Reachability and Controllability of Discrete Time Descriptor Systems

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***Abstract*—The causal and non-causal reachability Gramians are used, in order to define the reachability subspace which is the set of all states reachable from the origin. Controllability of the disrete-time regular descriptor system by means of the forward solution is examined and new criteria for testing reachability (controllability from the origin) and controllability to the origin in the forward sense are proved.**

I. Introduction

Discrete-time descriptor systems play an important role in system modeling, for example, in mechanical systems, in social dynamic procedures, in aerospace engineering, in financial problems etc. In previous years, there have been studies seeking for the generalization of existing definitions and theories, especially in the time domain, from normal systems to descriptor systems via different approaches. An approach often used is to transform the system into an equivalent form (e.g. Weierstrass canonical form), which offers prosperous knowledge of the underlying structure of singular systems. However these approaches, among the inconvinience in computation and numerical difficulties, also require a change of the internal variables which in many practical circumstances is not desirable, due to the structural significance of the original variables.

The parameters of the Laurent expansion of the generalized resolvent matrix $(zE-A)^{-1}$ are used in [2], where the forward solution (the initial condition x_0 and the input sequence are given and it is desired to determine x_k in a forward fashion), the backward solution (the final condition x_N and the input sequence are prescribed in order to determine x_k in a backward fashion) and the symmetric solution (the two-point boundary value problem where the inputs and the initial and final values of the state are given and it is desired to find the intermediate state values) are discussed depending on how the discrete singular equations are interpreted. In the same work there is a brief discussion about systems properties, which makes clear that reachability and observability are different depending on how the discrete singular equations are interpreted. The Laurent parameters where also used in [5] for the definition of the reachable subspace and the discrete-time reachability Gramians, the observable/unobservable subspace and the discrete-time observability Gramians.

In this note we use the forward fundamental matrix sequence and the causal and non-causal discrete-time reachability Gramians, in order to define the reachable subspace of desriptor systems and therefore to give a rank criterio for reachability. The definition of controllability in the forward sense of a descriptor system is given, along with a new criterio for testing controllability.

II. Preliminary results

Consider the linear time-invariant discrete time descriptor system given by the equations

$$\begin{aligned} Ex(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned} \tag{1}$$

where $E, A \in R^{r\times r}$ with E generally singular $B \in R^{r\times m}, C \in R^{p\times r}, u(k) \in R^m$, $x(k) \in R^r$, $y(k) \in R^p$. The system is assumed to be regular, i.e., $\det(zE-A) \neq 0$. For regular systems, there exists the Laurent series expansion about infinity of the resolvent matrix i.e.

$$(zE-A)^{-1} = z^{-1}\sum_{k=-\mu}^{\infty} \Phi_k z^{-k} \tag{2}$$

which is unique for some minimum μ, which is called the index of nillpotence of $zE-A$. If E is nonsingular then $\mu = 0$.

The sequence Φ_k is known as the (forward) fundamental matrix sequence of $(zE-A)$.The matrices $\Phi_k \in R^{r\times r}$ $(k = -\mu, \dots, -2, -1, 0, 1, \dots)$ are uniquely defined by the following properties :

Theorem 1: [1] With $(zE-A)$ regular and Φ_i defined by (2), the following properties between the fundamental matrices Φ_i holds :

1) $\Phi_i E - \Phi_{i-1}A = I\delta_i$,with δ_i the Kronecker delta.
2) $E\Phi_i - A\Phi_{i-1} = I\delta_i$.
3) $\Phi_i = 0$ for $i < -\mu$.
4) $\Phi_i = (\Phi_0 A)^i \Phi_0$, $i \geq 0$.
5) $\Phi_i = (\Phi_0 A)^i \Phi_0 = \Phi_{i-1}A\Phi_0$, $i = 1, 2, 3, \dots$
6) $\Phi_i = \Phi_0 (A\Phi_0)^i = \Phi_0 A\Phi_{i-1}$, $i = 1, 2, 3, \dots$
7) $\Phi_i = (-\Phi_{-1}E)^{-i-1}\Phi_{-1}$, $i < 0$.
8) $\Phi_{-i} = -\Phi_{-i+1}E\Phi_{-1} = (-\Phi_{-1}E)^{i-1}\Phi_{-1}$, $i = 2, 3, \dots \mu$.

9) $\Phi_{-i} = -\Phi_{-1}E\Phi_{-i+1} = \Phi_{-1}\left(-E\Phi_{-1}\right)^{i-1}, \ i = 2, 3, \dots \mu.$

10) $\Phi_i E\Phi_j = \Phi_j E\Phi_i$ for every i, j.

11) $\Phi_i E\Phi_j = \begin{cases} -\Phi_{i+j} & i<0, j<0 \\ \Phi_{i+j} & i\geq 0, j\geq 0 \\ 0 & ij\leq 0 \text{ and } |i|+|j|\neq 0 \end{cases}$
where $|.|$ is the absolute value of the argument number.

12) $\Phi_i A\Phi_j = \begin{cases} -\Phi_{i+j+1} & i<0, j<0 \\ \Phi_{i+j+1} & i\geq 0, j\geq 0 \\ 0 & ij\leq 0 \text{ and } |i|+|j|\neq 0 \end{cases}$
where $|.|$ is the absolute value of the argument number.

13) $\Phi_0 E\Phi_i = \begin{cases} \Phi_i & i\geq 0 \\ 0 & i<0 \end{cases}$

14) $-\Phi_{-1}A\Phi_i = \begin{cases} 0 & i\geq 0 \\ \Phi_i & i<0 \end{cases}$

15) $-\Phi_{-1}E\Phi_i = \begin{cases} 0 & i\geq 0 \\ \Phi_{i-1} & i<0 \end{cases}$

16) $\Phi_0 A\Phi_i = \begin{cases} \Phi_{i+1} & i\geq 0 \\ 0 & i<0 \end{cases}$

An extension of the Cayley Hamilton Theorem is given below.

Theorem 2: [2] The fundamental matrix sequence Φ_k satisfies the *generalized Cayley-Hamilton Theorem*

$$q_n\Phi_k + q_{n-1}\Phi_{k-1} + \dots + q_0\Phi_{k-n} = 0 \quad \text{for } k\geq n$$

where $n = \deg\left(\det\left(zE - A\right)\right)$ and q_i are defined by the expansion of $\det\left(zE - A\right)$ as in

$$q\left(z\right) = \det\left(zE - A\right) = \sum_{i=0}^{n} q_i z^i.$$

Based on the above Theorem we can now prove the following Lemma.

Lemma 3: If $x \in \ker B^T\Phi_i^T$ for $i = 0, 1, \dots, n-1$ then $x \in \ker B^T\Phi_i^T$ for $i \geq n$.

Proof: According to the *generalized Cayley-Hamilton Theorem* for $k = n$ we have

$$q_n\Phi_n + q_{n-1}\Phi_{n-1} + \dots + q_0\Phi_0 = 0 \overset{()^T}{\Rightarrow}$$
$$q_n\Phi_n^T + q_{n-1}\Phi_{n-1}^T + \dots + q_0\Phi_0^T = 0 \overset{\times B^T}{\Rightarrow}$$
$$q_nB^T\Phi_n^T + q_{n-1}B^T\Phi_{n-1}^T + \dots + q_0B^T\Phi_0^T = 0 \overset{\times x}{\Rightarrow}$$
$$q_nB^T\Phi_n^Tx + q_{n-1}B^T\Phi_{n-1}^Tx + \dots + q_0B^T\Phi_0^Tx = 0 \Rightarrow$$
$$\underset{i=0,1,\dots,n-1}{\overset{x\in\ker B^T\Phi_i^T}{\Rightarrow}} q_nB^T\Phi_n^Tx = 0$$

which means that $x \in \ker B^T\Phi_n^T$. Similarly, we can show that $x \in \ker B^T\Phi_i^T$ holds for every $i \geq n$. ■

III. On the reachability and controllability of discrete-time descriptor systems

There have been several interpretations of the equations (1). In this note we are concerned with the forward solution in terms of the fundamental matrix Φ_i and the properties of reachability and controllability presented by [2].

By assuming that the initial semistate value $x\left(0\right)$ is prescribed and $k \in Z_0^+$, the forward solution $x\left(k\right)$ is given by

$$x(k) = \Phi_k Ex(0) + \sum_{i=0}^{k+\mu-1}\Phi_{k-i-1}Bu(i) \tag{3}$$

In contrast to the state space case, the semistate $x\left(k\right)$, at time k, depends not only on the initial conditions and the previous inputs $u\left(i\right)$, $i<k$, but also on at most $\mu-1$ future inputs.

For every $x\left(0\right) \in R^r$ we have

$$x\left(0\right) = I_r x\left(0\right) \overset{\text{Theorem 1 (1)}}{=} \left(\Phi_0E - \Phi_{-1}A\right)x\left(0\right) =$$
$$= \Phi_0Ex\left(0\right) - \Phi_{-1}Ax\left(0\right) =$$
$$= \Phi_0Ex\left(0\right) - \Phi_{-1}\left(Ex\left(1\right) - Bu\left(0\right)\right) =$$
$$\overset{\text{Theorem 1 (1)}}{=} \Phi_0Ex\left(0\right) - \underbrace{\Phi_{-1}E}_{\Phi_{-2}A}x\left(1\right) + \Phi_{-1}Bu\left(0\right) =$$
$$= \Phi_0Ex\left(0\right) - \Phi_{-2}Ax\left(1\right) + \Phi_{-1}Bu\left(0\right)$$
$$= \Phi_0Ex\left(0\right) - \Phi_{-2}\left(Ex\left(2\right) - Bu\left(1\right)\right) + \Phi_{-1}Bu\left(0\right) =$$
$$= \Phi_0Ex\left(0\right) - \Phi_{-2}Ex\left(2\right) + \Phi_{-2}Bu\left(1\right) + \Phi_{-1}Bu\left(0\right)$$
$$\vdots$$
$$= \Phi_0Ex\left(0\right) - \underbrace{\Phi_{-\mu}E}_{0}x\left(\mu\right) + \Phi_{-\mu}Bu\left(\mu-1\right) +$$
$$+\Phi_{-\mu+1}Bu\left(\mu-2\right) + ... + \Phi_{-2}Bu\left(1\right) + \Phi_{-1}Bu\left(0\right)$$
$$= \Phi_0Ex\left(0\right) + \sum_{i=0}^{\mu-1}\Phi_{-i-1}Bu\left(i\right)$$

and thus

$$x(0) = \Phi_0Ex(0) + \sum_{i=0}^{\mu-1}\Phi_{-i-1}Bu(i)$$

which coincides with (3) if we set $k = 0$.

Definition 4: [6] The set H_{Iu} of consistent initial conditions with inputs is defined as

$$H_{Iu} = \left\{x(0) \in R^r / x(0) = \Phi_0Ex(0) + \sum_{i=0}^{\mu-1}\Phi_{-i-1}Bu(i)\right\}$$

It is obvious that $x(0) = 0_r \in H_{Iu}$ for $u(i) = 0, i = 0, 1, ..., \mu-1$.

Definition 5: [2] A **state** $z \in R^r$ is called *reachable in the forward sense* if there exists a $k>0$ and a control sequence $u_{0,k+\mu} = \left\{u\left(0\right), u\left(1\right), \dots, u\left(k+\mu-1\right)\right\}$ such that $x\left(k\right) = z$ when $x\left(0\right) = 0 \in H_{Iu}$.

Definition 6: [2] The **regular system** (1) is *reachable in the forward sense* if, for every $z \in R^r$, there exists a $k>0$ and a control $u_{0,k+\mu}$ such that $x\left(k\right) = z$ when $x\left(0\right) = 0$ (*controllability from the origin*).

Motivated by the above definitions we examine the set of all states $z \in R^r$ that are reachable in the forward sense.

Definition 7: Let $R(0)$ be the set of all states $z \in R^r$ that are reachable from 0 in the forward sense i.e.

$$R(0) = \{z \in R^r / \exists\ u_{0,k+\mu} \text{ such that } x(k) = z\}.$$

We define the *causal discrete-time descriptor system reachability Gramian* as

$$W_c(0,k) = \sum_{i=0}^{k-1} \Phi_{k-i-1} B B^T \Phi_{k-i-1}^T = \ell_{c,k} \ell_{c,k}^T$$

where

$$\ell_{c,k} = \begin{pmatrix} \Phi_0 B & \cdots & \Phi_{k-2} B & \Phi_{k-1} B \end{pmatrix} = \\ = \begin{pmatrix} \Phi_0 B & \cdots & (\Phi_0 A)^{k-1} \Phi_0 B \end{pmatrix} \in R^{r \times km}$$

and the *non-causal discrete-time descriptor system reachability Gramian* as

$$W_{nc}(-1,-\mu) = \sum_{i=-\mu}^{-1} \Phi_i B B^T \Phi_i^T = \ell_{nc} \ell_{nc}^T$$

where

$$\ell_{nc} = \begin{pmatrix} \Phi_{-\mu} B & \Phi_{-\mu+1} B & \cdots & \Phi_{-1} B \end{pmatrix} = \\ = \begin{pmatrix} (-\Phi_{-1} E)^{-\mu-1} \Phi_{-1} B & \ldots & \Phi_{-1} B \end{pmatrix} \in R^{r \times \mu m}$$

We introduce the notation

$$\langle \Phi / \operatorname{Im} B \rangle = \Phi_{n-1} \operatorname{Im} B + \Phi_{n-2} \operatorname{Im} B + ... + \Phi_0 \operatorname{Im} B$$

$$\left\langle \bar{\Phi} / \operatorname{Im} B \right\rangle = \Phi_{-1} \operatorname{Im} B + \Phi_{-2} \operatorname{Im} B + ... + \Phi_{-\mu} \operatorname{Im} B$$

Lemma 8: The space $\langle \Phi / \operatorname{Im} B \rangle$ coincides with the range of the causal discrete-time descriptor system reachability Gramian, i.e.

$$\langle \Phi / \operatorname{Im} B \rangle = \operatorname{Im} W_c(0,k) \text{ for } k \geq n \tag{4}$$

Proof: The space $\langle \Phi / \operatorname{Im} B \rangle$ is the subspace of $\mathbb{R}^r$ spanned by the linear independent columns of the matrix $\ell_{c,n}$. Instead of showing (4), we can show the equivalent condition

$$Ker W_c^T(0,k) = Ker \ell_{c,n}^T \text{ for } k \geq n.$$

$W_c(0,k)$ is symmetric and therefore $W_c^T(0,k) = W_c(0,k)$. Thus

$$Ker W_c(0,k) = Ker \ell_{c,n}^T \text{ for } k \geq n \ \Leftrightarrow \\ Ker W_c(0,k) = \cap_{i=0}^{n-1} Ker B^T \Phi_i^T \text{ for } k \geq n.$$

We first show that $Ker W_c(0,k) \subseteq \cap_{i=0}^{n-1} Ker B^T \Phi_i^T$ for $k \geq n$. If $x \in Ker W_c(0,k)$ for $k \geq n$ and $x \neq 0$ then

$$\begin{aligned} x^T W_c(0,k) x &= 0 \Rightarrow \\ \sum_{i=0}^{k-1} x^T \Phi_{k-i-1} B B^T \Phi_{k-i-1}^T x &= 0 \Rightarrow \\ \sum_{i=0}^{k-1} \left\| B^T \Phi_{k-i-1}^T x \right\|^2 &= 0 \text{ for } k \geq n \end{aligned}$$

For $k = n$ we have

$$\sum_{i=0}^{n-1} \left\| B^T \Phi_{n-i-1}^T x \right\|^2 = 0 \ \Rightarrow B^T \Phi_{n-i-1}^T x = 0$$

for $i = 0, 1, ..., n-1$, which implies that $x \in \cap_{i=0}^{n-1} Ker B^T \Phi_i^T$ and $Ker W_c(0,k) \subseteq \cap_{i=0}^{n-1} Ker B^T \Phi_i^T$.

For the reverse inclusion i.e. $Ker W_c(0,k) \supseteq \cap_{i=0}^{n-1} Ker B^T \Phi_i^T$, suppose that $x \in \cap_{i=0}^{n-1} Ker B^T \Phi_i^T$. Then we have

$$B^T \Phi_0^T x = B^T \Phi_1^T x = \ldots = B^T \Phi_{n-1}^T x = 0$$

Thus from Lemma 3, $x \in Ker B^T \Phi_k^T$ for $k \geq n$. Then

$$W_c(0,k) x = \sum_{i=0}^{k-1} \Phi_{k-i-1} B B^T \Phi_{k-i-1}^T x = 0 \text{ for } k \geq n.$$

which means that $x \in Ker W_c(0,k)$ for $k \geq n$ and thus $Ker W_c(0,k) \supseteq \cap_{i=0}^{n-1} Ker B^T \Phi_i^T$. ■

Lemma 9: The space $\left\langle \bar{\Phi} / \operatorname{Im} B \right\rangle$ coincides with the range of the non-causal discrete-time descriptor system reachability Grammian, i.e.

$$\left\langle \bar{\Phi} / \operatorname{Im} B \right\rangle = \operatorname{Im} W_{nc}(-1,-\mu) \tag{5}$$

Proof: The space $\left\langle \bar{\Phi} / \operatorname{Im} B \right\rangle$ is the subspace of $\mathbb{R}^r$ spanned by the linear independent columns of the matrix ℓ_{nc}. Instead of showing (5), we can show the equivalent condition

$$\ker W_{nc}^T(-1,-\mu) = \ker \ell_{nc}^T$$

Since $W_{nc}(-1,-\mu)$ is symmetric, we have that $W_{nc}^T(-1,-\mu) = W_{nc}(-1,-\mu)$. Thus

$$\ker W_{nc}(-1,-\mu) = \ker \ell_{nc}^T \Leftrightarrow \\ \ker W_{nc}(-1,-\mu) = \cap_{i=-\mu}^{-1} \ker B^T \Phi_i^T$$

First we prove that $\ker W_{nc}(-1,-\mu) \subseteq \cap_{i=-\mu}^{-1} \ker B^T \Phi_i^T$. Suppose that $x \in \ker W_{nc}(-1,-\mu)$. Then we have

$$x^T W_{nc}(-1,-\mu) x = 0 \Rightarrow \sum_{i=-\mu}^{-1} x^T \Phi_i B B^T \Phi_i^T x = 0 \Rightarrow \\ \sum_{i=-\mu}^{-1} \left\| B^T \Phi_i^T x \right\|^2 = 0$$

which means that $B^T \Phi_{-\mu}^T x = B^T \Phi_{-\mu+1}^T x = \ldots = B^T \Phi_{-1}^T x = 0$. So $x \in \cap_{i=-\mu}^{-1} \ker B^T \Phi_i^T$.

For the inverse induction $\ker W_{nc}(-1,-\mu) \supseteq \cap_{i=-\mu}^{-1} \ker B^T \Phi_i^T$, suppose that $x \in \cap_{i=0}^{n-1} \ker B^T \Phi_i^T$. Then we have

$$W_{nc}(-1,-\mu) x = \sum_{i=-\mu}^{-1} \Phi_i B B^T \Phi_i^T x = 0$$

and thus $x \in \ker W_{nc}(-1,-\mu)$. ■

In what follows we show the relation between the reachable space and the spaces $\langle \Phi / \operatorname{Im} B \rangle$ and$\left\langle \bar{\Phi} / \operatorname{Im} B \right\rangle$.

Theorem 10: The set of all reachable states from 0 in the forward sense is

$$R(0) = \langle \Phi / \operatorname{Im} B \rangle \oplus \langle \bar{\Phi} / \operatorname{Im} B \rangle$$

Proof: First we prove that $R(0) \subseteq \langle \Phi / \operatorname{Im} B \rangle \oplus \langle \bar{\Phi} / \operatorname{Im} B \rangle$. Let $z \in R^r$ and assume that $z \in R(0)$. Then by the definition of $R(0)$, there exists a control sequence $u_{0,k+\mu}$ such that $z = x(k)$ when $x(0) = 0$. This means that

$$z = x(k) = \Phi_{k-1} Bu(0) + \ldots + \Phi_0 Bu(k-1) +$$
$$+ \Phi_{-1} Bu(k) + \Phi_{-2} Bu(k+1) + \ldots + \Phi_{-\mu} Bu(k+\mu-1)$$

By using Theorem 2, repeatedly for every $k \geq n$ we conclude that Φ_k for $k \geq n$ depend on Φ_i for $i = 0, 1, .., n-1$ and thus $z \in \langle \Phi / \operatorname{Im} B \rangle \oplus \langle \overline{\Phi} / \operatorname{Im} B \rangle$ or otherwise $R(0) \subseteq \langle \Phi / \operatorname{Im} B \rangle \oplus \langle \bar{\Phi} / \operatorname{Im} B \rangle$.

For the proof of the inverse induction $R(0) \supseteq \langle \Phi / \operatorname{Im} B \rangle \oplus \langle \bar{\Phi} / \operatorname{Im} B \rangle$, let $z \in \langle \Phi / \operatorname{Im} B \rangle \oplus \langle \bar{\Phi} / \operatorname{Im} B \rangle$. $z \in R^r$ is reachable in the forward sense if there exists a $k > 0$ and a control sequence $u_{0,k+\mu}$ such that $x(k) = z$ when $x(0) = 0$. $x(0) = 0 \in H_{Iu}$ if the first μ control input vectors $u(0) = \tilde{u}_0, u(1) = \tilde{u}_1, \ldots, u(\mu-1) = \tilde{u}_{\mu-1}$ satisfy the following equation of consistency

$$0 = \begin{pmatrix} \Phi_{-1} B & \Phi_{-2} B & \cdots & \Phi_{-\mu} B \end{pmatrix} \begin{pmatrix} \tilde{u}_0 \\ \tilde{u}_1 \\ \vdots \\ \tilde{u}_{\mu-1} \end{pmatrix}$$

In case where ℓ_{nc} doesn't lose rank we select $\tilde{u}_0 = \tilde{u}_1 = \cdots = \tilde{u}_{\mu-1} = 0$, otherwise, we select

$$\tilde{u}_0 = B^T \Phi_{-1}^T w_0, \tilde{u}_1 = B^T \Phi_{-2}^T w_0, .., \tilde{u}_{\mu-1} = B^T \Phi_{-\mu}^T w_0$$

where $w_0 \in Ker W_{nc}(-1, -\mu)$. We write z as

$$z = z_c + z_{nc} \in \langle \Phi / \operatorname{Im} B \rangle \oplus \langle \bar{\Phi} / \operatorname{Im} B \rangle$$

where

$$z_c \in \langle \Phi / \operatorname{Im} B \rangle \quad \text{and} \quad z_{nc} \in \langle \bar{\Phi} / \operatorname{Im} B \rangle .$$

We will determine a control input $u_{0,n+\mu}$ such that

$$x(n+\mu) = z = \underbrace{\sum_{i=0}^{n+\mu-1} \Phi_{n+\mu-i-1} Bu(i) +}_{z_c}$$
$$+ \underbrace{\sum_{i=n+\mu}^{n+2\mu-1} \Phi_{n+\mu-i-1} Bu(i)}_{z_{nc}}$$

For the causal part z_c we shall have

$$z_c = \sum_{i=0}^{n+\mu-1} \Phi_{n+\mu-i-1} Bu(i) =$$
$$= \underbrace{\sum_{i=0}^{\mu-1} \Phi_{n+\mu-i-1} Bu(i)}_{z_c'} + \sum_{i=\mu}^{n+\mu-1} \Phi_{n+\mu-i-1} Bu(i)$$

According to the generalized Cayley-Hamilton theorem

$$z_c' = \sum_{i=0}^{\mu-1} \Phi_{n+\mu-i-1} Bu(i) \in \langle \Phi / \operatorname{Im} B \rangle$$

or otherwise according to Lemma 8, $z_c' \in \operatorname{Im}(W_c(0,k))$ for $k \geq n$. Based on the same Lemma we also have that

$$z_c \in \langle \Phi / \operatorname{Im} B \rangle \equiv \operatorname{Im}(W_c(0,k)) \text{ for } k \geq n.$$

For $k = n$ we have $z_c \in \operatorname{Im}(W_c(0,n))$ and $z_c' \in \operatorname{Im}(W_c(0,n))$ or equivalently there exists a non-zero vector $w_1 \in R^r$ such that

$$\begin{aligned} z_c - z_c' &= W_c(0,n) w_1 = \sum_{i=0}^{n-1} \Phi_{n-i-1} BB^T \Phi_{n-i-1}^T w_1 = \\ &= \sum_{i=\mu}^{n+\mu-1} \Phi_{n+\mu-i-1} BB^T \Phi_{n+\mu-i-1}^T w_1 \end{aligned}$$

If we choose for $i \leq n+\mu-1$ the following input

$$u(i) = \begin{cases} B^T \Phi_{-i-1}^T w_0 & i \leq \mu-1 \\ B^T \Phi_{n+\mu-i-1}^T w_1 & \mu \leq i \leq n+\mu-1 \end{cases}$$

we get that

$$\sum_{i=0}^{n+\mu-1} \Phi_{n+\mu-i-1} Bu(i) =$$
$$= \sum_{i=0}^{\mu-1} \Phi_{n+\mu-i-1} B\tilde{u}_i + \sum_{i=\mu}^{n+\mu-1} \Phi_{n+\mu-i-1} BB^T \Phi_{n+\mu-i-1}^T w_1 =$$
$$= z_c' + (z_c - z_c') = z_c$$

For the non-causal part $z_{nc} = \sum_{i=n+\mu}^{n+2\mu-1} \Phi_{n-i-1} Bu(i)$, we have, according to Lemma 9, that

$$z_{nc} \in \langle \overline{\Phi} / \operatorname{Im} B \rangle \equiv \operatorname{Im}(W_{nc}(-1, -\mu))$$

and therefore there exists a non-zero vector $w_2 \in R^r$ such that

$$z_{nc} = W_{nc}(-1, -\mu) w_2 = \sum_{i=-\mu}^{-1} \Phi_i BB^T \Phi_i^T w_2$$

or otherwise

$$\begin{aligned} z_{nc} &= \Phi_{-1} Bu(n+\mu) + \ldots + \Phi_{-\mu} Bu(n+2\mu-1) \equiv \\ &\equiv \Phi_{-1} BB^T \Phi_{-1}^T w_2 + \ldots + \Phi_{-\mu} BB^T \Phi_{-\mu}^T w_2 \end{aligned}$$

Thus by choosing $u(i) = B^T \Phi_{n+\mu-i-1}^T w_2$ for $i \geq n+\mu$ we have that,

$$\sum_{i=n+\mu}^{n+2\mu-1} \Phi_{n-i-1} Bu(i) =$$
$$= \sum_{i=n+\mu}^{n+2\mu-1} \Phi_{n+\mu-i-1} BB^T \Phi_{n+\mu-i-1}^T w_2 = z_{nc}$$

Therefore the input that transfers $x(0) = 0$ to $x(n) = z$ is the following

$$u(i) = \begin{cases} B^T \Phi_{-i-1}^T w_0 & \text{for } i \leq \mu-1 \\ B^T \Phi_{n+\mu-i-1}^T w_1 & \text{for } i \leq n+\mu-1 \\ B^T \Phi_{n+\mu-i-1}^T w_2 & \text{for } i \leq n+2\mu-1 \end{cases}$$

where $w_i, i = 0,1,2$ have been defined above. ■

Note that both $x(0) = 0$, and $x(0) \in Ker(\Phi_0 E)$ gives rise (according to (3)) to the same solution, and therefore to the same reachable subspace. $\langle \Phi / \operatorname{Im} B \rangle$ (resp. $\langle \overline{\Phi} / \operatorname{Im} B \rangle$) is called [5] the *causal* (resp. *noncausal*) *reachable subspace*. From Theorem 10, we conclude that the system (1) is reachable iff

$$\langle \Phi / \operatorname{Im} B \rangle \oplus \langle \bar{\Phi} / \operatorname{Im} B \rangle = R^r \tag{6}$$

Therefore we conclude to the following Theorem.

Theorem 11: The regular system (1) is reachable in the forward sense if and only if

$$rank\left(U_n\right) = r$$

where the forward reachability matrix is defined as

$$U_k = \left[\begin{array}{cc} \ell_{nc} & \ell_{c,k} \end{array}\right]$$

with $n = \deg(\det(zE - A))$.

Note that the above Theorem has initially proved by [2]. By using (4), (5) and (6) we conclude that the system is reachable iff

$$\operatorname{Im}\left(W_c\left(0,k\right)\right) \oplus \operatorname{Im}(W_{nc}(-1,-\mu)) = R^r \text{ for } k \geq n$$

which means that

$$rank\left[\begin{array}{cc} W_c\left(0,k\right) & W_{nc}(-1,-\mu) \end{array}\right] = r \text{ for } k \geq n$$

Due to Theorem 2 we have that

$$\begin{gathered} rank\left[\begin{array}{cc} W_c\left(0,k\right) & W_{nc}(-1,-\mu) \end{array}\right] = \\ = rank\left[\begin{array}{cc} W_c\left(0,n\right) & W_{nc}(-1,-\mu) \end{array}\right] \text{ for } k \geq n \end{gathered} \tag{7}$$

Lemma 12: [8] Let $A, B \in \mathbb{R}^{r \times r}$ and assume that A and B are positive semidefinite. Then

$$rank\left(A + B\right) = rank\left[\begin{array}{cc} A & B \end{array}\right]$$

Since the matrices $W_c\left(0,n\right), W_{nc}(-1,-\mu)$ are positive semidefinite, from (7) and using Lemma 12 we conclude to the following Theorem.

Theorem 13: The regular system (1) is reachable in the forward sense if and only if

$$\begin{gathered} rank\left[W_c\left(0,n\right) + W_{nc}\left(-1,-\mu\right)\right] = \\ = rank\left[\begin{array}{cc} W_c\left(0,n\right) & W_{nc}\left(-1,-\mu\right) \end{array}\right] = r \end{gathered}$$

where $n = \deg(\det(zE - A))$.

Corollary 14: Note that in case of state space systems i.e. $E = I_r$, we have that $\Phi_i = 0 \ \forall i < 0$ and $\Phi_i = A^i, \forall i > 0$. Thus

$$W_{nc}(-1,-\mu) = 0, W_c\left(0,k\right) = \sum_{i=0}^{k-1} A^{k-i-1} B B^T \left(A^T\right)^{k-i-1}$$

and the rank criterio is reduced to

$$rank\left(\sum_{i=0}^{k-1} A^{k-i-1} B B^T \left(A^T\right)^{k-i-1}\right) = r$$

as given by [4].

Definition 15: A **state** $x_0 \in H_{Iu}$ is *controllable to the origin* in the forward sense if there exists a control sequence $u_{0,k+\mu}$ such that $x(k) = 0$.

Definition 16: The **regular system** (1) is *controllable to the origin* in the forward sense if, for every $x_0 \in H_{Iu}$, there exists a finite $k > 0$ and a control $u_{0,k+\mu}$ such that $x(k) = 0$.

A criterio for testing controllability to the origin is given below.

Theorem 17: The regular system (1) is controllable to the origin in the forward sense if

$$\left(\begin{array}{c} -\left(\Phi_0 A\right)^n \\ \Phi_{-1} A \end{array}\right) x_0 \in \operatorname{Im}\left[\left(\begin{array}{cccc} \Phi_0 B & \cdots & \Phi_{n-2} B & \Phi_{n-1} B \\ \Phi_{-n} B & \cdots & \Phi_{-2} B & \Phi_{-1} B \end{array}\right)\right] \tag{8}$$

Proof: Suppose that $x(0) = x_0 \in H_{Iu}$ i.e. the first μ control input vectors $u(0) = \tilde{u}_0, u(1) = \tilde{u}_1, \ldots, u(\mu-1) = \tilde{u}_{\mu-1}$ satisfy the following equation of consistency

$$x_0 = \Phi_0 E x_0 + \sum_{i=0}^{\mu-1} \Phi_{-i-1} B \tilde{u}_i \Leftrightarrow$$

$$\Phi_{-1} A x_0 = \left(I - \Phi_0 E\right) x_0 = \sum_{i=0}^{\mu-1} \Phi_{-i-1} B \tilde{u}_i \tag{9}$$

For $k \geq 1$ we have

$$\begin{gathered} x(k) = \Phi_k E x(0) + \sum_{i=0}^{k+\mu-1} \Phi_{k-i-1} B u(i) = \\ = \left[\left(\Phi_0 A\right)^k \Phi_0\right] E x\left(0\right) + \sum_{i=0}^{k+\mu-1} \Phi_{k-i-1} B u(i) = \\ = \left(\Phi_0 A\right)^k \left(\Phi_{-1} A + I\right) x\left(0\right) + \sum_{i=0}^{k+\mu-1} \Phi_{k-i-1} B u(i) = \\ = \left(\Phi_0 A\right)^k \Phi_{-1} A x\left(0\right) + \left(\Phi_0 A\right)^k x\left(0\right) + \sum_{i=0}^{k+\mu-1} \Phi_{k-i-1} B u(i) \\ \overset{\text{Theorem 1 (16)}}{=} \left(\Phi_0 A\right)^k x\left(0\right) + \sum_{i=0}^{k+\mu-1} \Phi_{k-i-1} B u(i) \end{gathered}$$

The system is controllable iff for every $x_0 \in H_{Iu}$, there exists a $k > 0$ and a control $u_{0,k+\mu}$ such that $x(k) = 0$ i.e.

$$x\left(k\right) = \left(\Phi_0 A\right)^k x_0 + \sum_{i=0}^{k+\mu-1} \Phi_{k-i-1} B u(i) = 0$$

or equivalently

$$\begin{gathered} 0 = \left(\Phi_0 A\right)^k x_0 + \Phi_{k-1} B u\left(0\right) + \ldots + \Phi_0 B u\left(k-1\right) + \\ + \Phi_{-1} B u\left(k\right) + \Phi_{-2} B u\left(k+1\right) + \ldots + \Phi_{-\mu} B u\left(k+\mu-1\right) \end{gathered}$$

From Theorem 2 and [2], it follows that it is not necessary to take more than $n + \mu - 1$ steps in the control sequence since

$$\Phi_k B = \tilde{q}_{k-1} \Phi_{k-1} B + \ldots + \tilde{q}_0 \Phi_{k-n} B \text{ for } k \geq n.$$

Thus for $k = n$ we have

$$\begin{gathered} 0_{r \times 1} = \left(\Phi_0 A\right)^n x_0 + \\ + \left[\begin{array}{cccc} \Phi_0 B & \ldots & \Phi_{n-2} B & \Phi_{n-1} B \end{array}\right] U_{n-1,0} + \\ + \left[\begin{array}{cccc} \Phi_{-1} B & \Phi_{-2} B & \ldots & \Phi_{-\mu} B \end{array}\right] U_{n,n+\mu-1} \end{gathered}$$

where

$$U_{n-1,0}=\begin{bmatrix} u(n-1)\\ \vdots \\ u(1)\\ u(0)\end{bmatrix}, U_{n,n+\mu-1}=\begin{bmatrix} u(n)\\ u(n+1)\\ \dots \\ u(n+\mu-1)\end{bmatrix}$$

Premultiplying the above equation by the full column rank matrix $\begin{bmatrix}\Phi_0 E\\ \Phi_{-1}A\end{bmatrix} \in \mathbb{R}^{2r\times r}$, it can be broken into two equations as follows

$$\begin{bmatrix}\Phi_0 E\\ \Phi_{-1}A\end{bmatrix} 0_{r\times 1} = \begin{bmatrix}\Phi_0 E\\ \Phi_{-1}A\end{bmatrix} (\Phi_0 A)^n x_0 +$$
$$+\begin{bmatrix}\Phi_0 E\\ \Phi_{-1}A\end{bmatrix}\begin{bmatrix}\Phi_0 B & \dots & \Phi_{n-2}B & \Phi_{n-1}B\end{bmatrix} U_{n-1,0}+$$
$$+\begin{bmatrix}\Phi_0 E\\ \Phi_{-1}A\end{bmatrix}\begin{bmatrix}\Phi_{-1} B & \Phi_{-2}B & \dots & \Phi_{-\mu}B\end{bmatrix} U_{n,n+\mu-1}$$

which is reduced to the following equations

$$\begin{aligned}0_{r\times 1} &= \Phi_0 E (\Phi_0 A)^n x_0 +\\ &+\Phi_0 E\begin{bmatrix}\Phi_0 B & \dots & \Phi_{n-2}B & \Phi_{n-1}B\end{bmatrix} U_{n-1,0}+\\ &+\Phi_0 E\begin{bmatrix}\Phi_{-1} B & \Phi_{-2}B & \dots & \Phi_{-\mu}B\end{bmatrix} U_{n,n+\mu-1}\end{aligned} \quad (10)$$

and

$$\begin{aligned}0_{r\times 1} &= \Phi_{-1} A (\Phi_0 A)^n x_0 +\\ &+\Phi_{-1} A\begin{bmatrix}\Phi_0 B & \dots & \Phi_{n-2}B & \Phi_{n-1}B\end{bmatrix} U_{n-1,0}+\\ &+\Phi_{-1} A\begin{bmatrix}\Phi_{-1} B & \Phi_{-2}B & \dots & \Phi_{-\mu}B\end{bmatrix} U_{n,n+\mu-1}\end{aligned} \quad (11)$$

According to the properties of the fundamental matrix sequence in Theorem 1, we have

$$\Phi_0 E (\Phi_0 A)^n x_0 = \underbrace{\Phi_0 E \Phi_0}_{\Phi_0} A (\Phi_0 A)^{n-1} x_0 =$$
$$= \Phi_0 A (\Phi_0 A)^{n-1} x_0 = (\Phi_0 A)^n x_0$$

and

$$\Phi_0 E\begin{bmatrix}\Phi_0 B & \dots & \Phi_{n-2}B & \Phi_{n-1}B\end{bmatrix} =$$
$$\overset{\text{Theorem 1 (13)}}{=} \begin{bmatrix}\Phi_0 B & \dots & \Phi_{n-2}B & \Phi_{n-1}B\end{bmatrix}$$

Similarly

$$\Phi_0 E\begin{bmatrix}\Phi_{-1} B & \Phi_{-2}B & \dots & \Phi_{-\mu}B\end{bmatrix} \overset{\text{Theorem 1 (13)}}{=} 0_{r\times 1}$$

and

$$\Phi_{-1} A (\Phi_0 A)^n x_0 = \underbrace{\Phi_{-1} A \Phi_0}_{0} A (\Phi_0 A)^{n-1} x_0 = 0_{r\times 1}$$

$$\Phi_{-1} A\begin{bmatrix}\Phi_0 B & \dots & \Phi_{n-2}B & \Phi_{n-1}B\end{bmatrix} \overset{\text{Th. 1 (14)}}{=} 0_{r\times 1}$$

$$\Phi_{-1} A\begin{bmatrix}\Phi_{-1} B & \Phi_{-2}B & \dots & \Phi_{-\mu}B\end{bmatrix} \overset{\text{Theorem 1 (14)}}{=}$$
$$= -\begin{bmatrix}\Phi_{-1} B & \Phi_{-2}B & \dots & \Phi_{-\mu}B\end{bmatrix}$$

Therefore, (10) and (11) are reduced to

$$(\Phi_0 A)^n x_0 + \begin{bmatrix}\Phi_0 B & \dots & \Phi_{n-1}B\end{bmatrix} U_{n-1,0} = 0 \quad (12)$$

$$\begin{bmatrix}\Phi_{-1} B & \Phi_{-2}B & \dots & \Phi_{-\mu}B\end{bmatrix} U_{n,n+\mu-1} = 0 \quad (13)$$

Equations (9) and (12) can be written together as follows

$$\begin{pmatrix} -(\Phi_0 A)^n \\ \Phi_{-1}A\end{pmatrix} x_0 = \begin{pmatrix}\Phi_0 B & \cdots & \Phi_{n-2}B & \Phi_{n-1}B\\ \Phi_{-n}B & \cdots & \Phi_{-2}B & \Phi_{-1}B\end{pmatrix} U_{n-1,0} \quad (14)$$

where $\Phi_k = 0$ for $k < -\mu$. Equation (14) is satisfied for every $x_0 \in H_{Iu}$ if the condition (8) is satisfied.

Suppose condition (8) is satisfied. Then for every $x_0 \in H_{Iu}$ there exists a control sequence $U_{0,n-1}$ such that equation (14) is satisfied. If we choose the input sequence as

$$u(k) = \begin{cases} u(k) & \text{for } 0 \le k \le n-1 \\ 0 & \text{for } n \le k \le n+\mu-1\end{cases}$$

then equation (13) is also satisfied. ■

Note that the results of Theorem 17 coincides with the results of Theorem 3.1 in [3] concerning reachability in the closed interval $[0, N]$ for the final value $x_N = x(N) = 0$.

Corollary 18: Note that in case of state space systems i.e. $E = I_r$, we have that $n = r$, $\Phi_i = 0\ \forall i < 0$ and $\Phi_i = A^i, \forall i > 0$. Thus $(\Phi_0 A)^n = A^n$ and we have that x_0 is controllable if $A^n x_0 \in \operatorname{Im}\begin{bmatrix} B & AB & \dots & A^{n-1}B\end{bmatrix}$ as given by ([4], pp.245)

Corollary 19: The initial condition x_0 is written as $x_0 = (\Phi_0 E - \Phi_{-1}A) x_0 = \Phi_0 E x_0 - \Phi_{-1} A x_0 = x_0^c + x_0^{nc}$. The second part of the initial condition vector i.e. x_0^{nc}, is always driven to the origin with zero input, since $x_k = \Phi_k E[-\Phi_{-1} A x_0] + \sum_{i=0}^{k+\mu-1} \Phi_{k-i-1} B \times 0 \overset{\text{Theorem 1 (11)}}{=} 0$. Therefore, we are interested to drive only $x_0^c = \Phi_0 E x_0 \in H_{iu}$ to the origin.

References

[1] C.E. Langenhop, The Laurent expansion for a nearly singular matrix, Linear Algebra Appl. 4 (1971) 329–340.

[2] F.L. Lewis, B.G. Mertzios, On the analysis of discrete linear time-invariant singular systems, IEEE Trans. Autom. Control 35 (4) (1990) 506–511.

[3] F.L. Lewis, Fundamental, reachability and observability matrices for discrete descriptor systems, IEEE Trans. Autom. Control, 30, (5), 502-505.

[4] P.J. Antsaklis and A.N. Michel, 2006, Linear Systems, Birkhauser, Boston, 2nd Corrected Printing.

[5] D.J. Bender, Lyapunov-Like Equations and Reachability/Observability Gramians for Descriptor Systems, IEEE Trans. Automat. Control, vol. AC-32, No.4, (1987) pp.343-348

[6] Karampetakis N. P., Jones J. and Antoniou S., 2001, Forward, backward and symmetric solutions of discrete ARMA-representations, Circuit Systems & Signal Process, Vol.20, No.1, pp.89-109.

[7] T.Kailath, 1980, Linear Systems, Prentice-Hall, INC.,Englewood Cliffs, N.J. 07632

[8] D. Bernstein, 2009, Matrix Mathematics, Theory, Facts and Formulas, Second Edition, Princeton University Press.