1 & -1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 2 & -1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & -1 & 0 & 0 & 0 \\ 4 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 4 & 8 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We can find, by completing the basis formed by the rows of S_2,

$$V = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

such that $\det \begin{pmatrix} S_2^T & V^T \end{pmatrix}^T = 2 \neq 0$.

Step 5. *We construct* P *as follows*

$$P = \begin{pmatrix} I_2 & 0 \\ 0 & J_\infty \end{pmatrix} S_3^{-1} \begin{pmatrix} I_6 \\ 0 \end{pmatrix} S_2 = \begin{pmatrix} \frac{89}{65} & \frac{12}{13} & -\frac{3}{65} & 0 & 0 & 0 & 0 & \frac{29}{65} \\ -\frac{32}{65} & -\frac{3}{13} & \frac{4}{65} & 0 & 0 & 0 & 0 & -\frac{17}{65} \\ \frac{72}{65} & \frac{36}{13} & \frac{56}{65} & 0 & 0 & 0 & 0 & -\frac{108}{65} \\ -\frac{32}{13} & -\frac{80}{13} & \frac{4}{13} & 1 & 0 & 0 & 0 & -\frac{4}{13} \\ -\frac{48}{65} & -\frac{24}{13} & \frac{6}{65} & 0 & 1 & 0 & 0 & \frac{7}{65} \\ -\frac{8}{65} & -\frac{4}{13} & \frac{1}{65} & 0 & 0 & 1 & 0 & \frac{12}{65} \\ \frac{128}{65} & \frac{64}{13} & -\frac{16}{65} & 0 & 0 & 0 & 1 & -\frac{62}{65} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We find the matrices $U_i, i = 0, 1, 2, 3$ *as follows*

$$\begin{pmatrix} U_0 & U_1 & U_2 & U_3 \end{pmatrix} := S_3^{-1} = \left(\begin{array}{cc|cc|cc|cc} \frac{29}{65} & \frac{2}{5} & 0 & \frac{29}{65} & -\frac{3}{65} & -\frac{3}{65} & 0 & -\frac{3}{65} \\ -\frac{17}{65} & -\frac{1}{5} & 0 & -\frac{17}{65} & \frac{4}{65} & \frac{4}{65} & 0 & \frac{4}{65} \\ -\frac{63}{65} & -\frac{9}{5} & 0 & \frac{2}{65} & -\frac{54}{65} & -\frac{54}{65} & 1 & \frac{11}{65} \\ -\frac{108}{65} & -\frac{4}{5} & 0 & -\frac{108}{65} & \frac{56}{65} & \frac{56}{65} & 0 & -\frac{9}{65} \\ -\frac{4}{13} & 0 & 0 & -\frac{4}{13} & \frac{4}{13} & -\frac{9}{13} & 0 & \frac{4}{13} \\ \frac{7}{65} & \frac{1}{5} & 0 & -\frac{58}{65} & \frac{6}{65} & \frac{6}{65} & 0 & \frac{6}{65} \\ \frac{12}{65} & -\frac{4}{5} & 0 & \frac{12}{65} & \frac{1}{65} & \frac{1}{65} & 0 & \frac{1}{65} \\ -\frac{62}{65} & -\frac{6}{5} & 1 & \frac{3}{65} & -\frac{16}{65} & \frac{49}{65} & 0 & -\frac{16}{65} \end{array}\right)$$

Step 6. *Define the polynomial matrix* $A(s)$ *as follows :*

$$A(s) = V(I_8 - P)\begin{pmatrix} sI_2 - J_f & 0 \\ 0 & sJ_\infty - I_6 \end{pmatrix}(U_0 + U_1 s + \cdots + U_{q-1}s^{q-1}) =$$

$$= \begin{pmatrix} -\frac{19}{65}s^3 + \frac{82}{65}s^2 - \frac{98}{65}s + \frac{9}{65} & -\frac{19}{65}s^4 + \frac{63}{65}s^3 - \frac{16}{65}s^2 - \frac{108}{65}s + \frac{7}{5} \\ \frac{1}{65}s^3 - \frac{6}{65}s^2 + \frac{12}{65}s - \frac{7}{65} & \frac{1}{65}s^4 - \frac{1}{13}s^3 + \frac{6}{65}s^2 + \frac{6}{65}s - \frac{1}{5} \end{pmatrix}$$

It is easily checked that $S^{\mathbb{C}}_{A(s)}(s) = diag\left(1, (s-2)^2\right)$ *and* $S^{\infty}_{A(s)}(s) = diag\left(s^4, \frac{1}{s^2}\right)$. *Different choices of the variables* x_i *gives rise to different polynomial matrices* $A(s)$ *with the same finite and infinite zero structure and the same solution vector spaces. All these polynomial matrices are connected by the full unimodular equivalence relation [4], [5].*

4. Conclusions. An algorithm has been proposed for the construction of a system of algebraic and differential equations with prescribed smooth and impulsive behavior. The existing results extends the ones presented in [2] to the case where both the smooth and the impulsive behavior of systems is under consideration. Note that although, the results for the decomposable pairs where known, Gohberg never study the impulsive behavior of systems and this was the reason that these results are presented in this work for the first time. The proposed algorithm always gives rise to a regular polynomial matrix. However, the prescribed smooth and impulsive behavior, may be a solution of a non-regular AR-representation. Therefore, certain questions still arise as concerns the construction of a non-regular AR-representation with prescribed smooth-impulsive behavior.

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