Construction of Algebraic-Differential Equations with given Smooth and Impulsive Behavior

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Abstract

The paper is consisted of two parts. In the first part, we are presenting the smooth and impulsive solution space B of the system of algebraic and differential equations $A(\rho)\beta(t)=0, \rho:=dt$ where $A(\rho)\in\mathbb{R}[\rho]^{r\times r}$ (with $\det A(\rho)\neq 0$) is given. The solution space is given in terms of the finite and infinite Jordan pairs of the polynomial matrix $A(\rho)$, and therefore a necessary introduction is given for the finite and infinite zero structure of the polynomial matrix $A(\rho)$. The second part, studies the inverse problem : Given a smooth and impulsive solution space B, try to find out a system of algebraic and differential equations $A(\rho)\beta(t)=0, \rho:=dt$ with the given solution space.

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1 Introduction

Consider a system of algebraic and differential equations $A(\rho)\beta(t)=0, \rho:=dt$ where $A(\rho)=A_0+A_1\rho+\cdots+A_q\rho^q\in\mathbb{R}[\rho]^{r\times r}$ and $\det[A(\rho)]\neq 0$ ($A(\rho)$ is regular). The solution space of this system is consisted by both smooth and impulsive solutions. The construction of its smooth (resp. impulsive) solution space or behavior B is based on the finite (resp. infinite) zero structure of the polynomial matrix $A(\rho)$ and the finite (resp. infinite) Jordan pairs that becomes from its structure [6] (resp. [1]).

According to [15], it is the behavior, the solution set of the behavioral equations, not the behavioral equations themselves, which is the essential result of a modeling procedure. More particularly, whereas to a model or equivalently to a system of algebraic-differential equations corresponds one specific solution space, to a specific solution space corresponds more than one models in the form of system of algebraic-differential equations. In [16], [15] and [17] a recursive method is proposed, known as Most Powerful Unfalsified Model (MPUM), which constructs the system of algebraic and differential equations $A(\rho)\beta(t) = 0$ if the *smooth* behavior of the system is known. A different approach to the same problem is given in [6]. Therefore, we observe an attempt to the solution of the **inverse** problem : Given the solution space B of a system (a physical process), find the system of algebraic and differential equations that have B as a solution space. Since, the inverse problem has already been investigated for the smooth behavior case, our main aim in this work is to solve the inverse problem for the case where we know: a) the *impulsive* solution space B of a system, and b) both the *smooth* and *impulsive* solution space B. The constructed models are not unique, but are connected by the full unimodular equivalence that preserves both the finite and infinite zero structure of the polynomial matrices that describes the systems [12]. In the second section we present some preliminary results that concern the finite and infinite zero structure of a polynomial matrix $A(\rho)$ and its connection with the smooth and impulsive behavior of the system of algebraic and differential equations $A(\rho)\beta(t) = 0$. In section 3, we study the problem of finding the system $A(\rho)\beta(t) = 0$ if we know the *impulsive* solution space B. We show, that this problem is equivalent to a problem where we are looking for a system $\tilde{A}(\rho)\tilde{\beta}(t) = 0$ with $\tilde{A}(\rho) = \rho^q A\left(\frac{1}{\rho}\right) = A_0\rho^q + A_1\rho^{q-1} + \cdots + A_q$ if we know the *smooth* solution space $\tilde{B}$ which is constructed from the *impulsive* solution space B. Finally, in the last section, we propose a method for the more general inverse problem, of finding the system $A(\rho)\beta(t) = 0$ if we know both the *smooth* and *impulsive* solution space B.

2 Preliminary results

Consider a regular polynomial matrix

$$A(s) = A_q s^q + \cdots + A_1 s + A_0 \tag{1}$$

where $A_i \in \mathbb{R}^{r\times r}$, $A_q \neq 0$ and $\det A(s) \neq 0$.

Definition 1 *[1] Let $A(s)$ defined in (1). Then there exist unimodular matrices $U_L(s) \in \mathbb{R}[s]^{r\times r}$, $U_R(s) \in \mathbb{R}[s]^{r\times r}$ (i.e. $\det[U_L(s)], \det[U_R(s)] \in \mathbb{R}\backslash\{0\}$), such that*

$$\begin{gathered} U_L(s)A(s)U_R(s) = S^{\mathbb{C}}_{A(s)}(s) \equiv \\ \equiv \mathit{blockdiag}\left[\underbrace{1,1,...,1}_{z-1}, f_z(s), f_{z+1}(s), ..., f_r(s)\right] \end{gathered} \tag{2}$$

with $1 \leq z \leq r$ and $f_i(s)/f_{i+1}(s)$ $i = z, z+1, ..., r-1$. $S^{\mathbb{C}}_{A(s)}(s)$ is called the Smith form of $A(s)$ (in $\mathbb{C}$), where $f_i(s) \in \mathbb{R}[s]$ are the invariant polynomials *of* $A(s)$. *The zeros* $s_i \in \mathbb{C}$ *of* $f_j(s) \in \mathbb{R}[s]$, $j = z, z+1, ..., r$ *are called finite zeros of* $A(s)$. *Assume that the partial multiplicities of the zeros* $s_i \in \mathbb{C}$, $i \in k$ *are* $0 \leq \sigma_{i,z} \leq \sigma_{i,z+1} \leq \cdots \leq \sigma_{i,r}$ *i.e.*

$$f_j(s) = (s - s_i)^{\sigma_{i,j}} \hat{f}_j(s), j = z, z+1, .., r \text{ with } \hat{f}_j(s_i) \neq 0$$

The terms $(s - s_i)^{\sigma_{i,j}}$ *are called finite elementary divisors of* $A(s)$ *at* $s = s_i$. *We also denote by* n *the sum of the degrees of the finite elementary divisors of* $A(s)$ *i.e.*

$$n := \deg\left[\prod_{j=z}^{r} f_j(s)\right] = \sum_{i=1}^{k}\sum_{j=z}^{r}\sigma_{i,j}$$

Similarly we can find $U_L(s) \in \mathbb{R}(s)^{r\times r}, U_R(s) \in \mathbb{R}(s)^{r\times r}$ *having no poles and zeros at the point* $s = s_0$ *such that*

$$S_{A(s)}^{s_0}(s) := U_L(s)\,A(s)\,U_R(s) =$$
$$= blockdiag\left[1, 1, .., 1, (s-s_0)^{\sigma_z}, (s-s_0)^{\sigma_{z+1}}, ..., (s-s_0)^{\sigma_r}\right]$$

In that case, $S_{A(s)}^{s_0}(s)$ *is called the Smith form of* $A(s)$ *at the local point* $s = s_0$.

The definition of a *finite Jordan pair* of a polynomial matrix is given below.

Theorem 2 *([6], pp.184) Let* $(C_{s_0} \in \mathbb{R}^{r\times n}, J_{s_0} \in \mathbb{R}^{n\times n})$ *be a matrix pair where* J_{s_0} *is a Jordan matrix with unique eigenvalue* s_0 *i.e.*

$$J_{s_0} := \begin{pmatrix} s_0 & 1 & 0 & \cdots & 0 \\ 0 & s_0 & 1 & \cdots & 0 \\ 0 & 0 & s_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & s_0 \end{pmatrix} \in \mathbb{R}^{n\times n}$$

Then the following conditions are necessary and sufficient in order that (C_{s_0}, J_{s_0}) *be a Jordan pair of* $A(s)$ *(given in (1)) corresponding to* s_0 *:*

(a) $\det A(s)$ *has a zero at* s_0 *of multiplicity* n,

(b)

$$rank\begin{pmatrix} C_{s_0} \\ C_{s_0}J_{s_0} \\ \vdots \\ C_{s_0}J_{s_0}^{n-1} \end{pmatrix} = n \tag{3}$$

(c)

$$A_qC_{s_0}J_{s_0}^q + \cdots + A_1C_{s_0}J_{s_0} + A_0C_{s_0} = 0 \tag{4}$$

Taking a Jordan pair $(C_{s_i} \in \mathbb{R}^{r\times n_i}, J_{s_i} \in \mathbb{R}^{n_i\times n_i})$ for every zero $s_i, i = 1, 2, .., k$ of $A(s)$ we define a *finite Jordan pair* $(C \in \mathbb{R}^{r\times n}, J \in \mathbb{R}^{n\times n})$ of $A(s)$ as

$$C = \begin{pmatrix} C_{s_1} & C_{s_2} & \cdots & C_{s_k} \end{pmatrix} \; ; \; J = blockdiag\begin{pmatrix} J_{s_1} & J_{s_2} & \cdots & J_{s_k} \end{pmatrix}$$

where $n = n_1 + n_2 + \cdots n_k$ is the total number of finite zeros of $A(s)$ (order accounted for). The column span of $\Psi = Ce^{Jt}$ defines a solution space $B^{\mathbb{C}}$ of

$$A(\rho)\beta(t) = 0, \rho := d/dt \tag{5}$$

with dimension :

$$\dim B^{\mathbb{C}} = n := \text{total number of finite zeros of } A(s) \text{ (order accounted for)}$$

$B^{\mathbb{C}}$ is known as the *smooth solution space* of (5) defined by [6] and [1].

The infinite zero structure of polynomial matrices has been studied by many authors, among them by Rosenbrock [14], Kailath [11], Verghese [4], Verghese and Kailath [5], Pugh and Ratcliffe [13], Vardulakis [1], [2], Vardulakis et. al. [3], Hayton et. al. [7]. The definition of the infinite zeros of a polynomial matrix $A(s)$ and its connection with the local Smith form at $s = 0$ of its dual polynomial matrix is given below.

Definition 3 *[1], [7] Let $A(s)$ defined in (1). Then there exist biproper matrices $V_L(s), V_R(s) \in \mathbb{R}(s)^{r\times r}$ (i.e. having no poles and zeros at infinity), such that*

$$V_L(s)A(s)V_R(s) = S^{\infty}_{A(s)}(s) \equiv$$
$$\equiv blockdiag\left[\overbrace{\underbrace{s^{q_1}, s^{q_2}, ..., s^{q_k}}_{k}, I_{v-k}}^{v}, \overbrace{\frac{1}{s^{\widehat{q}_{v+1}}}, \frac{1}{s^{\widehat{q}_{v+2}}}, ..., \frac{1}{s^{\widehat{q}_r}}}^{r-v}\right] \tag{6}$$

with $1 \le v \le r$ and $q_1 \ge q_2 \ge \cdots \ge q_v \ge 0$ and $0 < \hat{q}_{v+1} \le \hat{q}_{v+2} \le \cdots \le \hat{q}_r$. $S^{\infty}_{A(s)}(s)$ is called the Smith-McMillan form of $A(s)$ at $s = \infty$, where q_i ($\hat{q}_i$) are the orders of the poles (zeros) of $A(s)$ at $s = \infty$. We also denote by $\hat{q}$ the sum of the orders of the infinite zeros of $A(s)$ (order accounted for) i.e.

$$\hat{q} := \sum_{i=v+1}^{r} \hat{q}_i$$

Define the dual polynomial matrix of $A(s)$

$$\tilde{A}(s) = s^q A\left(\frac{1}{s}\right) = A_0 s^q + A_1 s^{q-1} + \cdots + A_q$$

Since $rank\tilde{A}(0) = rankA_q$ the dual matrix $\tilde{A}(s)$ of $A(s)$ has zeros at $s = 0$ iff $rankA_q < r$. Let $rankA_q < r$ and let

$$S^0_{\tilde{A}(s)}(s) = diag\{s^{\mu_1}, ..., s^{\mu_r}\} \tag{7}$$

be the local Smith form of $\tilde{A}(s)$ at $s = 0$ where $\mu_j \in \mathbb{Z}^+$ and $0 \le \mu_1 \le \mu_2 \le ... \le \mu_r$. The *infinite elementary divisors* (*IED*) of $A(s)$ are defined as the f.e.d.

s^{μ_j} of its dual $\tilde{A}(s)$ at $s = 0$. The total number μ of the i.e.d. of $A(s)$ is $\mu = \sum_{i=1}^{r} \mu_i$. The infinite elementary divisors of $A(s)$ are given by [7], [1] :

$$s^{\mu_j}, j = 1, 2, 3, .., r$$
$$\mu_j = q - q_j, j = 1, 2, ..., k \text{ with } q_1 = q \geq q_2 \geq \cdots \geq q_k$$
$$\mu_j = q, j = k+1, k+2, .., v$$
$$\mu_j = q + \hat{q}_j, j = v+1, \tau+2, ..., r \text{ with } \hat{q}_{v+1} \leq \hat{q}_{v+2} \leq \cdots \leq \hat{q}_r$$

where q_i $(\hat{q}_i)$ are the orders of the *poles* (*zeros*) of $A(s)$ at $s = \infty$ defined above. The first kind of IED, i.e. the ones with degrees $\mu_j = q - q_j > 0, j = 2, 3, .., k$ are called *infinite pole IEDs*, since are connected with the orders of the poles q_j, $j = 1, 2, ..., k$ of $A(s)$ at $s = \infty$. The second kind of IED, i.e. the ones with degrees $\mu_j = q + \hat{q}_j > 0, j = v+1, v+2, ..., r$ are called *infinite zero IEDs*, since are connected with the orders of the zeros $\hat{q}_j$, $j = v+1, v+2, ..., r$ of $A(s)$ at $s = \infty$.

Based on the definition of a finite Jordan pair we can now give the definition of an *infinite Jordan pair.*

Theorem 4 *([6],pp.185) Let $(C_\infty \in \mathbb{R}^{r\times\mu}, J_\infty \in \mathbb{R}^{\mu\times\mu})$ be a matrix pair where J_∞ is a Jordan matrix with unique eigenvalue $s_0 = 0$. Then, the following conditions are necessary and sufficient in order that (C_∞, J_∞) be an infinite Jordan pair of $A(s) = A_q s^q + \cdots + A_1 s + A_0$:*

(a) $\det \tilde{A}(s)$ *has a zero at $s_0 = 0$ of multiplicity μ, where $\tilde{A}(s) = s^q A\left(\frac{1}{s}\right) = A_0 s^q + A_1 s^{q-1} + \cdots + A_q$,*

(b)

$$rank \begin{pmatrix} C_\infty \\ C_\infty J_\infty \\ \vdots \\ C_\infty J_\infty^{\mu-1} \end{pmatrix} = \mu$$

(c)

$$A_0 C_\infty J_\infty^q + \cdots + A_{q-1} C_\infty J_\infty + A_q C_\infty = 0$$

Taking a finite Jordan pair $(C_{\infty i} \in \mathbb{R}^{r\times\mu_i}, J_{\infty i} \in \mathbb{R}^{\mu_i\times\mu_i})$ for different algebraic multiplicities of the zero $s_i = 0, i = 1, 2, .., r$ of $\tilde{A}(s)$ we define an *infinite Jordan pair* $(C_\infty \in \mathbb{R}^{r\times\mu}, J_\infty \in \mathbb{R}^{\mu\times\mu})$ of $A(s)$ as

$$C_\infty = \begin{pmatrix} C_{\infty 1} & C_{\infty 2} & \cdots & C_{\infty r} \end{pmatrix} \in \mathbb{R}^{r\times\mu}$$
$$J_\infty = diag\begin{pmatrix} J_{\infty 1} & J_{\infty 2} & \cdots & J_{\infty r} \end{pmatrix} \in \mathbb{R}^{\mu\times\mu}$$

where $\mu = \mu_1 + \mu_2 + \cdots + \mu_r$ is the total number of infinite elementary divisors (eigenvalues) of $A(s)$ (order accounted for). It is easily seen from the above that (C_∞, J_∞) is an infinite Jordan pair of $A(s)$ iff it is a finite Jordan pair of its dual polynomial matrix $\tilde{A}(s)$ corresponding to its zero at $s = 0$.

Whereas the finite Jordan pairs of a polynomial matrix are connected with the smooth behavior of the Auto-Regressive (AR) representation (5), the infinite

Jordan pairs are connected with its impulsive solution space. More specifically the column span of

$$\Psi_{\infty} = \sum_{i=0}^{\hat{q}_r - 1} \tilde{C}_{\infty,z} \tilde{J}_{\infty,z}^{i} \delta^{(i)}(t)$$

where

$$\tilde{C}_{\infty,z} := \left[\tilde{C}_{\infty,v+1}, \tilde{C}_{\infty,v+2}, ..., \tilde{C}_{\infty,r}\right] \in \mathbb{R}^{r\times \mu_z}$$
$$\tilde{J}_{\infty,z} := blockdiag\left[\tilde{J}_{\infty,v+1}, \tilde{J}_{\infty,v+2}, ..., \tilde{J}_{\infty,r}\right] \in \mathbb{R}^{\mu_z \times \mu_z}$$

and $\tilde{C}_{\infty,l}$ (resp. $\tilde{J}_{\infty,l}$), with $l = v+1, v+2, ..., r$, consists of the first $\hat{q}_l$ columns (resp. the first $\hat{q}_l \times \hat{q}_l$ block) of the matrix $C_{\infty,l}$ (resp. $J_{\infty,l}$), defines a solution space B^{∞} of (5) with dimension :

$$\dim B^{\infty} = \mu_z = \sum_{i=v+1}^{r} \hat{q}_i$$

Therefore to each infinite Jordan block of order $\mu_j = q + \hat{q}_j$ with $j = v+1, v+2, ..., r$ corresponds $\hat{q}_j$ linear independent impulsive solutions. B^{∞} is known as the *impulsive solution space* of (5) defined by [1]. The whole solution space $B = B^{\mathbb{C}} + B^{\infty}$ of (5) has dimension

$$\dim B = \dim B^{\mathbb{C}} + \dim B^{\infty} = n + \mu_z = \deg\det A(s) + \sum_{i=v+1}^{r} \hat{q}_i \overset{([1])}{=} \sum_{i=1}^{k} q_i = \delta_M(A(s))$$

where $\delta_M(\cdot)$ denotes the McMillan degree.

3 Construction of a System of Algebraic-Differential Equations with Given Impulsive Solution Space

In the previous section we have shown how to construct the smooth and impulsive behavior of a given homogeneous system of algebraic-differential equations $A(\rho)\beta(t) = 0, A(\rho) \in \mathbb{R}[\rho]^{r\times r}$ with $\det A(\rho) \neq 0$ and $\rho := d/dt$ by using the finite and infinite Jordan pairs of the polynomial matrix $A(s)$. In this section, we study the **inverse problem** : Given a specific smooth and impulsive behavior $B = B^{\mathbb{C}} + B^{\infty}$, is it possible to construct a polynomial matrix $A(\rho)$ and therefore a homogeneous system of algebraic-differential equations $A(\rho)\beta(t) = 0$ with the prescribed behavior ? The answer to the case where only the smooth solution space $B^{\mathbb{C}}$ is known, is given by ([6], pp.227). However, certain questions still remaining concerning the case where only the impulsive solution space B^{∞} (or the smooth and impulsive solution $B = B^{\mathbb{C}} + B^{\infty}$) is known. In the following Theorem we present the know results presented in [6] that concerns the smooth solution space $B^{\mathbb{C}}$.

Theorem 5 *[6] Let* $\beta_j(t) = \left(\sum_{k=0}^{\sigma_j-1} \beta_{j,k} t^k\right) e^{s_j t}$ *where each* $\beta_{j,k}$ *is a vector in* $\mathbb{C}^r, 0 \le k \le \sigma_j - 1, 1 \le j \le \ell \le r$. *Define*

$$C_j = \begin{pmatrix} \beta_{j,0} & \cdots & (\sigma_j-2)!\beta_{j,\sigma_j-2} & (\sigma_j-1)!\beta_{j,\sigma_j-1} \end{pmatrix}$$

where $j = 1,2,...,\ell$ *and let*

$$C = \begin{pmatrix} C_1 & C_2 & \cdots & C_\ell \end{pmatrix} \in \mathbb{R}^{r\times n}, J = \begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_\ell \end{pmatrix} \in \mathbb{R}^{n\times n}$$

with J_i *be the Jordan block of order* σ_i *with eigenvalue* s_i *and* $n = \sum_{j=1}^{\ell} \sigma_j$. *Let* a *be a complex number different from* $s_1, s_2, ..., s_\ell$, *and define*

$$A(s) = I_r - C(J - aI_n)^{-q} \left\{ (s-a) V_q + (s-a)^2 V_{q-1} + \cdots + (s-a)^q V_1 \right\}$$

where $q = ind(C, J)$ *i.e.the least integer such that the matrix*

$$S_{q-1} = \begin{pmatrix} C \\ CJ \\ \vdots \\ CJ^{q-1} \end{pmatrix}$$

has full column rank, and $V = \begin{pmatrix} V_1 & V_2 & \cdots & V_q \end{pmatrix}$ *is a left inverse of*

$$S_{1-q} = \begin{pmatrix} C \\ C(J-aI_n)^{-1} \\ \vdots \\ C(J-aI_n)^{1-q} \end{pmatrix}$$

i.e. $VS_{1-q} = I_{rq}$. *Then* $\beta_j(t), j = 1,2,...,\ell$ *are solutions of the equation*

$$A(\rho)\beta(t) = 0$$

and every solution of this equation is a linear combination of $\beta_j(t)$ *and their derivatives. Further,* q *is the minimal possible degree of any* $r \times r$ *matrix polynomial with this property.*

In what follows, we are trying to extend the previous Theorem to the impulsive behavior case. More specifically we are trying to give an answer to the following problem.

Problem 6 *Given the impulsive solution vector space spanned by the vectors* $B^{\infty} = \left\langle \beta_j(t) = \sum_{k=0}^{\hat{q}_j-1} x_{j,k} \delta^{(\hat{q}_j-1-k)}(t) \right\rangle$ *where* $x_{j,k} \in \mathbb{C}^r, 0 \le k \le \hat{q}_j - 1, 1 \le j \le \ell$, *find* $q \in \mathbb{N}$ *and a polynomial matrix* $A(\rho) = A_0 + A_1\rho + \cdots + A_q\rho^q$ *where* $A_i \in \mathbb{R}^{r\times r}$, $i = 1,2,...,q$ *such that all the vectors* $\beta_j(t) \in B^{\infty}$ *are solutions of the AR-representation (5) i.e.* $A(\rho)\beta_j(t) = 0$.

As we shall see in the following Theorem, the answer to the above problem can be reduced to a problem that has already been solved by Theorem 5 but which concerns the dual polynomial matrix of $A(s)$ that we are looking for.

Theorem 7 *The vector*

$$\beta_j^{\infty}(t) = x_{j,0}\delta^{(\hat{q}_j-1)}(t) + x_{j,1}\delta^{(\hat{q}_j-2)}(t) + \cdots + x_{j,\hat{q}_j-2}\delta^{(1)}(t) + x_{j,\hat{q}_j-1}\delta(t) \quad (8)$$

with $j = 1, 2, ..., l$, is a solution of (5), iff

$$\tilde{\beta}_j(t) = x_{j,0}\frac{t^{q+\hat{q}_j-1}}{(q+\hat{q}_j-1)!} + x_{j,1}\frac{t^{q+\hat{q}_j-2}}{(q+\hat{q}_j-2)!} + \cdots + x_{j,q+\hat{q}_j-2}t + x_{j,q+\hat{q}_j-1} \quad (9)$$

is a solution of the dual homogeneous system

$$\tilde{A}(\rho)\tilde{\beta}(t) = 0 \quad (10)$$

where $\tilde{A}(\rho) = \rho^q A\left(\frac{1}{\rho}\right)$.

Proof. First we will show that if (8) is a solution of (5) then (9) is the solution of (10). Since (8) is a solution of (5), then according to [1] there exist vectors $x_{j,\hat{q}_j}, x_{j,\hat{q}_j+1}, ..., x_{j,q+\hat{q}_j-1}$ such that

$$A(s)\left(x_{j,0}s^{\hat{q}_j-1} + x_{j,1}s^{\hat{q}_j-2} + \cdots + x_{j,\hat{q}_j-2}s + x_{j,\hat{q}_j-1}\right) =$$
$$= \begin{pmatrix} s^{q-1}I_r & \cdots & sI_r & I_r \end{pmatrix} \begin{pmatrix} A_q & 0 & \cdots & 0 \\ A_{q-1} & A_q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \cdots & A_q \end{pmatrix} \begin{pmatrix} -x_{j,\hat{q}_j} \\ -x_{j,\hat{q}_j+1} \\ \vdots \\ -x_{j,q+\hat{q}_j-1} \end{pmatrix}$$

where $x_{j,i}, i = 0, 1, .., q+\hat{q}_j-1$ are satisfying the relations

$$\underbrace{\begin{pmatrix} A_q & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ A_{q-1} & A_q & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & & & \vdots \\ A_1 & A_2 & A_3 & \cdots & A_q & 0 & 0 & \cdots & 0 \\ A_0 & A_1 & A_2 & \cdots & A_{q-1} & A_q & 0 & \cdots & 0 \\ 0 & A_0 & A_1 & \cdots & A_{q-2} & A_{q-1} & A_q & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & & & & \\ 0 & 0 & A_0 & A_1 & A_2 & \cdots & \cdots & A_{q-1} & A_q \end{pmatrix}}_{r\mu_j \times r\mu_j} \underbrace{\begin{pmatrix} x_{j,0} \\ x_{j,1} \\ \vdots \\ x_{j,q-1} \\ x_{j,q} \\ x_{j,q+1} \\ \vdots \\ x_{j,\mu_j-1} \end{pmatrix}}_{r\mu_j \times 1} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{r\mu_j \times 1} \quad (11)$$

with $\mu_j = q + \hat{q}_j$. By taking Laplace transforms in (10) and using the Laplace transform of (9), we obtain that

$$\tilde{A}(s)(x_{j,0}\frac{1}{s^{q+\hat{q}_j}} + x_{j,1}\frac{1}{s^{q+\hat{q}_j-1}} + \cdots + x_{j,\hat{q}_j-2}\frac{1}{s^{q+2}} + x_{j,\hat{q}_j-1}\frac{1}{s^{q+1}} +$$
$$+ \cdots + x_{j,\hat{q}_j+q-2}\frac{1}{s^2} + x_{j,q+\hat{q}_j-1}\frac{1}{s}) =$$

$$= (A_0 s^q + \cdots + sA_{q-1} + A_q)\times$$
$$\times(x_{j,0}\frac{1}{s^{q+\hat{q}_j}} + x_{j,1}\frac{1}{s^{q+\hat{q}_j-1}} + \cdots + x_{j,\hat{q}_j-2}\frac{1}{s^{q+2}} + x_{j,\hat{q}_j-1}\frac{1}{s^{q+1}} +$$
$$+ + \cdots + x_{j,\hat{q}_j+q-2}\frac{1}{s^2} + x_{j,q+\hat{q}_j-1}\frac{1}{s}) =$$

$$= \begin{pmatrix} s^{q-1}I_r & \cdots & sI_r & I_r & \cdots & \frac{1}{s^{q+\hat{q}_j-1}}I_r & \frac{1}{s^{q+\hat{q}_j}}I_r \end{pmatrix}\times$$
$$\times \underbrace{\begin{pmatrix} A_0 & 0 & \cdots & 0 & 0 \\ A_1 & A_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_q & A_{q-1} & \ddots & A_1 & \ddots \\ 0 & A_q & \ddots & A_2 & \ddots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_q & A_{q-1} \\ 0 & 0 & \cdots & 0 & A_q \end{pmatrix}}_{r(q+l+1)\times r(q+l+1)} \begin{pmatrix} x_{j,q+\hat{q}_j-1} \\ x_{j,q+\hat{q}_j-2} \\ \vdots \\ x_{j,1} \\ x_{j,0} \end{pmatrix} =$$

$$= \begin{pmatrix} s^{q-1}I_r & \cdots & sI_r & I_r \end{pmatrix} \begin{pmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{pmatrix} \begin{pmatrix} x_{j,q+\hat{q}_j-1} \\ x_{j,q+\hat{q}_j-2} \\ \vdots \\ x_{j,\hat{q}_j} \end{pmatrix} +$$
$$+ \begin{pmatrix} \frac{1}{s}I_r & \cdots & \frac{1}{s^{q+\hat{q}_j-1}}I_r & \frac{1}{s^{q+\hat{q}_j}}I_r \end{pmatrix} \begin{pmatrix} A_q & A_{q-1} & \cdots & A_0 & \cdots \\ 0 & A_q & \ddots & A_1 & \ddots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & A_2 & \ddots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_q & A_{q-1} \\ 0 & 0 & \cdots & 0 & A_q \end{pmatrix} \begin{pmatrix} x_{j,q+\hat{q}_j-1} \\ x_{j,q+\hat{q}_j-2} \\ \vdots \\ x_{j,1} \\ x_{j,0} \end{pmatrix} \overset{(11)}{=}$$
$$= \begin{pmatrix} s^{q-1}I_r & \cdots & sI_r & I_r \end{pmatrix} \begin{pmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{pmatrix} \begin{pmatrix} x_{j,q+\hat{q}_j-1} \\ x_{j,q+\hat{q}_j-2} \\ \vdots \\ x_{j,\hat{q}_j} \end{pmatrix} \quad (12)$$

Thus $\tilde{\beta}_j(t)$ given in (9) is a solution of (5) for the initial conditions $\tilde{\beta}(0) = x_{j,q+\hat{q}_j-1}, \tilde{\beta}^{(1)}(0) = x_{j,q+\hat{q}_j-2}, .., \tilde{\beta}^{(q-1)}(0) = x_{j,\hat{q}_j}$.

Now, we show that if (9) is the solution of (10) then (8) is the solution of (5). By using the same reasoning if (9) is a solution of (10) we have that

$$\tilde{A}(s)\,(x_{j,0}\frac{1}{s^{q+\hat{q}_j}}+x_{j,1}\frac{1}{s^{q+\hat{q}_j-1}}+\cdots+x_{j,\hat{q}_j-2}\frac{1}{s^{q+2}}+x_{j,\hat{q}_j-1}\frac{1}{s^{q+1}}+$$
$$+\cdots+x_{j,\hat{q}_j+q-2}\frac{1}{s^2}+x_{j,q+\hat{q}_j-1}\frac{1}{s})=$$
$$=\begin{pmatrix} s^{q-1}I_r & \cdots & sI_r & I_r \end{pmatrix}\begin{pmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{pmatrix}\begin{pmatrix} x_{j,q+\hat{q}_j-1} \\ x_{j,q+\hat{q}_j-2} \\ \vdots \\ x_{j,\hat{q}_j} \end{pmatrix} \tag{13}$$

By taking Laplace transforms in (5) and (8), we have that

$$A(s)\left(x_{j,0}s^{\hat{q}_j-1}+x_{j,1}s^{\hat{q}_j-2}+\cdots+x_{j,\hat{q}_j-2}s+x_{j,\hat{q}_j-1}\right)=$$
$$=\left(A_qs^q+A_{q-1}s^{q-1}+\cdots+A_1s+A_0\right)\left(x_{j,0}s^{\hat{q}_j-1}+x_{j,1}s^{\hat{q}_j-2}+\cdots+x_{j,\hat{q}_j-2}s+x_{j,\hat{q}_j-1}\right)=$$
$$=\begin{pmatrix} s^{q+\hat{q}_j-1}I_r & \cdots & sI_r & I_r \end{pmatrix}\underbrace{\begin{pmatrix} A_q & 0 & \cdots & 0 & 0 \\ A_{q-1} & A_q & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_0 & A_1 & \ddots & A_q & \ddots \\ 0 & A_0 & \ddots & A_{q-1} & \ddots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_0 & A_1 \\ 0 & 0 & \cdots & 0 & A_0 \end{pmatrix}}_{r(q+\hat{q}_j)\times r\hat{q}_j}\begin{pmatrix} x_{j,0} \\ x_{j,1} \\ \vdots \\ x_{j,\hat{q}_j-2} \\ x_{j,\hat{q}_j-1} \end{pmatrix}$$

Taking into account that the term in (12) must be zero, in order the relation (13) to be satisfied, we have that

$$A(s)\left(x_{j,0}s^{\hat{q}_j-1}+x_{j,1}s^{\hat{q}_j-2}+\cdots+x_{j,\hat{q}_j-2}s+x_{j,\hat{q}_j-1}\right)=$$
$$=\begin{pmatrix} s^{q+\hat{q}_j-1}I_r & \cdots & sI_r & I_r \end{pmatrix}\underbrace{\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_q & 0 & \ddots & 0 \\ A_{q-1} & A_q & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \cdots & A_q \end{pmatrix}}_{r(q+\hat{q}_j)\times rq}\begin{pmatrix} -x_{j,\hat{q}_j} \\ -x_{j,\hat{q}_j+1} \\ \vdots \\ -x_{j,q+\hat{q}_j-1} \end{pmatrix}=$$

$$= \left(\begin{array}{cccc} s^{q-1}I_r & \cdots & sI_r & I_r \end{array} \right) \left(\begin{array}{cccc} A_q & 0 & \cdots & 0 \\ A_{q-1} & A_q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \cdots & A_q \end{array} \right) \left(\begin{array}{c} -x_{j,\hat{q}_j} \\ -x_{j,\hat{q}_j+1} \\ \vdots \\ -x_{j,q+\hat{q}_j-1} \end{array} \right)$$

Therefore the system (5) will have the solution defined in (8). ■

According to the above Theorem the problem of finding a system of the form (5) that gives rise to the specific solutions

$$\beta_j(t) = x_{j,0}\delta^{(\hat{q}_j-1)}(t) + x_{j,1}\delta^{(\hat{q}_j-2)}(t) + \cdots + x_{j,\hat{q}_j-2}\delta^{(1)}(t) + x_{j,\hat{q}_j-1}\delta(t)$$

where each $x_{j,k}$ is a vector in $\mathbb{C}^r, 0 \leq k \leq \hat{q}_j - 1, 1 \leq j \leq \ell$, is equivalent to the problem of finding a system of the form (10) that gives rise to the solutions

$$\tilde{\beta}_j(t) = x_{j,0}\frac{t^{q+\hat{q}_j-1}}{(q+\hat{q}_j-1)!} + x_{j,1}\frac{t^{q+\hat{q}_j-2}}{(q+\hat{q}_j-2)!} + \cdots + x_{j,q+\hat{q}_j-2}t + x_{j,q+\hat{q}_j-1}$$

where the unknown vectors $x_{j,i}, i = \hat{q}_j, \hat{q}_j + 1, .., q + \hat{q}_j - 1$ must satisfy the relations (11). Equivalently we can correspond to each known pair of matrices of the form

$$C_j = \left(\begin{array}{cccc} x_{j,0} & x_{j,1} & \cdots & x_{j,\hat{q}_j-1} \end{array} \right) \in \mathbb{R}^{r\times\hat{q}_j}, J_j = \left(\begin{array}{ccccc} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{array} \right) \in \mathbb{R}^{\hat{q}_j\times\hat{q}_j} \tag{14}$$

a finite Jordan pair of the form

$$C_{\infty j} = \left(\begin{array}{cccccccc} x_{j,0} & x_{j,1} & \cdots & x_{j,\hat{q}_j-1} & x_{j,\hat{q}_j} & x_{j,\hat{q}_j+1} & \cdots & x_{j,q+\hat{q}_j-1} \end{array} \right) \in \mathbb{R}^{r\times(q+\hat{q}_j)} \tag{15}$$

$$J_{\infty j} = \left(\begin{array}{ccccc} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{array} \right) \in \mathbb{R}^{(q+\hat{q}_j)\times(q+\hat{q}_j)}$$

where the vectors $x_{j,\hat{q}_j}, x_{j,\hat{q}_j+1}, ..., x_{j,q+\hat{q}_j-1}$ are unknowns, and the matrix pair $(C_{\infty j}, J_{\infty j})$ is a finite Jordan pair of the dual polynomial matrix $\tilde{A}(s)$ we are looking for. However, the last problem can be easily solved by using the results from Theorem 5, as we can easily see in the following Example. Based on the above we conclude to the following Theorem.

Theorem 8 *Let* $\beta_j^{\infty}(t) = \sum_{k=0}^{\hat{q}_j-1} x_{j,k}\delta^{(\hat{q}_j-1-k)}$ *where each* $x_{j,k}$ *is a vector in* $\mathbb{C}^r, 0 \leq k \leq \hat{q}_j - 1, 1 \leq j \leq \ell < r$. *Define*

$$C_{\infty j} = \left(\begin{array}{cccccccc} x_{j,0} & x_{j,1} & \cdots & x_{j,\hat{q}_j-1} & x_{j,\hat{q}_j} & x_{j,\hat{q}_j+1} & \cdots & x_{j,q+\hat{q}_j-1} \end{array} \right)$$

where $j = 1, 2, ..., \ell$ *and let*

$$C = \begin{pmatrix} C_1 & C_2 & \cdots & C_\ell \end{pmatrix} \in \mathbb{R}^{r \times \mu}, J = \begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_\ell \end{pmatrix} \in \mathbb{R}^{\mu \times \mu}$$

with J_j *be the Jordan block of order* $\mu_j = q + \hat{q}_j$ *with eigenvalue* 0 *and* $\mu = \sum_{j=1}^{\ell} \mu_j$. *Let* $a \neq 0$ *be a complex number, and define*

$$\tilde{A}(s) = I_r - C(J - aI_n)^{-q} \left\{ (s-a) V_q + (s-a)^2 V_{q-1} + \cdots + (s-a)^q V_1 \right\}$$

where $q = ind(C, J)$ *i.e.the least integer such that the matrix*

$$S_{q-1} = \begin{pmatrix} C \\ CJ \\ \vdots \\ CJ^{q-1} \end{pmatrix}$$

has full column rank, and $V = \begin{pmatrix} V_1 & V_2 & \cdots & V_q \end{pmatrix}$ *is a left inverse of*

$$S_{1-q} = \begin{pmatrix} C \\ C(J - aI_n)^{-1} \\ \vdots \\ C(J - aI_n)^{1-q} \end{pmatrix}$$

i.e. $VS_{1-q} = I_{rq}$. *Then* $\beta_j^{\infty}(t)$, $j = 1, 2, ..., \ell$ *are solutions of the equation*

$$A(\rho)\beta(t) = 0$$

where $A(\rho) = \rho^q \tilde{A}\left(\frac{1}{\rho}\right)$. *Further,* q *is the minimal possible degree of any* $r \times r$ *matrix polynomial with this property.*

Note that any unimodular equivalent polynomial matrix $\bar{A}(s)$ i.e. $\bar{A}(s) = U(s)\tilde{A}(s)$ with $\det U(s) \in \mathbb{R} - \{0\}$, gives rise to a solution of our problem.

Remark 9 *Similarly with the above Theorem, someone can easily show that the vector*

$$\beta_j^{\infty}(t) = x_{j,0}\delta^{(\hat{q}_j - 1)}(t) + x_{j,1}\delta^{(\hat{q}_j - 2)}(t) + \cdots + x_{j,\hat{q}_j - 2}\delta^{(1)}(t) + x_{j,\hat{q}_j - 1}\delta(t)$$

is a solution of (5), iff

$$\tilde{\beta}_j(t) = \left(x_{j,0}\frac{t^{q+\hat{q}_j-1}}{(q+\hat{q}_j-1)!} + x_{j,1}\frac{t^{q+\hat{q}_j-2}}{(q+\hat{q}_j-2)!} + \cdots + x_{j,q+\hat{q}_j-2}t + x_{j,q+\hat{q}_j-1} \right) e^{at}$$

is a solution of the homogeneous system $\tilde{A}(\rho)\tilde{\beta}(t) = 0$ *where* $\tilde{A}(\rho) = (\rho - a)^q A\left(\frac{1}{\rho - a}\right)$.

Example 10 *Suppose that we want to find a polynomial matrix $A(s)$ such that the AR-representation*

$$A(\rho)\beta(t) = 0 \tag{16}$$

has the following solution

$$\beta_1(t) = \underbrace{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{x_{1,1}} \delta(t) + \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{x_{1,0}} \delta^{(1)}(t) \tag{17}$$

and thus $\hat{q}_2 = 2$.

Let $q = 1$. Then we have that

$$C = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} \end{pmatrix} = \begin{pmatrix} 1 & 1 & x_1 \\ 0 & -1 & x_2 \end{pmatrix} \in \mathbb{R}^{2\times 3}, J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{3\times 3}$$

where the vector $x_{1,2}$ is unknown, and

$$S_{q-1} = S_{1-1} = S_0 = C = \begin{pmatrix} 1 & 1 & x_1 \\ 0 & -1 & x_2 \end{pmatrix}$$

has not full column rank and thus $q \neq 1$. Another explanation for the rejection of this order is that the only possible Smith form at infinity of a 2×2 matrix pencil, that includes infinite zeros, is the following :

$$S^{\infty}_{A(s)}(s) = \begin{pmatrix} s & 0 \\ 0 & \frac{1}{s} \end{pmatrix}$$

and thus the greatest dimension of the impulsive solution space is $q = \hat{q}_2 = 1$ and not 2.

Let $q = q + 1 = 2$. Then

$$C = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} \end{pmatrix} = \begin{pmatrix} 1 & 1 & x_1 & x_3 \\ 0 & -1 & x_2 & x_4 \end{pmatrix} \in \mathbb{R}^{2\times 4}$$

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{4\times 4}$$

where the vectors $x_{1,2}, x_{1,3}$ are unknowns and

$$S_{2-1} = S_1 = \begin{pmatrix} C \\ CJ^{2-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & x_1 & x_3 \\ 0 & -1 & x_2 & x_4 \\ 0 & 1 & 1 & x_1 \\ 0 & 0 & -1 & x_2 \end{pmatrix}$$

which has full column rank when $-x_2^2 - x_2 - x_1 - x_4 \neq 0$. Therefore, under the assumption that $-x_2^2 - x_2 - x_1 - x_4 \neq 0$, we have that $q = ind(C, J) = 2$.

Suppose for example that $x_1 = -1$ *and* $x_2 = x_3 = x_4 = 0$. *Let* $a \neq 0$ *be some complex number i.e.* $a = 1$, *and define*

$$\tilde{A}(s) \overset{a=1}{=} I_2 - C(J - aI_4)^{-2}\left\{(s-a)V_2 + (s-a)^2 V_1\right\}$$

where $\begin{pmatrix} V_1 & V_2 \end{pmatrix}$ *is a left inverse of*

$$\begin{pmatrix} C \\ C(J - aI_4)^{-1} \end{pmatrix} \overset{a=1}{=} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & -2 & -1 & -1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

Therefore

$$\begin{pmatrix} V_1 & V_2 \end{pmatrix} = \begin{pmatrix} C \\ C(J - aI_4)^{-1} \end{pmatrix}^{-1} \overset{a=1}{=} \begin{pmatrix} 0 & 1 & -1 & -1 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & -1 & -1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$$

or otherwise

$$V_1 = \begin{pmatrix} 0 & 1 \\ 0 & -1 \\ -1 & 0 \\ 1 & 1 \end{pmatrix} \quad ; \quad V_2 = \begin{pmatrix} -1 & -1 \\ 0 & 0 \\ -1 & -1 \\ 1 & 2 \end{pmatrix}$$

Thus

$$\tilde{A}(s) \overset{a=1}{=} I_2 - C(J - aI_4)^{-2}\left((s-a)V_2 + (s-a)^2 V_1\right) =$$
$$= \begin{pmatrix} 2s - s^2 & -3s^2 + s + 2 \\ s^2 - s & 2s^2 - 1 \end{pmatrix}$$

with

$$S^{\mathbb{C}}_{\tilde{A}(s)}(s) = \begin{pmatrix} 1 & 0 \\ 0 & s^4 \end{pmatrix}$$

Therefore

$$A(s) = s^2 \tilde{A}\left(\frac{1}{s}\right) = \begin{pmatrix} 2s - 1 & 2s^2 + s - 3 \\ 1 - s & 2 - s^2 \end{pmatrix}$$

with

$$S^{\infty}_{A(s)}(s) = \begin{pmatrix} s^2 & 0 \\ 0 & \frac{1}{s^2} \end{pmatrix}$$

Any other unimodular equivalent polynomial matrix of the dual polynomial matrix $\tilde{A}(s)$ *i.e.* $\bar{A}(s) = U(s)\tilde{A}(s)$ *with* $\det U(s) = a \in \mathbb{R} - \{0\}$, *is also a solution to our problem.*

An interesting question is the following : Can we always find a homogeneous system of the form $A(\rho)\beta(t) = 0$ that will achieve any specific impulsive behavior ? The answer is no, since the elementary divisor structure of $A(\rho)$ must meet certain conditions like the ones given in the following Theorem.

Theorem 11 *[8] The total number $n+\mu$ of f.e.d. and i.e.d. (orders accounted for) of the square polynomial matrix $A(s)$ defined in (1) is equal to rq.*

According to the above Theorem, it is impossible to have two linear independent impulsive solutions of order 2, since in that case the sum of the orders of the infinite elementary divisors $(q+\hat{q}_2)+(q+\hat{q}_3)$ will exceed the number $rq=2q$ or equivalently it is impossible for a 2×2 square polynomial matrix to have 2 infinite zeros (only one pole and one zero at $s=\infty$).

4 Construction of a System of Algebraic-Differential Equations with Given Smooth and Impulsive Behavior

In the previous section we propose a method for finding directly an AR-representation of the form (5) with given **either** *smooth* **or** *impulsive behavior*. In the last case, it was necessary first to construct a dual AR-representation (10) with given *smooth behavior* that is coming from the impulsive behavior of the AR-representation (5) we are looking for. An interesting question is, if it is possible to reduce the problem of finding the AR-representation of the form (5) with given *smooth behavior* to an equivalent problem of finding a dual AR-representation (10) with given a different *smooth behavior*. If this is possible, then instead of finding an AR-representation with given smooth-impulsive behavior it will be enough to find a dual AR-representation with given smooth behavior only. The next Theorem proposes an answer to this problem.

Theorem 12 *Let $A(s)=A_0+A_1s+\cdots+A_qs^q\in\mathbb{R}[s]^{r\times r}$ with $rank_{\mathbb{R}(s)}A(s)=r$ and $s_j\neq 0$ such that $\det A(s_j)=0$ (s_j is a zero of $A(s)$). If $\beta(t)=C_je^{J_jt}x_0$ (where $(C_j\in\mathbb{R}^{r\times n_j},J_j\in\mathbb{R}^{n_j\times n_j})$ is a finite Jordan pair that corresponds to a zero s_j of the polynomial matrix $A(s)$) is a solution of the AR-representation $A(\rho)\beta(t)=0$ then $\tilde{\beta}_j(t)=\left(C_jJ_j^{-1}\right)e^{J_j^{-1}t}x_0$ is a solution of the dual representation $\tilde{A}(\rho)\tilde{\beta}(t)=0$.*

Proof. Since $\beta_j(t)=C_je^{J_jt}x_0$ is a solution of the AR-representation $A(\rho)\beta(t)=0$, we have that

$$\begin{pmatrix}\beta_j(0)\\ \beta_j^{(1)}(0)\\ \vdots\\ \beta_j^{(q-1)}(0)\end{pmatrix}=\begin{pmatrix}C_j\\ C_jJ_j\\ \vdots\\ C_jJ_j^{q-1}\end{pmatrix}x_0$$

and thus by taking Laplace transforms in $A(\rho)\beta(t)=0$ we obtain

$$A(s)\underbrace{C_j\left(sI_{n_j}-J_j\right)^{-1}x_0}_{\beta(s)}=$$

$$=\begin{pmatrix} s^{q-1}I_r & \cdots & sI_r & I_r \end{pmatrix}\begin{pmatrix} A_q & 0 & \cdots & 0 \\ A_{q-1} & A_q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \cdots & A_q \end{pmatrix}\begin{pmatrix} C_j \\ C_jJ_j \\ \vdots \\ C_jJ_j^{q-1} \end{pmatrix}x_0$$

or equivalently by replacing s by $\frac{1}{s}$ and multiplying both sides by s^q

$$\tilde{A}(s)\,C_j\left(\frac{1}{s}I_{n_j}-J_j\right)^{-1}x_0=$$

$$=s^q\begin{pmatrix} \frac{1}{s^{q-1}}I_r & \cdots & \frac{1}{s}I_r & I_r \end{pmatrix}\begin{pmatrix} A_q & 0 & \cdots & 0 \\ A_{q-1} & A_q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \cdots & A_q \end{pmatrix}\begin{pmatrix} C_j \\ C_jJ_j \\ \vdots \\ C_jJ_j^{q-1} \end{pmatrix}x_0=$$

$$=\begin{pmatrix} s^qI_r & \cdots & s^2I_r & sI_r \end{pmatrix}\begin{pmatrix} A_1 & A_2 & \cdots & A_q \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_q & \cdots & 0 \\ A_q & 0 & \cdots & 0 \end{pmatrix}\begin{pmatrix} C_j \\ C_jJ_j \\ \vdots \\ C_jJ_j^{q-1} \end{pmatrix}x_0$$

and multiplying both sides by $\frac{1}{s}$

$$\tilde{A}(s)\,C_j\frac{1}{s}\left(\frac{1}{s}I_{n_j}-J_j\right)^{-1}x_0=$$

$$=\begin{pmatrix} s^{q-1}I_r & \cdots & sI_r & I_r \end{pmatrix}\begin{pmatrix} A_1 & A_2 & \cdots & A_q \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_q & \cdots & 0 \\ A_q & 0 & \cdots & 0 \end{pmatrix}\begin{pmatrix} C_j \\ C_jJ_j \\ \vdots \\ C_jJ_j^{q-1} \end{pmatrix}x_0\Leftrightarrow$$

$$\tilde{A}(s)\,C_j\frac{1}{s}\left(\frac{1}{s}\left(I_{n_j}-sJ_j\right)\right)^{-1}x_0=$$

$$=\begin{pmatrix} s^{q-1}I_r & \cdots & sI_r & I_r \end{pmatrix}\begin{pmatrix} A_1 & A_2 & \cdots & A_q \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_q & \cdots & 0 \\ A_q & 0 & \cdots & 0 \end{pmatrix}\begin{pmatrix} C_j \\ C_jJ_j \\ \vdots \\ C_jJ_j^{q-1} \end{pmatrix}x_0\Leftrightarrow$$

$$\tilde{A}(s)\, C_j \left(I_{n_j} - sJ_j\right)^{-1} x_0 =$$

$$= \begin{pmatrix} s^{q-1}I_r & \cdots & sI_r & I_r \end{pmatrix} \begin{pmatrix} A_1 & A_2 & \cdots & A_q \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_q & \cdots & 0 \\ A_q & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} C_j \\ C_j J_j \\ \vdots \\ C_j J_j^{q-1} \end{pmatrix} x_0$$

Since $s_j \neq 0$ the matrix J_j is invertible and thus

$$\tilde{A}(s)\, C_j \left(\left(J_j^{-1} - sI_{n_j}\right) J_j\right)^{-1} x_0 =$$

$$= \begin{pmatrix} s^{q-1}I_r & \cdots & sI_r & I_r \end{pmatrix} \begin{pmatrix} A_1 & A_2 & \cdots & A_q \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_q & \cdots & 0 \\ A_q & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} C_j J_j \\ C_j J_j^2 \\ \vdots \\ C_j J_j^q \end{pmatrix} J_j^{-1} x_0$$

Since the matrix pair (C_j, J_j) is a finite matrix Jordan pair we have that $A_0 C_j + A_1 C_j J_j + \cdots + A_q C_j J_j^{q-1} = 0$ and therefore

$$-\tilde{A}(s)\, C_j J_j^{-1} \left(sI_{n_j} - J_j^{-1}\right)^{-1} x_0 =$$

$$= -\begin{pmatrix} s^{q-1}I_r & \cdots & sI_r & I_r \end{pmatrix} \begin{pmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \ddots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{pmatrix} \begin{pmatrix} C_j \\ C_j J_j^{-1} \\ \vdots \\ C_j J_j^{-q+1} \end{pmatrix} J_j^{-1} x_0 \Leftrightarrow$$

$$\tilde{A}(s) \underbrace{C_j J_j^{-1} \left(sI_{n_j} - J_j^{-1}\right)^{-1} x_0}_{\tilde{\beta}(s)} =$$

$$= \begin{pmatrix} s^{q-1}I_r & \cdots & sI_r & I_r \end{pmatrix} \begin{pmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \ddots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{pmatrix} \begin{pmatrix} C_j J_j^{-1} \\ C_j J_j^{-2} \\ \vdots \\ C_j J_j^{-q} \end{pmatrix} x_0$$

Therefore

$$\tilde{\beta}(t) = L^{-1}\left\{\tilde{\beta}(s)\right\} = L^{-1}\left\{C_j J_j^{-1} \left(sI_{n_j} - J_j^{-1}\right)^{-1} x_0\right\} = \left(C_j J_j^{-1}\right) e^{J_j^{-1} t} x_0$$

is a solution of the dual AR-representation $\tilde{A}(\rho)\,\tilde{\beta}(t) = 0$ for the initial conditions

$$\begin{pmatrix} \beta_j(0) \\ \beta_j^{(1)}(0) \\ \vdots \\ \beta_j^{(q-1)}(0) \end{pmatrix} = \begin{pmatrix} C_j J_j^{-1} \\ C_j J_j^{-2} \\ \vdots \\ C_j J_j^{-q} \end{pmatrix} x_0$$

■

Since the matrix J_j^{-1} is not in a Jordan form, we can always find a nonsingular constant matrix $U \in \mathbb{R}^{n_j \times n_j}$ such that $J_j^{-1} = U\tilde{J}_j U^{-1}$ where the matrix $\tilde{J}_j$ is in Jordan form. In that case, we have that the solution of the dual AR-representation $\tilde{A}(\rho)\tilde{\beta}(t) = 0$ can be written also as follows :

$$\begin{aligned}
\tilde{\beta}(t) &= C_j J_j^{-1} e^{J_j^{-1}} x_0 = C_j J_j^{-1}\left(I + \frac{t}{1!}J_j^{-1} + \frac{t^2}{2!}J_j^{-2} + \cdots\right) x_0 \overset{J_j^{-1}=U\tilde{J}_jU^{-1}}{=} \\
&= C_j U\tilde{J}_j U^{-1}\left(UU^{-1} + \frac{t}{1!}\left(U\tilde{J}_jU^{-1}\right) + \frac{t^2}{2!}\left(U\tilde{J}_jU^{-1}\right)^2 + \cdots\right) x_0 = \\
&= C_j U\tilde{J}_j U^{-1}U\left(I + \frac{t}{1!}\tilde{J}_j + \frac{t^2}{2!}\tilde{J}_j^2 + \cdots\right) U^{-1}x_0 = \\
&= \left(C_j U\tilde{J}_j\right) e^{\tilde{J}_j} U^{-1} x_0 \overset{\tilde{C}_j = C_j J_j^{-1} U}{=} \tilde{C}_j e^{\tilde{J}_j}\left(U^{-1}x_0\right)
\end{aligned}$$

Therefore, instead of using the matrix pair $\left(C_j J_j^{-1} \in \mathbb{R}^{r\times n_j}, J_j^{-1} \in \mathbb{R}^{n_j\times n_j}\right)$ where J_j^{-1} is not in Jordan form, we can use the matrix pair $(\tilde{C}_j = C_jU\tilde{J}_j \in \mathbb{R}^{r\times n_j},\ \tilde{J}_j = U^{-1}J_j^{-1}U \in \mathbb{R}^{n_j\times n_j})$ with $\tilde{J}_j$ in Jordan form.

$$\begin{gathered}
\left(C_j \in \mathbb{R}^{r\times n_j}, J_j \in \mathbb{R}^{n_j\times n_j}\right) \\
\updownarrow \\
\left(\tilde{C}_j = C_jU\tilde{J}_j \in \mathbb{R}^{r\times n_j}, \tilde{J}_j = U^{-1}J_j^{-1}U \in \mathbb{R}^{n_j\times n_j}\right)
\end{gathered}$$

Example 13 *Consider the AR-representation*

$$\underbrace{\begin{pmatrix} \frac{5}{2} - \frac{3}{2}\rho & \frac{1}{2}\rho - \frac{1}{2} \\ \frac{1}{2} - \frac{1}{2}\rho & \frac{3}{2} - \frac{1}{2}\rho \end{pmatrix}}_{A(\rho)}\underbrace{\begin{pmatrix} \beta_1(t) \\ \beta_2(t) \end{pmatrix}}_{\beta(t)} = 0_{2,1}$$

with a solution space generated by the following linear independent vector functions

$$\begin{aligned}
\begin{pmatrix} \beta_1(t) & \beta_2(t) \end{pmatrix} &= \underbrace{\begin{pmatrix} \beta_{1,0} & \beta_{1,1} \end{pmatrix}}_{C} e^{Jt} = \underbrace{\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}_{C} e^{\overbrace{\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}}^{J} t} = \\
&= \begin{pmatrix} e^{2t} & e^{2t}(t-1) \\ e^{2t} & e^{2t}(t+1) \end{pmatrix}
\end{aligned}$$

for initial conditions given by

$$\beta(0) = \begin{pmatrix} \beta_1(0) & | & \beta_2(0) \end{pmatrix} = C = \left(\begin{array}{c|c} 1 & -1 \\ 1 & 1 \end{array}\right)$$

Consider the matrices

$$J^{-1}=\left(\begin{array}{cc}2&1\\0&2\end{array}\right)^{-1}=\left(\begin{array}{cc}\frac{1}{2}&-\frac{1}{4}\\0&\frac{1}{2}\end{array}\right)=\underbrace{\left(\begin{array}{cc}1&0\\0&-4\end{array}\right)}_{U}\underbrace{\left(\begin{array}{cc}\frac{1}{2}&1\\0&\frac{1}{2}\end{array}\right)}_{\tilde{J}}\underbrace{\left(\begin{array}{cc}1&0\\0&-4\end{array}\right)^{-1}}_{U^{-1}}\Longrightarrow$$

$$\tilde{J}=\left(\begin{array}{cc}\frac{1}{2}&1\\0&\frac{1}{2}\end{array}\right)$$

$$\tilde{C}=\underbrace{\left(\begin{array}{cc}1&-1\\1&1\end{array}\right)}_{C}\underbrace{\left(\begin{array}{cc}2&1\\0&2\end{array}\right)^{-1}}_{J^{-1}}\underbrace{\left(\begin{array}{cc}1&0\\0&-4\end{array}\right)}_{U}=\left(\begin{array}{cc}\frac{1}{2}&3\\\frac{1}{2}&-1\end{array}\right)$$

Then the dual AR-representation

$$\underbrace{\left(\begin{array}{cc}\frac{5}{2}\rho-\frac{3}{2}&\frac{1}{2}-\frac{1}{2}\rho\\\frac{1}{2}\rho-\frac{1}{2}&\frac{3}{2}\rho-\frac{1}{2}\end{array}\right)}_{\tilde{A}(\rho)}\underbrace{\left(\begin{array}{c}\tilde{\beta}_1(t)\\\tilde{\beta}_2(t)\end{array}\right)}_{\tilde{\beta}(t)}=0_{2,1}$$

according to the above Theorem has a solution space generated by the following linear independent vector functions

$$\left(\begin{array}{cc}\tilde{\beta}_1(t)&\tilde{\beta}_2(t)\end{array}\right)=\tilde{C}e^{\tilde{J}t}=$$

$$=\left(\begin{array}{cc}\frac{1}{2}&3\\\frac{1}{2}&-1\end{array}\right)e^{\left(\begin{array}{cc}\frac{1}{2}&1\\0&\frac{1}{2}\end{array}\right)t}=\left(\begin{array}{cc}\frac{1}{2}e^{\frac{1}{2}t}&\frac{1}{2}e^{\frac{1}{2}t}(t+6)\\\frac{1}{2}e^{\frac{1}{2}t}&\frac{1}{2}e^{\frac{1}{2}t}(t-2)\end{array}\right)$$

for initial conditions given by

$$\tilde{\beta}(0)=\left(\begin{array}{c|c}\tilde{\beta}_1(0)&\tilde{\beta}_2(0)\end{array}\right)=CJ^{-1}=\left(\begin{array}{cc}\frac{1}{2}&3\\\frac{1}{2}&-1\end{array}\right)$$

According to Theorems 7 and 12, in order to construct an AR-representation of the form (5) for a given smooth and impulsive solution space we need to apply the following Algorithm :

Algorithm 14 *Construction of an AR-representation with a given smooth-impulsive behavior (not including polynomial behaviour).*

1. Transform the finite Jordan pairs $(C_j\in\mathbb{R}^{r\times n_j},J_j\in\mathbb{R}^{n_j\times n_j})$ *that correspond to the solutions of the form* $\beta(t)=C_je^{J_jt}x_0$ *to the finite Jordan pairs of the form* $\left(C_jU\tilde{J}_j\in\mathbb{R}^{r\times n_j},\tilde{J}_j=U^{-1}J_j^{-1}U\in\mathbb{R}^{n_j\times n_j}\right)$ *that correspond to the solutions of the form* $\tilde{\beta}_j(t)=\left(C_jU\tilde{J}_j\right)e^{\tilde{J}_j^{-1}t}x_0$ *of the dual system that we are looking for.*

2. Transform the infinite Jordan pairs (14) that correspond to solutions of the form (8) to finite Jordan pairs (15) that correspond to solutions (9) of the dual system that we are looking for.

3. Construct the dual polynomial matrix $\tilde{A}(s)$ by using the method appearing in Theorem 5.

4. Get the polynomial matrix $A(s) = s^q \tilde{A}\left(\frac{1}{s}\right)$ that we are looking for and thus the AR-representation $A(\rho)\beta(t) = 0$.

According to the above algorithm, we are trying to map all the smooth and impulsive solutions of $A(\rho)\beta(t) = 0$ to smooth solutions of the dual system $\tilde{A}(\rho)\tilde{\beta}(t) = 0$ and then using Theorem 5 we construct the dual matrix $\tilde{A}(s)$. The above algorithm works well only when the smooth solution $\beta(t) = C_j e^{J_j t} x_0$ is not a polynomial of t, since in that case the polynomial matrix $A(s)$ that we are looking for, has a zero at $s_0 = 0$ and thus the Theorem 12 is not working.

Example 15 *Suppose that we want to find an AR-representation $A(\rho)\beta(t) = 0$ with the following smooth and impulsive solutions*

$$\beta_1(t) = \underbrace{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}_{\beta_{1,1}} e^{2t} + \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\beta_{1,0}} te^{2t} \quad ; \quad \beta_2(t) = \underbrace{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{x_{1,1}} \delta(t) + \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{x_{1,0}} \delta^{(1)}(t)$$

We can start by assuming that the order of the polynomial matrix is $q = 1$, then $q = 2$ e.t.c.. However, the form of the first solution inform us that the polynomial matrix $A(s)$ that we are looking for, has determinant at least equal to $(s-2)^2$ and thus a possible Smith form at $\mathbb{C}$ of $A(s)$ is equal to

$$S^{\mathbb{C}}_{A(s)}(s) = \begin{pmatrix} 1 & 0 \\ 0 & (s-2)^2 \end{pmatrix}$$

From the form of the second solution we can conclude that the Smith form at infinity of $A(s)$ must have a zero at infinity of order 2. Therefore, the polynomial matrix $A(s)$ must have at least order 4, since in that case we have

$$S^{\infty}_{A(s)}(s) = \begin{pmatrix} s^4 & 0 \\ 0 & \frac{1}{s^2} \end{pmatrix}$$

Let $q = 4$. Then according to Algorithm 14, we have

Step 1.

$$C_1 = \begin{pmatrix} \beta_{1,0} & \beta_{1,1} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad ; \quad J_1 = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

$$J_1^{-1} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} \\ 0 & \frac{1}{2} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{pmatrix}}_{\tilde{J}_1} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix}^{-1}}_{U^{-1}} \Longrightarrow$$

$$\tilde{J}_1 = \begin{pmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$\tilde{C}_1 = C_1 U \tilde{J}_1 = \underbrace{\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}_{C_1} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{pmatrix}}_{\tilde{J}_1} = \begin{pmatrix} \frac{1}{2} & 3 \\ \frac{1}{2} & -1 \end{pmatrix}$$

Step 2.

$$C_2 = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} \end{pmatrix} = \begin{pmatrix} 1 & 1 & x_1 & x_3 & x_5 & x_7 \\ 0 & -1 & x_2 & x_4 & x_6 & x_8 \end{pmatrix} \in \mathbb{R}^{2\times 6},$$

$$J_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{6\times 6}$$

where the vectors $x_{1,2}, x_{1,3}, x_{1,4}, x_{1,5}$ are unknown.

Step 3. Let

$$C = \begin{pmatrix} \tilde{C}_1 & C_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{3}{16} & 1 & 1 & x_1 & x_3 & x_5 & x_7 \\ \frac{1}{2} & -\frac{1}{16} & 0 & -1 & x_2 & x_4 & x_6 & x_8 \end{pmatrix}$$

$$J = \begin{pmatrix} \tilde{J}_1 & 0 \\ 0 & J_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Then

$$rank S_4 = rank \begin{pmatrix} C \\ CJ \\ CJ^2 \\ CJ^3 \end{pmatrix} = rank \begin{pmatrix} \frac{1}{2} & 3 & 1 & 1 & x_1 & x_3 & x_5 & x_7 \\ \frac{1}{2} & -1 & 0 & -1 & x_2 & x_4 & x_6 & x_8 \\ \frac{1}{4} & 2 & 0 & 1 & 1 & x_1 & x_3 & x_5 \\ \frac{1}{4} & 0 & 0 & 0 & -1 & x_2 & x_4 & x_6 \\ \frac{1}{8} & \frac{5}{4} & 0 & 0 & 1 & 1 & x_1 & x_3 \\ \frac{1}{8} & \frac{1}{4} & 0 & 0 & 0 & -1 & x_2 & x_4 \\ \frac{1}{16} & \frac{3}{4} & 0 & 0 & 0 & 1 & 1 & x_1 \\ \frac{1}{16} & \frac{1}{4} & 0 & 0 & 0 & 0 & -1 & x_2 \end{pmatrix} = 8$$

Therefore $q = ind\,(C, J) = 4$ which agrees with our assumption that $q = 4$ (note that the matrices $S_1 \in \mathbb{R}^{2\times 5}, S_2 \in \mathbb{R}^{4\times 6}, S_3 \in \mathbb{R}^{6\times 7}$ do not have full column rank and therefore q is different from $1, 2, 3$). Suppose for example that $x_1 = 4$ and $x_2 = 0, x_3 = 0, x_4 = 0, x_5 = 0, x_6 = 0, x_7 = 0, x_8 = 0$. Let a be some complex number different from 0 and $\frac{1}{2}$ i.e. $a = 1$ and define

$$\tilde{A}\,(s) \overset{a=1}{\equiv} I_2 - C\,(J - aI_8)^{-4} \times$$
$$\times \left\{ (s-a)\,V_4 + (s-a)^2\,V_3 + (s-a)^3\,V_2 + (s-a)^4\,V_1 \right\}$$

where $\begin{pmatrix} V_1 & V_2 & V_3 & V_4 \end{pmatrix}$ *is a left inverse (simple inverse since it is square) of*

$$\begin{pmatrix} C \\ C\left(J-aI_8\right)^{-1} \\ C\left(J-aI_8\right)^{-2} \\ C\left(J-aI_8\right)^{-3} \end{pmatrix} \overset{a=1}{=} \begin{pmatrix} \frac{1}{2} & 3 & 1 & 1 & 4 & 0 & 0 & 0 \\ \frac{1}{2} & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & -8 & -1 & -2 & -6 & -6 & -6 & -6 \\ -1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 2 & 20 & 1 & 3 & 9 & 15 & 21 & 27 \\ 2 & 4 & 0 & -1 & -2 & -3 & -4 & -5 \\ -4 & -48 & -1 & -4 & -13 & -28 & -49 & -76 \\ -4 & -16 & 0 & 1 & 3 & 6 & 10 & 15 \end{pmatrix}$$

Therefore

$$\begin{pmatrix} V_1 & V_2 & V_3 & V_4 \end{pmatrix} = \begin{pmatrix} C \\ C\left(J-aI_8\right)^{-1} \\ C\left(J-aI_8\right)^{-2} \\ C\left(J-aI_8\right)^{-3} \end{pmatrix}^{-1} \overset{a=1}{=}$$

$$= \begin{pmatrix} 0 & \frac{22}{9} & \frac{11}{18} & \frac{115}{18} & \frac{5}{6} & \frac{17}{3} & \frac{2}{9} & \frac{4}{3} \\ 0 & -\frac{1}{3} & -\frac{1}{12} & -\frac{11}{12} & -\frac{1}{8} & -\frac{7}{8} & -\frac{1}{24} & -\frac{1}{4} \\ 0 & -\frac{110}{9} & -\frac{55}{18} & -\frac{485}{18} & -\frac{8}{3} & -\frac{107}{6} & -\frac{11}{18} & -\frac{11}{3} \\ 0 & \frac{5}{9} & \frac{7}{18} & \frac{37}{9} & \frac{13}{24} & \frac{89}{24} & \frac{11}{72} & \frac{11}{12} \\ \frac{1}{4} & \frac{103}{36} & \frac{47}{72} & \frac{403}{72} & \frac{25}{48} & \frac{167}{48} & \frac{17}{144} & \frac{17}{24} \\ -\frac{3}{4} & -\frac{29}{36} & -\frac{109}{72} & -\frac{161}{72} & -\frac{47}{48} & -\frac{73}{48} & -\frac{31}{144} & -\frac{7}{24} \\ \frac{3}{4} & -\frac{19}{36} & \frac{121}{72} & -\frac{67}{72} & \frac{59}{48} & -\frac{35}{48} & \frac{43}{144} & -\frac{5}{24} \\ -\frac{1}{4} & \frac{13}{36} & -\frac{43}{72} & \frac{61}{72} & -\frac{23}{48} & \frac{35}{48} & -\frac{19}{144} & \frac{5}{24} \end{pmatrix}$$

Thus

$$V_1 = \begin{pmatrix} 0 & \frac{22}{9} \\ 0 & -\frac{1}{3} \\ 0 & -\frac{110}{9} \\ 0 & \frac{5}{9} \\ \frac{1}{4} & \frac{103}{36} \\ -\frac{3}{4} & -\frac{29}{36} \\ \frac{3}{4} & -\frac{19}{36} \\ -\frac{1}{4} & \frac{13}{36} \end{pmatrix} \quad ; \quad V_2 = \begin{pmatrix} \frac{11}{18} & \frac{115}{18} \\ -\frac{1}{12} & -\frac{11}{12} \\ -\frac{55}{18} & -\frac{485}{18} \\ \frac{7}{18} & \frac{37}{9} \\ \frac{47}{72} & \frac{403}{72} \\ -\frac{109}{72} & -\frac{161}{72} \\ \frac{121}{72} & -\frac{67}{72} \\ -\frac{43}{72} & \frac{61}{72} \end{pmatrix}$$

$$V_3 = \begin{pmatrix} \frac{5}{6} & \frac{17}{3} \\ -\frac{1}{8} & -\frac{7}{8} \\ -\frac{8}{3} & -\frac{107}{6} \\ \frac{13}{24} & \frac{89}{24} \\ \frac{25}{48} & \frac{167}{48} \\ -\frac{47}{48} & -\frac{73}{48} \\ \frac{59}{48} & -\frac{35}{48} \\ -\frac{23}{48} & \frac{35}{48} \end{pmatrix} \quad ; \quad V_4 = \begin{pmatrix} \frac{2}{9} & \frac{4}{3} \\ -\frac{1}{24} & -\frac{1}{4} \\ -\frac{11}{18} & -\frac{11}{3} \\ \frac{11}{72} & \frac{11}{12} \\ \frac{17}{144} & \frac{17}{24} \\ -\frac{31}{144} & -\frac{7}{24} \\ \frac{43}{144} & -\frac{5}{24} \\ -\frac{19}{144} & \frac{5}{24} \end{pmatrix}$$

The dual polynomial matrix $\tilde{A}(s)$ is given by

$$\tilde{A}(s) \overset{a=1}{\equiv} I_2 - C(J - aI_8)^{-4} \times$$
$$\times \left\{ (s-a) V_4 + (s-a)^2 V_3 + (s-a)^3 V_2 + (s-a)^4 V_1 \right\} =$$
$$= \begin{pmatrix} \frac{3}{2}s^4 - \frac{1}{12}s^3 - \frac{5}{8}s^2 + \frac{5}{24}s & \frac{7}{6}s^4 - \frac{13}{12}s^3 + \frac{1}{8}s^2 - \frac{5}{12}s + \frac{5}{24} \\ -\frac{1}{4}s^4 + \frac{49}{72}s^3 - \frac{9}{16}s^2 + \frac{19}{144}s & \frac{89}{36}s^4 - \frac{131}{72}s^3 + \frac{31}{48}s^2 - \frac{31}{72}s + \frac{19}{144} \end{pmatrix}$$

and therefore the polynomial matrix $A(s)$ is

$$A(s) = s^4 A\left(\frac{1}{s}\right) =$$
$$= \begin{pmatrix} \frac{3}{2} - \frac{1}{12}s - \frac{5}{8}s^2 + \frac{5}{24}s^3 & \frac{7}{6} - \frac{13}{12}s + \frac{1}{8}s^2 - \frac{5}{12}s^3 + \frac{5}{24}s^4 \\ -\frac{1}{4} + \frac{49}{72}s - \frac{9}{16}s^2 + \frac{19}{144}s^3 & \frac{89}{36} - \frac{131}{72}s + \frac{31}{48}s^2 - \frac{31}{72}s^3 + \frac{19}{144}s^4 \end{pmatrix}$$

We can easily check that

$$S^{\mathbb{C}}_{A(s)}(s) = \begin{pmatrix} 1 & 0 \\ 0 & (s-2)^2 \end{pmatrix}$$

and

$$S^{0}_{\tilde{A}(s)}(s) = \begin{pmatrix} 1 & 0 \\ 0 & s^6 \end{pmatrix} = \begin{pmatrix} s^{4-4} & 0 \\ 0 & s^{4+2} \end{pmatrix}$$

or otherwise

$$S^{\infty}_{A(s)}(s) = \begin{pmatrix} s^4 & 0 \\ 0 & \frac{1}{s^2} \end{pmatrix}$$

Any other unimodular equivalent polynomial matrix $\hat{A}(s)$ of $\tilde{A}(s)$ i.e. $\hat{A}(s) = U(s)\tilde{A}(s)$ with $\det U(s) \in \mathbb{R}$, provides us with a solution to this problem.

By using Algorithm 14, we have constructed a polynomial matrix $A(s)$ (or a family of polynomial matrices dependent on the parameter a and the parameters x_i) which describes the system of algebraic-differential equations that we are looking for. As we have noticed $A(s)$ is not the only polynomial matrix with this specific behavior. Any other unimodular equivalent polynomial matrix $\hat{A}(s)$ of $\tilde{A}(s)$ i.e. $\hat{A}(s) = U(s)\tilde{A}(s)$ with $\det U(s) \in \mathbb{R}$, give us a solution to this problem. Another answer to this question is given by [12]. More specifically, all the *full unimodular* equivalent polynomial matrices of $A(s)$ are having this specific property, where the definition of full unimodular equivalence is given below :

Definition 16 *[9] $A_1(s), A_2(s) \in \mathbb{R}[s]^{r\times m}$ are said to be fully unimodular equivalent if there exists a unimodular matrix $U(s) \in \mathbb{R}[s]^{r\times r}$ such that*

$$U(s) A_1(s) = A_2(s)$$

where the compound matrix

$$\begin{pmatrix} U(s) & A_2(s) \end{pmatrix}$$

(i) has no zeros at $s=\infty$,

(ii) $\delta_M\begin{pmatrix} U(s) & A_2(s)\end{pmatrix} = \delta_M(A_2(s))$ *where* $\delta_M(\cdot)$ *indicates the McMillan degree of the indicated matrix.*

Full unimodular equivalence is a special case of full system equivalence [7] and thus has the nice property of preserving both the finite and infinite zero structure of polynomial matrices (see [9]) in contrast to unimodular equivalence which preserves only the finite aspects. In a recent work [12] it was shown that two AR-representations that are described by the full row rank polynomial matrices $A_1(s), A_2(s) \in \mathbb{R}[s]^{r\times m}$ have the same behavior iff the polynomial matrices $A_1(s), A_2(s)$ are full unimodular equivalent.

Example 17 *Consider the polynomial matrix*

$$A_1(s) = \begin{pmatrix} \frac{3}{2} - \frac{1}{12}s - \frac{5}{8}s^2 + \frac{5}{24}s^3 & \frac{7}{6} - \frac{13}{12}s + \frac{1}{8}s^2 - \frac{5}{12}s^3 + \frac{5}{24}s^4 \\ -\frac{1}{4} + \frac{49}{72}s - \frac{9}{16}s^2 + \frac{19}{144}s^3 & \frac{89}{36} - \frac{131}{72}s + \frac{31}{48}s^2 - \frac{31}{72}s^3 + \frac{19}{144}s^4 \end{pmatrix}$$

that we find in Example 15 and the polynomial matrix

$$A_2(s) = \begin{pmatrix} -\frac{3}{119}s^3 + \frac{6}{119}s^2 + \frac{2}{17}s - \frac{5}{17} & -\frac{3}{119}s^4 + \frac{3}{119}s^3 + \frac{2}{17}s^2 - \frac{12}{119}s - \frac{1}{119} \\ -\frac{5}{119}s^3 + \frac{22}{119}s^2 - \frac{4}{17}s + \frac{1}{17} & -\frac{5}{119}s^4 + \frac{1}{7}s^3 - \frac{16}{119}s^2 + \frac{18}{119}s - \frac{27}{119} \end{pmatrix}$$

Then we can check that

$$\underbrace{\begin{pmatrix} -\frac{19}{2856}s^3 + \frac{5}{272}s^2 + \frac{331}{17\,136}s - \frac{781}{4284} & \frac{5}{476}s^3 - \frac{15}{952}s^2 - \frac{197}{2856}s + \frac{59}{714} \\ -\frac{95}{8568}s^3 + \frac{109}{1904}s^2 - \frac{1747}{17\,136}s + \frac{95}{4284} & \frac{25}{1428}s^3 - \frac{65}{952}s^2 + \frac{125}{2856}s - \frac{73}{714} \end{pmatrix}}_{U(s)}$$

$$\times\underbrace{\begin{pmatrix} \frac{3}{2} - \frac{1}{12}s - \frac{5}{8}s^2 + \frac{5}{24}s^3 & \frac{7}{6} - \frac{13}{12}s + \frac{1}{8}s^2 - \frac{5}{12}s^3 + \frac{5}{24}s^4 \\ -\frac{1}{4} + \frac{49}{72}s - \frac{9}{16}s^2 + \frac{19}{144}s^3 & \frac{89}{36} - \frac{131}{72}s + \frac{31}{48}s^2 - \frac{31}{72}s^3 + \frac{19}{144}s^4 \end{pmatrix}}_{A_1(s)} =$$

$$= \underbrace{\begin{pmatrix} -\frac{3}{119}s^3 + \frac{6}{119}s^2 + \frac{2}{17}s - \frac{5}{17} & -\frac{3}{119}s^4 + \frac{3}{119}s^3 + \frac{2}{17}s^2 - \frac{12}{119}s - \frac{1}{119} \\ -\frac{5}{119}s^3 + \frac{22}{119}s^2 - \frac{4}{17}s + \frac{1}{17} & -\frac{5}{119}s^4 + \frac{1}{7}s^3 - \frac{16}{119}s^2 + \frac{18}{119}s - \frac{27}{119} \end{pmatrix}}_{A_2(s)}$$

where $\det U(s) = \frac{2}{119} \in \mathbb{R}$ *(*$U(s)$ *is unimodular), the compound matrix* $\begin{pmatrix} U(s) & A_2(s)\end{pmatrix}$ *has no infinite zeros since*

$$S^{\infty}_{\begin{pmatrix} U(s) & A_2(s)\end{pmatrix}}(s) = \begin{pmatrix} s^4 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\delta_M\begin{pmatrix} U(s) & A_2(s)\end{pmatrix} = 4 = \delta_M(A_2(s))$$

Therefore $A_1(s), A_2(s)$ *are full unimodular equivalent and thus the AR-representations that are described by these matrices possess the same smooth and impulsive behavior, or equivalently* $A_2(s)$ *belongs to the class of polynomial matrices that we are looking for in Example 15.*

Theorem 12 and Algorithm 14 are working well only in case where we have no polynomial functions in the behavior space, since we are using the transformation $s \rightarrow \frac{1}{s}$. We may overcome this problem, and thus include polynomial functions in the behavior space, by using the transformation $s \rightarrow \frac{1}{s} + b$ instead of $s \rightarrow \frac{1}{s}$, and thus moving all possible zeros of $A(s)$ to non-zero places. In order to do this, we can use the following two remarks given by ([6], Proof of Theorem 7.3) :

- If (C_1, J_1) is a finite Jordan pair of $A(s)$ then $(C_1, J_1 + bI_n)$ is a finite Jordan pair of $A(s-b)$.
- If (C_2, J_2) is an infinite Jordan pair of $A(s)$ then $\left(C_2, J_2 (I_n + bJ_2)^{-1}\right)$ is an infinite Jordan pair of $A(s-b)$.

Example 18 *Suppose that we want to find an AR-representation $A(\rho)\beta(t) = 0$ with the following smooth and impulsive solutions*

$$\beta_1(t) = \underbrace{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}_{\beta_{1,1}} + \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\beta_{1,0}} t \quad ; \quad \beta_2(t) = \underbrace{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{x_{1,1}} \delta(t) + \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{x_{1,0}} \delta^{(1)}(t)$$

We can start by assuming that the order of the polynomial matrix is $q = 1$, then $q = 2$ e.t.c.. However, the form of the first solution inform us that the polynomial matrix $A(s)$ that we are looking for, has determinant at least equal to s^2 and thus a possible Smith form at $\mathbb{C}$ of $A(s)$ is equal to

$$S^{\mathbb{C}}_{A(s)}(s) = \begin{pmatrix} 1 & 0 \\ 0 & s^2 \end{pmatrix}$$

From the form of the second solution we can conclude that the Smith form at infinity of $A(s)$ must have a zero at infinity of order 2. Therefore, the polynomial matrix $A(s)$ must have at least order 4, since in that case we have

$$S^{\infty}_{A(s)}(s) = \begin{pmatrix} s^4 & 0 \\ 0 & \frac{1}{s^2} \end{pmatrix}$$

Let $q = 4$. Then according to Algorithm 14 we have

Step 1. Instead of using the finite Jordan pair

$$C_1 = \begin{pmatrix} \beta_{1,0} & \beta_{1,1} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad ; \quad J_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

with zero at $s = 0$, we select the finite Jordan pair

$$C_1 = \begin{pmatrix} \beta_{1,0} & \beta_{1,1} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad ; \quad J_1 := J_1 + bI_2 \overset{b=2}{=} \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

that will give us at the end the dual polynomial matrix of $A(s-2)$ if we use the Algorithm 14. Let also

$$\tilde{C}_1 = \underbrace{\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}_{C_1} \underbrace{\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}^{-1}}_{J_1^{-1}} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix}}_{U} = \begin{pmatrix} \frac{1}{2} & 3 \\ \frac{1}{2} & -1 \end{pmatrix}$$

$$J_1^{-1} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} \\ 0 & \frac{1}{2} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{pmatrix}}_{\tilde{J}_1} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix}^{-1}}_{U^{-1}} \Longrightarrow$$

$$\tilde{J}_1 = \begin{pmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{pmatrix}$$

Step 2. Instead of using the finite Jordan pair

$$C_2 = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} \end{pmatrix} = \begin{pmatrix} 1 & 1 & x_1 & x_3 & x_5 & x_7 \\ 0 & -1 & x_2 & x_4 & x_6 & x_8 \end{pmatrix} \in \mathbb{R}^{2\times 6},$$

$$J_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{6\times 6}$$

we use the same C_2 and the matrix

$$J_2 = J_2 (I_6 + aJ_2)^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^{-1} =$$

$$= \begin{pmatrix} 0 & 1 & -2 & 4 & -8 & 16 \\ 0 & 0 & 1 & -2 & 4 & -8 \\ 0 & 0 & 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

for the same reasons given above, where the vectors $x_{1,2}, x_{1,3}, x_{1,4}, x_{1,5}$ are unknown.

Step 3. Let

$$C = \begin{pmatrix} \tilde{C}_1 & C_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{3}{16} & 1 & 1 & x_1 & x_3 & x_5 & x_7 \\ \frac{1}{2} & -\frac{1}{16} & 0 & -1 & x_2 & x_4 & x_6 & x_8 \end{pmatrix}$$

$$J = \begin{pmatrix} \tilde{J}_1 & 0 \\ 0 & J_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 4 & -8 & 16 \\ 0 & 0 & 0 & 0 & 1 & -2 & 4 & -8 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Then

$$rank S_3 = rank \begin{pmatrix} C \\ CJ \\ CJ^2 \\ CJ^3 \end{pmatrix} =$$

$$= rank \begin{pmatrix} \frac{1}{2} & 3 & 1 & 1 & x_1 & x_3 & x_5 & x_7 \\ \frac{1}{2} & -1 & 0 & -1 & x_2 & x_4 & x_6 & x_8 \\ \frac{1}{4} & \frac{19}{32} & 0 & 1 & -1 & x_1+2 & x_3-2x_1-4 & 4x_1-2x_3+x_5+8 \\ \frac{1}{4} & \frac{15}{32} & 0 & 0 & -1 & x_2+2 & x_4-2x_2-4 & 4x_2-2x_4+x_6+8 \\ \frac{1}{8} & \frac{35}{64} & 0 & 0 & 1 & -3 & x_1+8 & x_3-4x_1-20 \\ \frac{1}{8} & \frac{31}{64} & 0 & 0 & 0 & -1 & x_2+4 & x_4-4x_2-12 \\ \frac{1}{16} & \frac{51}{128} & 0 & 0 & 0 & 1 & -5 & x_1+18 \\ \frac{1}{16} & \frac{47}{128} & 0 & 0 & 0 & 0 & -1 & x_2+6 \end{pmatrix} = 8$$

Therefore $q = ind\,(C, J) = 4$ *which agrees with our assumption that* $q = 4$ *(note that the matrices* $S_1 \in \mathbb{R}^{2\times 5}$, $S_2 \in \mathbb{R}^{4\times 6}$, $S_3 \in \mathbb{R}^{6\times 7}$ *have note full column rank and thus* $q > 3$*). Suppose for example that* $x_1 = 4$ *and* $x_2 = 0, x_3 = 0, x_4 = 0, x_5 = 0, x_6 = 0, x_7 = 0, x_8 = 0$*. Let* a *be some complex number different from* 0 *and* $1/2$ *i.e.* $a = 1$ *and define*

$$\tilde{A}\,(s) \overset{a=1}{\equiv} I_2 - C\,(J - aI_8)^{-4} \times$$
$$\times \left\{ (s-a)\,V_4 + (s-a)^2\,V_3 + (s-a)^3\,V_2 + (s-a)^4\,V_1 \right\}$$

where $\begin{pmatrix} V_1 & V_2 & V_3 & V_4 \end{pmatrix}$ *is a left inverse (simple inverse since it is square) of*

$$\begin{pmatrix} C \\ C\,(J-aI_8)^{-1} \\ C\,(J-aI_8)^{-2} \\ C\,(J-aI_8)^{-3} \end{pmatrix} \overset{a=1}{\equiv} \begin{pmatrix} \frac{1}{2} & \frac{3}{16} & 1 & 1 & 4 & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{16} & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & -\frac{19}{8} & -1 & -2 & -4 & -4 & 4 & -4 \\ -1 & -\frac{15}{8} & 0 & 1 & 1 & -1 & 1 & -1 \\ 2 & \frac{35}{4} & 1 & 3 & 5 & 7 & -3 & -1 \\ 2 & \frac{31}{4} & 0 & -1 & -2 & 1 & 0 & -1 \\ -4 & -\frac{51}{2} & -1 & -4 & -7 & -10 & -1 & 8 \\ -4 & -\frac{47}{2} & 0 & 1 & 3 & 0 & -2 & 3 \end{pmatrix}$$

Therefore

$$\begin{pmatrix} V_1 & V_2 & V_3 & V_4 \end{pmatrix} = \begin{pmatrix} C \\ C\left(J - aI_8\right)^{-1} \\ C\left(J - aI_8\right)^{-2} \\ C\left(J - aI_8\right)^{-3} \end{pmatrix}^{-1} \overset{a=1}{\equiv}$$

$$= \begin{pmatrix} -\frac{63}{16} & \frac{189}{16} & -\frac{59}{8} & \frac{177}{8} & -\frac{267}{64} & \frac{927}{64} & -\frac{47}{64} & \frac{47}{16} \\ \frac{1}{2} & -\frac{3}{2} & 1 & -3 & \frac{5}{8} & -\frac{17}{8} & \frac{1}{8} & -\frac{1}{2} \\ 7 & -19 & 10 & -31 & \frac{19}{4} & -\frac{67}{4} & \frac{3}{4} & -3 \\ -2 & 5 & -\frac{15}{4} & \frac{45}{4} & -\frac{17}{8} & \frac{59}{8} & -\frac{3}{8} & \frac{3}{2} \\ -\frac{17}{32} & \frac{67}{32} & -\frac{11}{16} & \frac{37}{16} & -\frac{21}{128} & \frac{81}{128} & -\frac{1}{128} & \frac{1}{32} \\ \frac{13}{32} & -\frac{55}{32} & \frac{7}{16} & -\frac{41}{16} & -\frac{31}{128} & -\frac{109}{128} & -\frac{35}{128} & \frac{3}{32} \\ -\frac{19}{32} & \frac{41}{32} & -\frac{33}{16} & \frac{63}{16} & -\frac{287}{128} & \frac{467}{128} & -\frac{99}{128} & \frac{35}{32} \\ -\frac{17}{32} & \frac{35}{32} & -\frac{23}{16} & \frac{41}{16} & -\frac{165}{128} & \frac{257}{128} & -\frac{49}{128} & \frac{17}{32} \end{pmatrix}$$

Thus

$$V_1 = \begin{pmatrix} -\frac{63}{16} & \frac{189}{16} \\ \frac{1}{2} & -\frac{3}{2} \\ 7 & -19 \\ -2 & 5 \\ -\frac{17}{32} & \frac{67}{32} \\ \frac{13}{32} & -\frac{55}{32} \\ -\frac{19}{32} & \frac{41}{32} \\ -\frac{17}{32} & \frac{35}{32} \end{pmatrix} \quad ; \quad V_2 = \begin{pmatrix} -\frac{59}{8} & \frac{177}{8} \\ 1 & -3 \\ 10 & -31 \\ -\frac{15}{4} & \frac{45}{4} \\ -\frac{11}{16} & \frac{37}{16} \\ \frac{7}{16} & -\frac{41}{16} \\ -\frac{33}{16} & \frac{63}{16} \\ -\frac{23}{16} & \frac{41}{16} \end{pmatrix}$$

$$V_3 = \begin{pmatrix} -\frac{267}{64} & \frac{927}{64} \\ \frac{5}{8} & -\frac{17}{8} \\ \frac{19}{4} & -\frac{67}{4} \\ -\frac{17}{8} & \frac{59}{8} \\ -\frac{21}{128} & \frac{81}{128} \\ -\frac{31}{128} & -\frac{109}{128} \\ -\frac{287}{128} & \frac{467}{128} \\ -\frac{165}{128} & \frac{257}{128} \end{pmatrix} \quad ; \quad V_4 = \begin{pmatrix} -\frac{47}{64} & \frac{47}{16} \\ \frac{1}{8} & -\frac{1}{2} \\ \frac{3}{4} & -3 \\ -\frac{3}{8} & \frac{3}{2} \\ -\frac{1}{128} & \frac{1}{32} \\ -\frac{35}{128} & \frac{3}{32} \\ -\frac{99}{128} & \frac{35}{32} \\ -\frac{49}{128} & \frac{17}{32} \end{pmatrix}$$

The dual polynomial matrix of $A(s-2)$, *let* $\tilde{B}(s)$ *is given by*

$$\tilde{B}(s) \overset{a=1}{\equiv} I_2 - C\left(J - aI_8\right)^{-4} \times$$
$$\times \left\{(s-a)V_4 + (s-a)^2 V_3 + (s-a)^3 V_2 + (s-a)^4 V_1\right\}$$

$$= \begin{pmatrix} -\frac{17}{8}s^4 + \frac{23}{4}s^3 - \frac{101}{32}s^2 + \frac{17}{32}s & \frac{67}{8}s^4 - \frac{65}{4}s^3 + \frac{353}{32}s^2 - \frac{59}{16}s + \frac{17}{32} \\ -\frac{81}{32}s^4 + \frac{71}{16}s^3 - \frac{293}{128}s^2 + \frac{49}{128}s & \frac{259}{32}s^4 - \frac{209}{16}s^3 + \frac{1057}{128}s^2 - \frac{171}{64}s + \frac{49}{128} \end{pmatrix}$$

and therefore

$$A(s-2) = s^4 \tilde{B}\left(\frac{1}{s}\right) = \begin{pmatrix} \frac{17}{32}s^3 - \frac{101}{32}s^2 + \frac{23}{4}s - \frac{17}{8} & \frac{17}{32}s^4 - \frac{59}{16}s^3 + \frac{353}{32}s^2 - \frac{65}{4}s + \frac{67}{8} \\ \frac{49}{128}s^3 - \frac{293}{128}s^2 + \frac{71}{16}s - \frac{81}{32} & \frac{49}{128}s^4 - \frac{171}{64}s^3 + \frac{1057}{128}s^2 - \frac{209}{16}s + \frac{259}{32} \end{pmatrix}$$

or otherwise

$$A(s)=\begin{pmatrix} \frac{17}{32}s^3+\frac{1}{32}s^2-\frac{1}{2}s+1 & \frac{17}{32}s^4+\frac{9}{16}s^3+\frac{53}{32}s^2+\frac{5}{8}s-1 \\ \frac{49}{128}s^3+\frac{1}{128}s^2-\frac{1}{8}s+\frac{1}{4} & \frac{49}{128}s^4+\frac{25}{64}s^3+\frac{181}{128}s^2+\frac{5}{32}s-\frac{1}{4} \end{pmatrix}$$

and

$$\tilde{A}(s)=\begin{pmatrix} s^4-\frac{1}{2}s^3+\frac{1}{32}s^2+\frac{17}{32}s & -s^4+\frac{5}{8}s^3+\frac{53}{32}s^2+\frac{9}{16}s+\frac{17}{32} \\ \frac{1}{4}s^4-\frac{1}{8}s^3+\frac{1}{128}s^2+\frac{49}{128}s & -\frac{1}{4}s^4+\frac{5}{32}s^3+\frac{181}{128}s^2+\frac{25}{64}s+\frac{49}{128} \end{pmatrix}$$

Any other unimodular equivalent polynomial matrix $\hat{A}(s)$ *of* $\tilde{A}(s)$ *i.e.* $\hat{A}(s)=U(s)\tilde{A}(s)$ *with* $\det U(s)\in\mathbb{R}$*, give us a solution to this problem. We can easily check that*

$$S^{\mathbb{C}}_{A(s)}(s)=\begin{pmatrix} 1 & 0 \\ 0 & s^2 \end{pmatrix}$$

and

$$S^{0}_{\tilde{A}(s)}(s)=\begin{pmatrix} 1 & 0 \\ 0 & s^6 \end{pmatrix}=\begin{pmatrix} s^{4-4} & 0 \\ 0 & s^{4+2} \end{pmatrix}$$

or otherwise

$$S^{\infty}_{A(s)}(s)=\begin{pmatrix} s^4 & 0 \\ 0 & \frac{1}{s^2} \end{pmatrix}$$

5 Conclusions

An algorithm has been presented for the construction of a system of algebraic-differential equations with prescribed smooth and impulsive behavior space. We extend in this way the results presented in Section 8.3 of [6]. The respective case of the smooth-impulsive behavior of continuous time AR-representations, is the forward-backward behavior to discrete-time AR-representations [18], [10]. Therefore, the suggested algorithms can be transferred to the discrete time case as well. We have to note here, that our aim in this work was to find a *square* system of algebraic-differential equations (i.e. regular polynomial matrix $A(\rho)$) with the prescribed behavior, although non-square systems of smaller order may exist (i.e. non-regular polynomial matrix $A(\rho)$). The answer to these questions is under further research.

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