Notions of equivalence for linear multivariable systems

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***Abstract*— The present paper is a survey on linear multivariable systems equivalences. We attempt a review of the most significant types of system equivalence having as a starting point matrix transformations preserving certain types of their spectral structure. From a system theoretic point of view, the need for a variety of forms of polynomial matrix equivalences, arises from the fact that different types of spectral invariants give rise to different types of dynamics of the underlying linear system. A historical perspective of the key results and their contributors is also given.**

***Index Terms*— Linear Systems, Polynomial Matrix, System Equivalence, Continuous Time System, Discrete Time System**

I. Introduction

In this survey work we consider linear time invariant systems and the respective equivalence relations amongst them. The class of linear multivariable systems has been studied through a variety of models. In what follows $\mathbb{R}, \mathbb{C}$ denote the fields of real and complex numbers, respectively, $\mathbb{R}[s]$ the ring of polynomials with real coefficients, $\mathbb{R}(s)$ the field of real rational functions and $\mathbb{R}_{pr}(s)$ the ring of real proper rational functions.

The classical time domain approach uses **state space** representations of the form

$$\begin{aligned} \rho x(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{p\times p}$, $B \in \mathbb{R}^{p\times m}$, $C \in \mathbb{R}^{l\times p}$, $D \in \mathbb{R}^{m\times l}$, $x(t) \in \mathbb{R}^p$ is the state vector and $u(t) \in \mathbb{R}^l$, $y(t) \in \mathbb{R}^m$ are respectively the input and output vectors. In view of a more general perspective, (1) is a special case of the **generalized state space** or **descriptor models**

$$\begin{aligned} \rho Ex(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t). \end{aligned} \tag{2}$$

From a frequency domain point of view, one usually studies the input/output description of the linear system expressed by the transfer function. In the multivariable case, transfer functions are essentially rational matrices. In the multiple input - multiple output (MIMO) case the factorization of a given transfer function $G(s) \in \mathbb{R}(s)^{m\times l}$ as a fraction of polynomial matrices, leads to the introduction of two distinct fractional representations, i.e.

$$G(s) = N_R(s)D_R(s)^{-1} = D_L(s)^{-1}N_L(s), \tag{3}$$

Namely, the **left matrix fraction description** (MFDs) of $G(s)$ can be seen as an input - output relation of the form

$$D_L(\rho)y(t) = N_L(\rho)u(t) \tag{4}$$

where $D_L(s) \in \mathbb{R}[s]^{m\times m}$, $N_L(s) \in \mathbb{R}[s]^{m\times l}$, whereas the **right matrix fraction description** can be considered as a model of the form

$$\begin{aligned} D_L(\rho)\xi(t) &= u(t) \\ y(t) &= N_L(\rho)\xi(t) \end{aligned} \tag{5}$$

where $D_R(s) \in \mathbb{R}[s]^{l\times l}$, $N_R(s) \in \mathbb{R}[s]^{m\times l}$ and $\xi(t) \in \mathbb{R}^l$ is the pseudostate vector. All the above representations can be considered as special cases of **polynomial matrix descriptions** (PMDs)

$$\begin{aligned} A(\rho)\xi(t) &= B(\rho)u(t) \\ y(t) &= C(\rho)\xi(t) + D(\rho)u(t) \end{aligned} \tag{6}$$

where $A(s) \in \mathbb{R}[s]^{r\times r}$, $B(s) \in \mathbb{R}[s]^{r\times l}$, $C(s) \in \mathbb{R}[s]^{m\times r}$, $N_R \in \mathbb{R}[s]^{m\times l}$.

In the continuous time case, ρ is to be interpreted as the differential operator in the distributional sense (see [10], [11], [4], [3]). According to this approach, certain classes of linear systems may exhibit impulsive solutions along with the smooth functional ones (see [35], [11], [3], [8], [9], [32], [31]). For a detailed presentation of the framework of impulsive-smooth distributions and related results, we encourage the reader to see the articles cited above. It is well known that the finite frequency modes of the above systems are associated to the the structure of the finite zeros of the matrix $A(s)$ in the case of (6), or the ones of the corresponding matrix in the rest of the models. On the other hand, impulsive modes or infinite frequency behavior is closely related to the presence of zeros at $s = \infty$ in the corresponding matrix.

When discrete time systems are under consideration, ρ is to be interpreted as the forward shift operator $\rho x(t) =$

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$x(t+1)$. While the regular discrete time case has no significant differences compared to the continuous one, when singularity comes into consideration, a radically different approach is required. Singular discrete time systems are in general non causal and hence not physically realizable. However, there are situations where singular models arise in a natural manner (see for instance [21]) or systems where the independent variable t is spatial rather than temporal. The framework proposed for such problems (see [21], [20], [2], [17], [16]) is to use a finite time interval and consider solutions propagating forward and backward in time given a set of admissible boundary conditions. The finite and infinite elementary divisors of the polynomial matrices involved in such models have been shown to play a central role. Notably, while the finite elementary divisors structure of a matrix, is completely determined by the finite zero structure, the corresponding infinite elementary divisor structure depends both on the poles and zeros at infinity.

In either case, the finite and infinite zero structure in continuous time systems, or the corresponding elementary divisor structure in discrete time systems, plays a central role in the determination of the behavior of a linear system. Thus, depending on the type of the model under investigation and the type of behavior to be preserved, it is natural to seek transformations between system representations leaving invariant the corresponding spectral characteristics of the matrices involved. Following this approach, in the next three sections we present matrix transformations preserving certain types of spectral structure accompanied by the induced system transformations preserving the corresponding type of dynamics.

II. Transformations preserving the finite structure

A. Matrix Similarity

The most common form of matrix equivalence related to linear system's theory is matrix similarity. Matrix similarity relates constant squares matrices of same dimensions. Similar matrices represent the same linear transformation expressed in two different bases.

Definition 1 (Similarity): [7] Two constant matrices $A_1, A_2 \in \mathbb{R}^{p\times p}$, are **similar** if there exists an invertible matrix M, such that

$$A_1 = MA_2M^{-1} \tag{7}$$

Notice that (7) can be also written in the form

$$(sI_p - A_1) = M(sI_p - A_2)M^{-1} \tag{8}$$

Similarity, as a relation on the set of $p \times p$ constant matrices, is easily shown to be an equivalence relation. The eigenstructures of two similar matrices, that is the eigenvalues along with their algebraic and geometric multiplicities, are identical. In particular, given a matrix A with v distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_v \in \mathbb{C}$, there exists an invertible matrix M (see for instance [7]), such that $A = MJM^{-1}$, where

$$J = \underset{\substack{i=1,\ldots,v\\ j=1,\ldots,v_i}}{diag}\ \{J_{ij}\}$$

is the Jordan canonical form of A, v_i is the geometric multiplicity of λ_i,

$$J_{ij} = \begin{bmatrix} \lambda_i & 1 & \cdots & 0 \\ 0 & \lambda_i & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & 0 & \lambda_i \end{bmatrix} \in \mathbb{C}^{v_{ij}\times v_{ij}}$$

and $\sum_{j=1}^{v_i} v_{ij}$ is the algebraic multiplicity of λ_i. In general, the above decomposition, even if A is a real matrix, may result in complex matrices M, J. However, a real alternative of J and M is also possible.

Given a linear system described by a set of state space equations of the form (1) the eigenstructure of the matrix $A \in \mathbb{R}^{n\times n}$ plays a crucial role to the dynamics of the system. A question naturally arising in the study of such systems, is how equation (1) changes, when a transformation of the state vector takes place, i.e. when

$$x(t) = M\hat{x}(t) \tag{9}$$

It is easy to see that the above change of coordinates results in a new description of the form

$$\begin{aligned} \rho\hat{x}(t) &= \hat{A}\hat{x}(t) + \hat{B}u(t) \\ y(t) &= \hat{C}\hat{x}(t) + Du(t) \end{aligned}$$

where $\hat{A} = M^{-1}AM$, $\hat{B} = M^{-1}B$ and $\hat{C} = CM$. This type of transformation is known in control literature as **system similarity**.

B. Unimodular Equivalence

Unimodular equivalence provides a transformation between rational matrices of the same dimensions. In what follows we focus on the polynomial case which is widely used in linear systems theory. Recall that unimodular matrices are polynomial matrices whose inverse is also polynomial, that is their determinant is a non-zero constant. Pre or post multiplication of a polynomial matrix by a unimodular one, corresponds to performing row or column elementary operations on the matrix. This naturally leads to the following definition.

Definition 2 (Unimodular equivalence): [7], [29] Two polynomial matrices $A_i(s) \in \mathbb{R}[s]^{p\times q}$, $i = 1, 2$ are **unimodular equivalent,** if there exist unimodular matrices $U(s) \in \mathbb{R}[s]^{p\times p}$, $V(s) \in \mathbb{R}[s]^{q\times q}$, such that

$$A_1(s) = U(s)A_2(s)V(s).$$

Unimodular equivalence is an equivalence relation and a canonical form is the well known Smith form of the polynomial matrix. It has been shown [7] that given a polynomial matrix $A(s)$, there exist unimodular matrices $U(s)$, $V(s)$ such that

$$A(s) = U(s)S^{\mathbb{C}}_{A(s)}(s)V(s)$$

where

$$S^{\mathbb{C}}_{A(s)}(s) = diag\{\varepsilon_1(s), \ldots, \varepsilon_r(s), \mathbf{0}\} \quad (10)$$

is the Smith form of $A(s)$ and $\varepsilon_i(s) = \prod_{j=1}^{v_i} (s-\lambda_j)^{m_{ij}}$ are monic polynomials such that $\varepsilon_i(s)|\varepsilon_{i+1}(s)$, $i = 1, \ldots, r-1$. The roots of $\varepsilon_i(s)$ are the **finite zeros** of $A(s)$, whereas the factors $(s-\lambda_j)^{m_{ij}}$ of $\varepsilon_i(s)$, are the **finite elementary divisors** of $A(s)$. The finite elementary divisor structure and therefore the zeros and their multiplicities form a complete set of invariants for unimodular equivalence. In the special case where $A_i(s) = sI_p - A_i$, unimodular equivalence reduces to matrix similarity of A_i (see [7]).

From a system point of view, unimodular equivalence relates matrix fractional descriptions (MFDs) of a system with a given transfer function $G(s) \in \mathbb{R}(s)^{m\times l}$. In particular, given right and left MFDs of the form (3), it is easy to see that choosing $V(s), U(s)$ unimodular and of appropriate dimensions, the pairs $\hat{N}_R(s) = N_R(s)V(s)$, $\hat{D}_R(s) = D_R(s)V(s)$ and $\hat{N}_L(s) = U(s)N_L(s)$, $\hat{D}_L(s) = U(s)D_L(s)$ give rise to the same transfer function $G(s)$, preserving input (resp. output) decoupling zeros of the MFDs, i.e. the zeros of

$$\begin{bmatrix} D_L(s) & N_L(s) \end{bmatrix}, \text{(resp. } \begin{bmatrix} N_R(s) \\ -D_R(s) \end{bmatrix}\text{)}. \quad (11)$$

On the other hand, it is natural to consider a left and a right MFD of the form (3) giving rise to the same transfer function as equivalent, if they also share input and output decoupling zeros. Since left MFDs have no input decoupling zeros and right MFDs have no output decoupling zeros, we can deduce that a left and a right MFD of the form (3) can be equivalent if and only if there are no input and output decoupling zeros in both descriptions. Notice that (3) can be written as

$$D_L(s)N_R(s) = N_L(s)D_R(s)$$

while the absence of input output decoupling zeros can be expressed by the coprimeness of the compound matrices in (11). Notice that the last two conditions essentially imply that the pairs of numerators and denominators share common finite zero structures, despite the fact that $D_L(s)$ and $D_R(s)$ may be of different dimensions.

Furthermore, a more general equivalence relation between systems expressed in Polynomial Matrix Descriptions (PMDs) has been introduced in [29] known as **strict system equivalence** (s.s.e)**.** In [37], a different approach using state space reduction was followed. Additionally, an alternative formulation of s.s.e. was proposed in [6] and was proven equivalent to s.s.e. in [19]. In what follows we will focus on the definition of s.s.e. given in [6]. Let

$$P_i(s) = \begin{bmatrix} A_i(s) & -B_i(s) \\ C_i(s) & D_i(s) \end{bmatrix}, i = 1, 2 \quad (12)$$

be the system matrices of two PMDs Σ_i, where $A_i(s) \in \mathbb{R}[s]^{r_i\times r_i}$, $B_i(s) \in \mathbb{R}[s]^{r_i\times l}$, $C_i(s) \in \mathbb{R}[s]^{m\times r_i}$, $D_i(s) \in \mathbb{R}[s]^{m\times l}$.

Definition 3 (Furhmann's s.s.e): [6] Σ_1 and Σ_2 are **Fuhrmann strictly system equivalent**, if there exist polynomial matrices $M_1(s)$, $M_2(s)$, $X_1(s)$, $X_2(s)$ such that

$$\begin{bmatrix} M_1(s) & \mathbf{0} \\ X_1(s) & I_m \end{bmatrix}\begin{bmatrix} A_1(s) & -B_1(s) \\ C_1(s) & D_1(s) \end{bmatrix} = \\ = \begin{bmatrix} A_2(s) & -B_2(s) \\ C_2(s) & D_2(s) \end{bmatrix}\begin{bmatrix} M_2(s) & X_2(s) \\ \mathbf{0} & I_l \end{bmatrix}$$

where

$$\begin{bmatrix} M_1(s) & A_2(s) \end{bmatrix}, \begin{bmatrix} A_1(s) \\ -M_2(s) \end{bmatrix}$$

have full rank and no zeros in $\mathbb{C}$.

A dynamic interpretation of strict system equivalence is given in terms of the existence of a bijective map between the solution sets of the systems in [23]. The above discussion, provides a motivation for the definition of extended unimodular equivalence. In order to overcome the difficulty of relating systems with different number of pseudostates having the same number of inputs/outputs, we introduce $\mathcal{P}(m,l)$ to be the class of polynomial matrices with dimensions $(r+m)\times(r+l)$, for all $r = 1, 2, \ldots$ and provide the following definition.

Definition 4 (Extended unimodular equivalence): [26] Two polynomial matrices $A_1(s)$, $A_2(s) \in \mathcal{P}(m,l)$ are **extended unimodular equivalent** (e.u.e.) if there exist polynomial matrices of appropriate dimensions $M(s)$, $N(s)$ such that

$$M(s)A_1(s) = A_2(s)N(s) \quad (13)$$

where the compound matrices

$$\begin{bmatrix} M(s) & A_2(s) \end{bmatrix}, \begin{bmatrix} A_1(s) \\ -N(s) \end{bmatrix} \quad (14)$$

have full rank and no zeros in $\mathbb{C}$.

Proving that e.u.e. is an equivalence relation is not a trivial task ([26], [30]). It has been shown in [26] that e.u.e. preserves the finite zero structure as well as the normal rank defect of the matrices involved. From a system point of view, if two PMDs with system matrices as in (12) are s.s.e., then it can be shown that any of the following matrix pairs

$$A_i(s), \begin{bmatrix} A_i(s) & -B_i(s) \end{bmatrix}, \begin{bmatrix} A_i(s) \\ C_i(s) \end{bmatrix}, P_i(s), i = 1, 2, \quad (15)$$

are e.u.e. Hence, s.s.e preserves the poles, the input and output decoupling zeros and system zeros of Σ_i, that are the zeros of the respective matrices in (15).

III. TRANSFORMATIONS PRESERVING THE STRUCTURE AT INFINITY

The class of singular linear systems received special attention by many authors (see [35], [4], [11], [31]). A distinguishing feature of this class of linear systems, is their infinite frequency behavior which is directly related to the so called pole/zero structure at infinity of the polynomial matrices involved.

As noted in [14], while unimodular equivalence operations preserve the finite zero structure of polynomial matrices, they will in general destroy the corresponding structure at infinity. A solution to this difficulty was given with the introduction of biproper equivalence of rational matrices.

Definition 5 (Equivalence at infinity): [22], [33] Two rational matrices $A_i(s) \in \mathbb{R}(s)^{p\times q},\ i = 1, 2$, are **equivalent at** $s = \infty$ if there exist biproper rational matrices $U(s) \in \mathbb{R}_{pr}(s)^{p\times p}, V(s) \in \mathbb{R}_{pr}(s)^{q\times q}$, such that

$$A_1(s) = U(s)A_2(s)V(s).$$

Recall that a proper rational matrix is called biproper, if its inverse is also proper, that is if its determinant is a biproper rational function. Equivalence at $s = \infty$ preserves the pole zero structure at $s = \infty$, which is exposed by the Smith McMillan form at ∞. Every $A(s) = \sum_{i=0}^{n} A_i s^i \in \mathbb{R}(s)^{p\times q}$ is equivalent at $s = \infty$ to a matrix of the form

$$S^{\infty}_{A(s)}(s) = diag\left\{s^{q_1}, \cdots, s^{q_k}, \frac{1}{s^{\hat{q}_k+1}}, \cdots, \frac{1}{s^{\hat{q}_r}}, 0_{p-r,q-r}\right\}$$

where $n = q_1 \geq q_2 \geq \cdots \geq q_k \geq 0$ are the orders of the **poles at** ∞ and $\hat{q}_r \geq \hat{q}_{r-1} \geq \cdots \geq \hat{q}_{k+1} \geq 0$ are the orders of the respective **zeros**.

On the other hand the **infinite elementary divisors** of $A(s)$, are defined as the finite elementary divisors of the form $s^{\mu_i},\ i = 1, \ldots, r$ of its dual, $\tilde{A}(s) = s^n A(\frac{1}{s})$. Notably the orders μ_i are related to the orders of poles and zeros at ∞ (see [13], [33], [32]) by

$$\mu_i = q_1 - q_i,\ i = 1, \ldots, k \quad (16)$$

$$\mu_i = q_1 + \hat{q}_i,\ i = k + 1, \ldots, r. \quad (17)$$

Obviously the infinite elementary divisors in (16) reflect the infinity pole structure, while the ones in (17) correspond to the zero structure at infinity.

IV. Transformations preserving the structure at $\mathbb{C} \cup \{\infty\}$

A. Strict Equivalence

The study of equivalences between matrix pencils dates back to [36] where Weierstrass considered regular pencils and introduced the concept of strict equivalence and discovered a canonical form that is named after him. An extension of strict equivalence to the non regular case was given by Kronecker in [18] where the Kronecker canonical form was established. Both results have significant system theoretic applications and/or interpretations and they are summarized in [7].

Definition 6 (Strict equivalence): [7] Two matrix pencils $sE_i - A_i,\ i = 1, 2$, with $E_i, A_i \in \mathbb{R}^{p\times q}$, are **strictly equivalent** if there exist non singular matrices $M \in \mathbb{R}^{p\times p}$, $N \in \mathbb{R}^{q\times q}$, such that

$$sE_1 - A_1 = M(sE_2 - A_2)N.$$

Strictly equivalent matrix pencils, share identical finite and infinite elementary divisor structure and minimal indices [7], rendered in the well known Kronecker canonical form. Every matrix pencil $sE - A \in \mathbb{R}[s]^{p\times q}$, is strictly equivalent to a pencil of the form

$$K(s) = diag\{sI_n - J_{\mathbb{C}}, sJ_\infty - I_\mu, L_\varepsilon(s), L_\eta(s)\}$$

where the $sI_n - J_{\mathbb{C}}$ corresponds to the finite zeros (elementary divisors) of the pencil, with $J_{\mathbb{C}}$ being in Jordan canonical form. The block $sJ_\infty - I_\mu$ corresponds to the infinite elementary divisors of the pencil, where J_∞ is in Jordan form, with all its diagonal elements equal to zero. Finally, the block $L_\varepsilon(s)$ (resp. $L_\eta(s)$) is a block diagonal matrix, comprised by non square blocks $L_{\varepsilon_i}(s),\ i = 1, \ldots, r$ (resp. $L_{\eta_i}(s)\ i = 1, \ldots, l$) of the form

$$L_{\varepsilon_i}(s) = sM_{\varepsilon_i} - N_{\varepsilon_i} \in \mathbb{R}[s]^{\varepsilon_i\times(\varepsilon_i+1)}$$

$$L_{\eta_i}(s) = sM^{\intercal}_{\eta_i} - N^{\intercal}_{\eta_i} \in \mathbb{R}[s]^{(\eta_i+1)\times\eta_i}$$

where

$$M_{\varepsilon_i} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix} \in \mathbb{R}^{\varepsilon_i\times(\varepsilon_i+1)}$$

$$N_{\eta_i} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{bmatrix} \in \mathbb{R}^{\eta_i\times(\eta_i+1)}.$$

The block $L_{\varepsilon_i}(s)$ ($L_{\eta_i}(s)$) is called the right (left) Kronecker block and the integers ε_i (η_i) are the right (left) Kronecker indices of the original pencil. Moreover, if we set $\varepsilon = \sum_{i=1}^{r} \varepsilon_i$ and $\eta = \sum_{i=1}^{l} \eta_i$, then $p = n + \mu + \varepsilon + \eta + l$ and $q = n + \mu + \varepsilon + \eta + r$.

As a result, strict equivalence of matrix pencils, serves as a tool to identify descriptor representations of the form (2) corresponding to the same underlying linear system. Given a descriptor system of the form (2) one may apply a change of coordinates on the descriptor vector similar to that in (9), that is $x(t) = M\hat{x}(t)$, along with a premultiplication of the first descriptor equation by a square invertible matrix N. This gives rise to a new descriptor system of the form

$$\rho\hat{E}\hat{x}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t)$$

$$y(t) = \hat{C}\hat{x}(t) + Du(t)$$

where $\hat{E} = NEM$, $\hat{A} = NAM$, $\hat{B} = NB$, $\hat{C} = CM$. It is easy to verify that the matrix pencils $sE - A$ and $s\hat{E} - \hat{A}$ are strictly equivalent in the sense of definition 6. Notably, the system matrices of the descriptor systems related as described above, are connected via

$$\begin{bmatrix} s\hat{E} - \hat{A} & -\hat{B} \\ \hat{C} & D \end{bmatrix} = \begin{bmatrix} N & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} sE - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} M & 0 \\ 0 & I_l \end{bmatrix}$$

which essentially states that the system matrix pencils are also strictly equivalent. Similar relations hold for the pairs $\begin{bmatrix} s\hat{E} - \hat{A} & -\hat{B} \end{bmatrix}$, $\begin{bmatrix} sE - A & -B \end{bmatrix}$ and $\begin{bmatrix} s\hat{E} - \hat{A} \\ \hat{C} \end{bmatrix}$, $\begin{bmatrix} sE - A \\ C \end{bmatrix}$.

B. Complete Equivalence

In an attempt to obtain a system equivalence that preserves the structural invariants of descriptor systems, of possibly different state dimensions, both in $\mathbb{C}$ and $s = \infty$ and hence their finite and infinite frequency behavior, **strong system**

equivalence was introduced in [35]. The definition of strong system equivalence in [35], can be seen as a collection of allowable operations on the descriptor system matrices. In [1] strong equivalence received a more compact formulation as **constant system equivalence**, and took on a closed form description in [25] as **complete system equivalence**. In the present note, due to size restrictions, we will focus only on the presentation of complete system equivalence.

Definition 7 (Complete equivalence): [25] Two matrix pencils $sE_i - A_i$, $i = 1, 2$, with $E_i, A_i \in \mathcal{P}(m, l)$, are **completely equivalent** if there exist constant matrices M, N of appropriate dimensions such that

$$M(sE_1 - A_1) = (sE_2 - A_2)N$$

where the compound matrices

$$\left[\begin{array}{cc} M & sE_2 - A_2 \end{array} \right], \left[\begin{array}{c} sE_1 - A_1 \\ -N \end{array} \right] \tag{18}$$

have full rank and no zeros in $\mathbb{C} \cup \{\infty\}$.

Complete system equivalence can be seen as a generalization of strict equivalence of matrix pencils, with the extra feature of being able to compare matrix pencils of different dimensions. It has been shown in [25] that two regular pencils are completely equivalent iff they possess the same finite and non-trivial infinite elementary divisors.

From system's point of view complete equivalence takes the form described bellow. Given two descriptor systems of the form

$$\begin{array}{rcl} \rho E_i x_i(t) & = & A_i x_i(t) + B_i u(t) \\ y(t) & = & C_i x_i(t) + D_i u(t) \end{array} \tag{19}$$

for $i = 1, 2$, where $A_i, E_i \in \mathbb{R}^{p_i \times p}$, $B_i \in \mathbb{R}^{p_i \times m}$, $C_i \in \mathbb{R}^{l \times p_i}$, $D_i \in \mathbb{R}^{m \times l}$ and $x_i(t) \in \mathbb{R}^{p_i}$, we have the following

Definition 8 (Complete System Equivalence): [25] Two descriptor systems of the form (19) are **completely system equivalent** if there exist constant matrices M, N, X, Y of appropriate dimensions such that

$$\begin{bmatrix} M & \mathbf{0} \\ X & I_m \end{bmatrix} \begin{bmatrix} sE_1 - A_1 & -B_1 \\ C_1 & D_1 \end{bmatrix} = \\ = \begin{bmatrix} sE_2 - A_2 & -B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} N & Y \\ \mathbf{0} & I_l \end{bmatrix}$$

where the compound matrices

$$\left[\begin{array}{cc} M & sE_2 - A_2 \end{array} \right], \left[\begin{array}{c} sE_1 - A_1 \\ -N \end{array} \right] \tag{20}$$

have full rank and no zeros in $\mathbb{C} \cup \{\infty\}$.

Notice that the above definition does not require the two descriptor vectors $x_i(t)$ to be of the same dimension. It was shown in [25] that two descriptor systems are completely system equivalent iff they are strongly system equivalent. The corresponding relation between strong and constant system equivalence was established in [1]. Furthermore, in [12] it was shown that two descriptor systems are completely equivalent iff there exist bijective maps that leave input/output behavior and essential dynamics unchanged. This type of equivalence is termed **fundamental equivalence** in [12].

C. Full equivalence

In [1] a generalization of strong system equivalence to polynomial matrix descriptions of arbitrary degree was proposed and the dynamic interpretation of the corresponding invariants was studied in [5]. However, in its original formulation, strong system equivalence of PMDs, suffered from a technical point of view a serious drawback: two PMDs were termed strongly system equivalent if they are both polynomially system equivalent and system equivalent at infinity. In order to overcome the difficulty of checking two separate conditions, a more compact form of matrix equivalence, known as **full equivalence** was proposed in [27].

Definition 9 (Full equivalence): [27] Let $A_1(s), A_2(s) \in \mathcal{P}(m, l)$. Then $A_1(s), A_2(s)$ are **fully equivalent** (FE) if there exist polynomial matrices $M(s), N(s)$ of appropriate dimensions, such that

$$M(s)A_1(s) = A_2(s)N(s) \tag{21}$$

is satisfied and the compound matrices

$$\left[\begin{array}{cc} M(s) & A_2(s) \end{array} \right], \left[\begin{array}{c} A_1(s) \\ -N(s) \end{array} \right] \tag{22}$$

(i) have full rank and no zeros in $\mathbb{C} \cup \{\infty\}$

(ii) $\delta_M \left[\begin{array}{cc} M(s) & A_2(s) \end{array} \right] = \delta_M A_2(s)$, $\delta_M \left[\begin{array}{c} A_1(s) \\ -N(s) \end{array} \right] = \delta_M A_1(s)$

Notice that δ_M denotes the McMillan degree i.e. the total number of poles of the rational matrix involved. Some of the invariants of FE are (see [28])

- The McMillan degree of $A_i(s)$.
- The finite and infinite zero structures of $A_i(s)$.

The dynamical interpretation of the conditions appearing in definition 9 is given bellow. A generalized version of fundamental equivalence introduced in [12] for descriptor systems, was proposed in [28] as more complete form of the equivalence proposed in [23]. Given two PMD's of the form (12), we form the normalized system matrices

$$\mathcal{P}_i(s) = \left[\begin{array}{ccc|c} A_i(s) & -B_i(s) & 0 & 0 \\ C_i(s) & D_i(s) & -I_m & 0 \\ 0 & I_l & 0 & -I_l \\ \hline 0 & 0 & I_m & 0 \end{array} \right] = \\ = \left[\begin{array}{cc} \mathcal{T}_i(s) & -\mathcal{U}_i \\ \mathcal{V}_i & 0 \end{array} \right] \tag{23}$$

which has the advantage over (12) of allowing uniform treatment of finite and infinite frequency characteristics to be made. Then,

Definition 10 (Normal Full System Equivalence): [28] The normalized system matrices $\mathcal{P}_i(s)$ $i = 1, 2$ are **normal full system equivalent** if there exist polynomial matrices $M(s)$, $N(s)$, $X(s)$, $Y(s)$ such that the relation

$$\begin{bmatrix} M(s) & \mathbf{0} \\ X(s) & I \end{bmatrix} \left[\begin{array}{cc} \mathcal{T}_1(s) & -\mathcal{U}_1 \\ \mathcal{V}_1 & 0 \end{array} \right] = \\ = \left[\begin{array}{cc} \mathcal{T}_2(s) & -\mathcal{U}_2 \\ \mathcal{V}_2 & 0 \end{array} \right] \begin{bmatrix} N(s) & Y(s) \\ \mathbf{0} & I_l \end{bmatrix}$$

is a full equivalence relation.

With the above setup fundamental equivalence is in turn defined as follows:

Definition 11 (Fundamental Equivalence of PMDs): [28] Let $\mathcal{P}_i(s)$, $i = 1, 2$ be the normalized form of two PMDs described by (12). The PMDs are said to be **fundamentally equivalent** if there exists bijective polynomial differential map of the form

$$\xi_2(t) = N(\rho)\xi_1(t) + Y(\rho)u(t)$$

and they have the same output $y(t)$.

It has been shown in [28], that two PMDs are fundamentally equivalent if and only if they are normal full system equivalent.

Proceeding a step further in the dynamic interpretation of full matrix equivalence, fundamental equivalence has been extended to fit to the behavioral framework for higher order implicit systems introduced in [9]. Given two auto regressive (AR) representations

$$A_i(\rho)\xi_i = 0 \tag{24}$$

where $A_i(s) \in \mathbb{R}^{p_i \times q_i}[s]$, for $i = 1, 2$, their behaviors are

$$\mathcal{B}_i = \{\xi_i \in \ell_{imp}^{q_i} : A_i(\rho)\xi_i = 0\} \tag{25}$$

where ℓ_{imp} is the space of impulsive - smooth distributions. In view of the above setting we have the following

Definition 12 (Fundamental equivalence of AR representations): [24] The systems described by (24) are **fundamentally equivalent** if there exists a bijective polynomial differential map $N(\rho) : \mathcal{B}_1 \to \mathcal{B}_2$.

The connection between fundamental system equivalence and full matrix equivalence is given by the following

Theorem 13: [24] The systems described by the AR - representations (24) are fundamentally equivalent iff there exists a polynomial differential operator $N(s) \in \mathbb{R}^{q_2 \times q_1}[s]$ satisfying the following conditions

(i) $\exists M(s) \in \mathbb{R}^{p_2 \times p_1}[s] : M(s)A_1(s) = A_2(s)N(s)$.

(ii) $\delta_M \begin{bmatrix} A_1(s) \\ N(s) \end{bmatrix} = \delta_M(A_1(s))$ and $\delta_M \begin{bmatrix} M(s) & A_2(s) \end{bmatrix} = \delta_M(A_2(s))$.

(iii) $\begin{bmatrix} A_1(s) \\ N(s) \end{bmatrix}, \begin{bmatrix} M(s) & A_2(s) \end{bmatrix}$ have full rank and no zeros in $\mathbb{C} \cup \{\infty\}$.

(iv) $q_1 - p_1 = q_2 - p_2$.

Notice that conditions (i)-(iii) in the above theorem, coincide with the requirements of full matrix equivalence, while condition (iv) is essentially an alternative formulation of the assumption $A_1(s),\ A_2(s) \in \mathcal{P}(m, l)$ in definition 9. Hence, the polynomial matrices $A_1(s),\ A_2(s)$ involved in two fundamentally equivalent AR representations, are fully equivalent.

D. Divisor Equivalence

In the discrete time case, the behavior of systems of the form (24) is considered over a finite time interval (see [20], [2], [17]). According to this approach the finite and infinite elementary divisors give rise to trajectories that can be seen as the forward or backward propagation of the initial and final conditions of the pseudostate respectively. Thus, in order to preserve this type of behavior, a new kind of matrix equivalence preserving finite and infinite elementary divisors of a polynomial matrix has to be introduced.

Definition 14 (Divisor equivalence): [15] Two regular matrices $A_i(s) \in \mathbb{R}[s]^{r_i \times r_i}$, $i = 1, 2$ with $r_1 \deg A_1(s) = r_2 \deg A_2(s)$, are said to be **divisor equivalent** if there exist polynomial matrices $M(s), N(s)$ of appropriate dimensions, such that

$$M(s)A_1(s) = A_2(s)N(s) \tag{26}$$

is satisfied and the compound matrices

$$\begin{bmatrix} M(s) & A_2(s) \end{bmatrix}, \begin{bmatrix} A_1(s) \\ -N(s) \end{bmatrix}$$

have no finite and infinite elementary divisors.

The key property of the above defined equivalence, is that if $A_1(s)$, $A_2(s)$ are divisor equivalent then they share identical finite and infinite elementary divisors structure. In the special case where $A_i(s)$ are matrix pencils, it has been shown that $A_i(s)$ are strictly equivalent (see definition 6) if and only if they are divisor equivalent.

From a system point of view one may define a form of equivalence between two systems, seen as a polynomial isomorphism between their behaviors.

Definition 15 (Fundamental equivalence): [34] Two AR-representations

$$A_i(\rho)\,\xi_i = 0, \tag{27}$$

where $A_i(s) \in \mathbb{R}[s]^{r_i \times r_i}$, $i = 1, 2$ with $r_1 \deg A_1(s) = r_2 \deg A_2(s)$ will be *called* ***fundamentally equivalent*** *over the finite time interval* $k = 0, 1, 2, ..., N$, iff there exists a bijective polynomial map between $N(\sigma) : \mathcal{B}^N_{A_1(\sigma)} \longrightarrow \mathcal{B}^N_{A_2(\sigma)}$.

It has been shown that if $A_1(\sigma), A_2(\sigma) \in R_c[\sigma]^{r_i \times r_i}$ are divisor equivalent, then the AR-representations (27) are fundamentally equivalent.

V. Conclusions

Equivalence transformations of polynomial matrices and their system theoretic counterparts, are without doubt key concepts in the theory of linear multivariable systems developed during the last four decades. In the present note we have attempted a review of the main results both from a methodological and a historical point of view. However, mainly due to space limitations, it is virtually impossible to cover every aspect of the subject. For instance, a very important subject closely related to equivalences of matrices and systems, is the problem of linearization/realization of a polynomial matrix/model, which was not touched here. We hope that the present work will serve as a reference point for further studies which will extend the existing theory and address open questions.

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