Error analysis for the discretization of singular systems

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***Abstract*—The main purpose of this work is to propose upper bounds for the local truncation error between the continuous time response of a state space (resp. generalized state space) system and the response of its discretized model that came from either a zero-order hold or a triangle order hold discretization.**

I. Introduction

Consider the linear time-invariant state space system

$$\dot{x}_c(t) = Ax_c(t) + Bu(t) \tag{1}$$

with solution given by

$$x_c(t) = e^{At}x_c(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau \tag{2}$$

Some known discretization methods for state-space systems, which are based on *numerical integration*, are the Euler forward (backward) rectangular method and the trapezoidal (bilinear or Tustin's) method. By assuming that the input $u(t)$ is constant in the interval $[kT, kT+T)$ i.e.

$$u(t) = u(kT) \qquad \forall t \in [kT, kT+T)$$

a *zero-order hold* discretized model of (1) is given [11] by :

$$x_d((k+1)T) = \tilde{A}x_d(kT) + \tilde{B}u(kT) \tag{3}$$

where

$$\tilde{A} = e^{AT} \quad ; \quad \tilde{B} = \left[\int_0^T e^{Aw}dw\right]B$$

The connection between the transfer function of (2) and the ones in (3) is given by

$$\left(zI_n - \tilde{A}\right)^{-1}\tilde{B} = \left(1 - z^{-1}\right)Z\left\{L^{-1}\left\{\frac{H_{spr}(s)}{s}\right\}\right\} \tag{4}$$
$$H_{spr}(s) = (sI_n - A)^{-1}B$$

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where $L^{-1}[\cdot]$ denotes the inverse Laplace transform and $Z[\cdot]$ denotes the Z-transform. In case the input $u(t)$ is approximated by the first order polynomial

$$u(t) = u(kT) + \frac{u(kT) - u((k-1)T)}{T}(t - kT)$$

$\forall \tau \in (kT, kT+T]$ and $u(kT)$ stands for $u(kT+0)$ (backward Euler approximation of the derivative of the input) we get the causal *first-order hold* discretized model of (8) as given by [4]

$$x((k+1)T) = \hat{A}x(kT) + \hat{B}_0u(kT) + \hat{B}_{-1}u((k-1)T) \tag{5}$$

where

$$\hat{A} = e^{AT} \ ; \ \hat{B}_0 = \left[\int_0^T e^{Aw}\left(2 - \frac{w}{T}\right)dw\right]B$$
$$\hat{B}_{-1} = \left[\int_0^T e^{Aw}\left(\frac{w}{T} - 1\right)dw\right]B$$

The connection between the transfer function of (2) and the ones in (5) is given by

$$\left(zI_n - \tilde{A}\right)^{-1}\left(\hat{B}_0 + \hat{B}_{-1}z^{-1}\right) = \tag{6}$$
$$= \left(1 - z^{-1}\right)^2 Z\left\{L^{-1}\left[\frac{(Ts+1)H_{spr}(s)}{Ts^2}\right]\right\}$$

Alternatively, we can approximate the input $u(t)$ by the first order polynomial

$$u(t) = u(kT) + \frac{u((k+1)T) - u(kT)}{T}(\tau - kT)$$

$\forall t \in [kT, kT+T)$ (forward Euler approximation of the derivative of the input) and get the discretized model

$$x((k+1)T) = \hat{A}x(kT) + \hat{B}_0u(kT) + \hat{B}_1u((k+1)T) \tag{7}$$

where

$$\hat{A} = e^{AT} \quad ; \quad \hat{B}_0 = \left[\int_0^T e^{Aw}\frac{w}{T}dw\right]B$$
$$\hat{B}_1 = \left[\int_0^T e^{Aw}(1 - \frac{w}{T})dw\right]B$$

The above system is called *triangle-hold* equivalent model [3]. Its transfer function is given by

$$\left(zI_n-\tilde{A}\right)^{-1}\left(\hat{B}_0+\hat{B}_1 z\right)=\frac{(z-1)^2}{Tz}Z\left\{L^{-1}\left[\frac{H_{spr}(s)}{s^2}\right]\right\}$$

A more general class of systems, are the linear time-invariant non-homogeneous singular systems of the form

$$E\dot{x}(t)=Ax(t)+Bu(t) \tag{8}$$

where $E,A\in R^{n\times n}, B\in R^{m\times n}$,$C\in R^{p\times n}$ and $D\in R^{p\times m}$ and E is not necessary nonsingular. (8) is assumed to be regular $\det[sE-A]\neq 0$. Systems of the above form are usually called singular systems, descriptor systems, generalized state space systems, semistate systems etc. Descriptor systems appear in the modelling of many physical phenomena, such as engineering systems (power systems, electrical networks, aerospace engineering, chemical processes), social economic systems, network analysis, biological systems, etc. An extended reference on descriptor systems may be found in [1]. The main differences of singular systems from the state space systems are : a) that include both algebraic and differential equations, and b) that may include impulses in their solutions due to inconsistent initial conditions i.e. the role of the impulses is to transfer the inconsistent initial conditions at $t=0^-$ to consistent initial conditions at $t=0^+$.

In case where E is singular, we may use the forward or backward Euler method, or even the Gears method [12] in order to get a discretized singular model of (8). A zero-order (resp. triangle order or otherwise interpolating first-order) hold discretization method, based on the Laurent expansion terms of $(sE-A)^{-1}$, has been recently proposed by [7] (resp. [8]) and is used in the latest version of Wolfram Mathematica 9. A zero-order (resp. triangle-order) hold discretization approach for singular systems, based on the Weierstrass form of the matrix pencil $sE-A$ is given in [5], [6] (resp. [9]). All the proposed discretization methods produce local truncation errors between the continuous time solution and the discretized ones i.e. $\|x(kT)-x_k\|$. In case we select a discretization method (zero or triangle order hold) that is based on the Weierstrass method an upper bound for this error was provided by [5], [6] and [9]. In this paper, we provide an upper bound for the norm of the difference between the continuous time solution and the discretized ones $\|x(kT)-x_k\|$ for the case of zero-order and triangle-order hold approximation, of state space systems and singular models by using the Laurent expansion terms of $(sE-A)^{-1}$. The proposed bounds penalize our choice for the sampling period T and thus we can estimate a maximum period T if we demand the error to not exceed a given value.

II. Preliminary results

Consider the singular system described by (8). Its resolvent matrix can be expressed in a power series expansion of s as follows

$$\Phi(s)=(sE-A)^{-1}=\sum_{k=-\mu}^{\infty}\Phi_k(E,A)s^{-k-1} \tag{9}$$

where μ is the index of nilpotency of the pencil $sE-A$.

Lemma 1: The solution of (8) in terms of the resolvent matrix of the singular systems is given by [10] :

$$\begin{aligned}x(t)&=e^{\Phi_0 At}\Phi_0 Ex(0-)+\\&+\int_0^t e^{\Phi_0 A(t-\tau)}\Phi_0 Bu(\tau)d\tau+\sum_{i=0}^{\mu-1}\Phi_{-i-1}Bu^{(i)}(t)\\&+\sum_{i=0}^{\mu-1}\Phi_{-i-1}\left(\delta^{(i)}(t)Ex(0-)+\sum_{j=0}^{i-1}\delta^{(i-j-1)}Bu^{(j)}(0-)\right)\end{aligned} \tag{10}$$

The notation $u^{(i)}(t)$ describes the i^{th} derivative of $u(t)$. Some preliminary results concerning norms of vector valued functions are given in the sequel.

Theorem 2: [2] Let $f:[a,b]\to\mathbb{R}^n$ be continuous in $[a,b]$ and differentiable in (a,b). Then there exists $c\in(a,b)$ such that $\|f(b)-f(a)\|\leq(b-a)\left\|f^{(1)}(c)\right\|$, where $f^{(i)}$ describes the i^{th} derivative of f.

Proposition 3: [2] Let $f:[a,b]\to\mathbb{R}^n$ belongs to $C^1[a,b]$. Then $\|f(b)-f(a)\|\leq(b-a)\left\|f^{(1)}\right\|_\infty$ where $\|f\|_\infty=\max\limits_{a\leq x\leq b}\|f(x)\|_2=\max\limits_{a\leq x\leq b}\sqrt{\sum_{i=1}^n f_i(x)^2}$.

Theorem 4: Let $f\in C^{n+1}[a,b]$ and $\left|f^{(n+1)}(x)\right|\leq M$, $\forall x\in[a,b]$. Let $p_n(x)$ be the unique interpolation polynomial of degree less than equal to n that interpolates f in $n+1$ equidistant points in $[a,b]$ (included a,b). If $x\in[a,b]$ then

$$|f(x)-p_n(x)|\leq\frac{1}{4(n+1)}M\left(\frac{b-a}{n}\right)^{n+1}$$

III. Local errors on state-space discretizations

In this section we provide upper bounds for the zero-order and triangle order hold discretization of state space systems.

Theorem 5: An upper bound for the local error e_{zoh} between the continuous time response $x_c(t)$ of the state space system (1) and the discrete time response $x_d(kT)$ of its zero order hold discretization given in (3) at the discrete time $t=kT,k=0,1,...$ is given by

$$e_{zoh}\leq\frac{\left(e^{\|A\|T}-\|A\|T-1\right)\left(e^{\|A\|kT}-1\right)}{\|A\|^2\left(e^{\|A\|T}-1\right)}\|B\|M_1 \tag{11}$$

where $M_1=\left\|u^{(1)}(t)\right\|_\infty, t\in[0,kT]$.

Proof: Let $t=kT,k=0,1,\ldots$ where T is the sampling period. Then

$$\begin{aligned}x_c(kT)&=e^{AkT}x_c(0)+\int_0^{kT}e^{A(kT-\tau)}Bu(\tau)d\tau=\\&=e^{AkT}x_c(0)+\sum_{i=0}^{k-1}\int_{iT}^{iT+T}e^{A(kT-\tau)}Bu(\tau)d\tau\end{aligned} \tag{12}$$

Given that $u(\tau) = u(iT)$ for $iT \le \tau < iT + T$ we have the local trunction error

$$e_{zoh} = \|x_c(kT) - x_d(kT)\| =$$

$$= \left\| \begin{array}{c} \left\{ e^{AkT} x_c(0) + \sum\limits_{i=0}^{k-1} \int_{iT}^{iT+T} e^{A(kT-\tau)} B u(\tau)\, d\tau \right\} - \\ -\left\{ e^{AkT} x_c(0) + \sum\limits_{i=0}^{k-1} \int_{iT}^{iT+T} e^{A(kT-\tau)} B u(iT)\, d\tau \right\} \end{array} \right\| =$$

$$= \left\| \sum_{i=0}^{k-1} \int_{iT}^{iT+T} e^{A(kT-\tau)} B \left(u(\tau) - u(iT)\right) d\tau \right\| \le$$

$$\le \sum_{i=0}^{k-1} \int_{iT}^{iT+T} \left\| e^{A(kT-\tau)} \right\| \|B\| \left\| (u(\tau) - u(iT)) \right\| d\tau$$

According to Theorem 2 if $u : [a, b] \to \mathbb{R}^m$ belongs to $C^1[a, b]$ then ■

$$\|u(b) - u(a)\| \le (b-a) \left\| u^{(1)} \right\|_\infty$$

where $\left\| u^{(1)} \right\|_\infty = \max\limits_{a \le x \le b} \left\| u^{(1)}(x) \right\|_2 = \max\limits_{a \le x \le b} \sqrt{\sum\limits_{i=1}^{m} \left| u_i^{(1)}(x) \right|^2}$. For $\tau \le iT + T$ we have that

$$\|(u(\tau) - u(iT))\| \le (\tau - iT)\, M_1$$

where $M_1 = \left\| u^{(1)}(t) \right\|_\infty$, $t \in [0, kT]$. Therefore an upper bound of the local error is given by

$$e_{zoh} \le \sum_{i=0}^{k-1} \int_{iT}^{iT+T} e^{\|A\|(kT-\tau)} \|B\| (\tau - iT)\, M_1 d\tau =$$

$$= e^{\|A\|kT} \|B\| M_1 \sum_{i=0}^{k-1} \int_{iT}^{iT+T} e^{-\|A\|\tau} (\tau - iT)\, d\tau \overset{w=\tau-iT}{=}$$

$$= e^{\|A\|kT} \|B\| M_1 \sum_{i=0}^{k-1} e^{\|A\|(-iT)} \int_0^T e^{-\|A\|w} w d\tau =$$

$$= e^{\|A\|kT} \|B\| M_1 \sum_{i=0}^{k-1} e^{\|A\|(-iT)} \frac{1 - e^{-\|A\|T}(1 + \|A\|T)}{\|A\|^2}$$

$$= e^{\|A\|kT} \frac{1 - e^{-\|A\|T}(1 + \|A\|T)}{\|A\|^2} \|B\| \times$$

$$\times M_1 \sum_{i=0}^{k-1} e^{\|A\|(-iT)} =$$

$$= \frac{\left(e^{\|A\|T} - \|A\| T - 1\right)\left(e^{\|A\|kT} - 1\right)}{\|A\|^2 \left(e^{\|A\|T} - 1\right)} \|B\| M_1$$

It is easily seen from (11) that given a specific k and a specific upper bound e_{zoh} we can always estimate the corresponding sampling period T.

Theorem 6: An upper bound for the local truncation error e_{toh} between the continuous time response $x_c(t)$ of the state space system (1) and the discrete time response $x_d(kT)$ of the its triangle order hold discretization given in (7) at the discrete time $t = kT, k = 0, 1, ...$ is given by

$$e_{toh} \le \frac{\left(1 - e^{-\|A\|kT}\right)}{\|A\|} \|B\| \frac{M_2 T^2}{8} \tag{13}$$

where $M_2 = \left\| u^{(2)}(t) \right\|_\infty$, $t \in [0, kT]$

Proof: According to (12) and using

$$u(\tau) \cong u(iT) + \frac{u(iT+T) - u(iT)}{T} (\tau - iT) =$$

$$= q^i(\tau), \tau \in [iT, iT + T)$$

the local truncation error between the continuous time response $x_c(t)$ of the state space system (1) and the discrete time response $x_d(kT)$ of its triangle order hold discretization is given by

$$e_{toh} = \|x_c(kT) - x_d(kT)\| =$$

$$\le \sum_{i=0}^{k-1} \int_{iT}^{iT+T} \left\| e^{A(kT-\tau)} \right\| \|B\| \left\| \left(u(\tau) - q^i(\tau)\right) \right\| d\tau$$

Using Theorem 4 we get

$$\left\| u(\tau) - q^i(\tau) \right\| \le \frac{1}{4 \cdot 2} M_2 \left(\frac{iT + T - iT}{1} \right)^2 = \frac{M_2 T^2}{8}$$

and therefore

$$e_{toh} \le \sum_{i=0}^{k-1} \int_{iT}^{iT+T} \left\| e^{A(kT-\tau)} \right\| \|B\| \tfrac{M_2 T^2}{8} d\tau \le$$

$$\le \sum_{i=0}^{k-1} e^{\|A\|kT} \|B\| \tfrac{M_2 T^2}{8} \int_{iT}^{iT+T} e^{-\|A\|\tau} d\tau =$$

$$= e^{\|A\|kT} \|B\| \tfrac{M_2 T^2}{8} \sum_{i=0}^{k-1} \int_{iT}^{iT+T} e^{-\|A\|\tau} d\tau =$$

$$= \frac{\left(1 - e^{-\|A\|kT}\right)}{\|A\|} \|B\| \tfrac{M_2 T^2}{8}$$

which verifies the proof. ■

It is easily seen from (13), that for a specific k, the upper bound for the local truncation error e_{toh} is analogue to the square of the sampling period T and the second derivative of the input, whereas e_{zoh} is analogue to the sampling period T and the first derivative of the input. For small values of T the local truncation error e_{toh} is analogue to T^3 whereas e_{zoh} is analogue to T^2.

IV. Discretization of singular systems

By assuming that the input $u(t)$ is constant in the interval $[kT, kT + T)$ i.e.

$$u(t) = u(kT) \qquad \forall t \in [kT, kT + T)$$

and by using a *forward Euler approximation* of the derivatives of the inputs i.e.

$$u^{(i+1)}(kT) \simeq \frac{\left(u^{(i)}((k+1)T) - u^{(i)}(kT)\right)}{T} \tag{14}$$

a *zero-order hold* state-space discretization of (8) is given by [7] :

$$x((k+1)T) = \tilde{A}x(kT) + \tilde{B}(\sigma)\, u(kT) \tag{15}$$

$$x(0) = \Phi_0 E x(0-) + \sum_{i=0}^{\mu-1} \left(-\Phi_{-1} E\right)^i \Phi_{-1} B u^{(i)}(0+)$$

with

$$\tilde{A} = e^{\Phi_0 A T} \quad ; \quad \tilde{B}(\sigma) = \sum_{i=0}^{\mu} \tilde{B}_i \sigma^i \tag{16}$$

$$\tilde{B}_0 = \left[\int_0^T e^{\Phi_0 A w} dw \right] \Phi_0 B + \sum_{i=1}^{\mu} (-1)^i \Phi_{-i} B T^{1-i}$$

$$\tilde{B}_\ell = \sum_{j=\ell}^{\mu} (-1)^{j-\ell} \Phi_{-j} B T^{1-j} \begin{pmatrix} j \\ j-\ell \end{pmatrix}$$

where $\ell = 1, 2, .., \mu$ and $\sigma^i u(kT) = u((k+i)T)$. Note that according to [7]

$$\left(zI - \tilde{A}\right)^{-1} \tilde{B}(z) = \tag{17}$$

$$= \left(1 - z^{-1}\right) Z\left\{L^{-1}\left\{\frac{H_{spr}(s)}{s}\right\}\right\} + H_{pol}\left(\frac{z-1}{T}\right)$$

where $H_{spr}(s)$ (resp. $H_{pol}(s)$) denotes the strictly proper part (resp. polynomial part) of the transfer function matrix $H(s) = (sE - A)^{-1} B$.

In order to obtain a *triangle-order hold* discretization of (8) we assume that the input $u(t)$ is approximated in the interval $[kT, kT+T)$ by a first order polynomial of the form

$$u(\tau) = u(kT) + \frac{u((k+1)T) - u(kT)}{T}(\tau - kT) \tag{18}$$

$\forall \tau \in [kT, kT+T)$.

Theorem 7: [8] Using a *triangle-order hold* approximation of the input $u(t)$ and its derivatives, the continuous time nonhomogeneous singular system (8) is discretized to yield the state space system

$$x((k+1)T) = \tilde{A}x(kT) + \tilde{B}(\sigma)u(kT) \tag{19}$$

$$x(0) = \Phi_0 E x(0-) + \sum_{i=0}^{\mu-1} (-\Phi_{-1}E)^i \Phi_{-1} B u^{(i)}(0+) \tag{20}$$

where

$$\tilde{A} = e^{\Phi_0 A T} \tag{21}$$

$$\tilde{B}(\sigma) = \sum_{i=0}^{\mu} \tilde{B}_i \sigma^i \quad \text{with} \quad \sigma^i u(kT) = u((k+i)T) \tag{22}$$

$$\tilde{B}_0 = \left[\int_0^T e^{\Phi_0 A w} \frac{w}{T} dw\right] \Phi_0 B + \left(\sum_{i=1}^{\mu} (-1)^i \Phi_{-i} B T^{1-i}\right)$$

$$\tilde{B}_1 = \left[\int_0^T e^{\Phi_0 A w}\left(1 - \frac{w}{T}\right) dw\right] \Phi_0 B + \sum_{i=1}^{\mu} (-1)^{i-1} i \Phi_{-i} B T^{1-i}$$

$$\tilde{B}_\ell = \sum_{j=\ell}^{\mu} (-1)^{j-\ell} \Phi_{-j} B T^{1-j} \begin{pmatrix} j \\ j-\ell \end{pmatrix} \quad \ell = 2, 3, .., \mu$$

Note that in the state-space case where $E = I_n, \Phi_0 = I_n, \Phi_{-i} = 0, i > 0$, (19) coincides with (7).

Theorem 8: The connection between the transfer function matrix of the continuous time system $H(s) = H_{spr}(s) + H_{pol}(s)$ and the ones of the discrete time system $G(z) = \left(zI_n - \tilde{A}\right)^{-1} \tilde{B}(z)$ is given by

$$\left(zI_n - \tilde{A}\right)^{-1} \tilde{B}(z) =$$

$$= \frac{(z-1)^2}{Tz} Z\left\{L^{-1}\left[\frac{H_{spr}(s)}{s^2}\right]\right\} + H_{pol}\left(\frac{z-1}{T}\right)$$

Proof: The transfer function of (19) is given by

$$\left(zI_n - \tilde{A}\right)^{-1} \tilde{B}(z) =$$

$$= \left(zI_n - \tilde{A}\right)^{-1} \cdot$$

$$\cdot \left(\underbrace{\left\{\left[\int_0^T e^{\Phi_0 A w} \frac{w}{T} dw\right] \Phi_0 B + \sum_{i=1}^{\mu} (-1)^i \Phi_{-i} B T^{1-i}\right\}}_{\tilde{B}_0} z^0 \right) +$$

$$+ \left(zI_n - \tilde{A}\right)^{-1} \left(\begin{array}{c} \underbrace{\left\{ \begin{array}{c} \left[\int_0^T e^{\Phi_0 A w}(1 - \frac{w}{T}) dw\right] \Phi_0 B + \\ + \sum_{i=1}^{\mu} (-1)^i i \Phi_{-i} B T^{1-i} \end{array} \right\}}_{\tilde{B}_1} z^1 + \\ + \sum_{i=2}^{\mu} \tilde{B}_i z^i \end{array} \right) =$$

$$= \left(zI_n - \tilde{A}\right)^{-1} \left(\begin{array}{c} \left[\int_0^T e^{\Phi_0 A w} \frac{w}{T} dw\right] \Phi_0 B \ z^0 + \\ + \left[\int_0^T e^{\Phi_0 A w}(1 - \frac{w}{T}) dw\right] \Phi_0 B z^1 \end{array} \right) +$$

$$+ \left(zI_n - \tilde{A}\right)^{-1} \left(\sum_{i=1}^{\mu} (-1)^i \Phi_{-i} B T^{1-i} + \right.$$

$$\left. \sum_{i=1}^{\mu} (-1)^i i \Phi_{-i} B T^{1-i} z^1 + \sum_{i=2}^{\mu} \tilde{B}_i z^i \right)$$

$$= \frac{(z-1)^2}{Tz} Z\left\{\frac{H_{spr}(s)}{s^2}\right\} +$$

$$+ \left(zI_n - \tilde{A}\right)^{-1} \left(\sum_{i=1}^{\mu} (-1)^i \Phi_{-i} B T^{1-i} + \right.$$

$$\left. \sum_{i=1}^{\mu} (-1)^i \begin{pmatrix} i \\ i-1 \end{pmatrix} \Phi_{-i} B T^{1-i} z^1 + \sum_{i=2}^{\mu} \tilde{B}_i z^i \right) =$$

$$= \frac{(z-1)^2}{Tz} Z\left\{\frac{H_{spr}(s)}{s^2}\right\} -$$

$$- \left(zI_n - \tilde{A}\right)^{-1} \left[\int_0^T e^{\Phi_0 A w} dw\right] \Phi_0 B +$$

$$+ \left(zI_n - \tilde{A}\right)^{-1} \left(\begin{array}{c} \left[\int_0^T e^{\Phi_0 A w} dw\right] \Phi_0 B + \\ + \sum_{i=1}^{\mu} (-1)^i \Phi_{-i} B T^{1-i} + \sum_{i=1}^{\mu} \tilde{B}_i z^i \end{array} \right)$$

and thus by (17) we have that

$$\left(zI_n - \tilde{A}\right)^{-1} \tilde{B}(z) = \frac{(z-1)^2}{Tz} Z\left\{\frac{H_{spr}(s)}{s^2}\right\} -$$

$$- \left(zI_n - \tilde{A}\right)^{-1} \left[\int_0^T e^{\Phi_0 A w} dw\right] \Phi_0 B +$$

$$+ \left(1 - z^{-1}\right) Z\left\{L^{-1}\left\{\frac{H_{spr}(s)}{s}\right\}\right\} + H_{pol}\left(\frac{z-1}{T}\right) =$$

$$= \frac{(z-1)^2}{Tz} Z\left\{\frac{H_{spr}(s)}{s^2}\right\} + H_{pol}\left(\frac{z-1}{T}\right) +$$

$$+ \left\{ \begin{array}{c} \left(1 - z^{-1}\right) Z\left\{L^{-1}\left\{\frac{H_{spr}(s)}{s}\right\}\right\} - \\ - \left(zI_n - \tilde{A}\right)^{-1} \left[\int_0^T e^{\Phi_0 A w} dw\right] \Phi_0 B \end{array} \right\} =$$

$$\overset{(4)}{=} \frac{(z-1)^2}{Tz} Z\left\{\frac{H_{spr}(s)}{s^2}\right\} + H_{pol}\left(\frac{z-1}{T}\right)$$

■

V. Local errors on singular system discretizations

Theorem 9: An upper bound for the local error $serror_{zoh}$ between the continuous time response $x_c(t)$ of the singular system (8) and the discrete time response $x_d(kT)$ of its zero order hold discretization given in (15) at the discrete time instants $t = kT$ with $k = 0, 1, ...$ is given by

$$serror_{zoh} \leq \frac{e^{\|\Phi_0 A\|T} - \|\Phi_0 A\| T - 1}{\|\Phi_0 A\|} \times \\ \times \frac{\left(e^{\|\Phi_0 A\|kT} - 1\right)}{e^{\|\Phi_0 A\|T} - 1} \|\Phi_0 B\| M_1 + \\ + \sum_{i=1}^{\mu-1} \|\Phi_{-i-1} B\| \left(\sum_{l=0}^{i-1} \frac{\begin{pmatrix} i \\ l \end{pmatrix} (i-l)^{i+1}}{(i+1)!}\right) \tilde{M}_{i+1} T$$

where $M_1 = \left\|u^{(1)}(t)\right\|_\infty$, $t \in [0, kT]$ and $\tilde{M}_i = \max \left\|u^{(i)}(t)\right\|_\infty$, $t \in [kT, kT + (\mu-1)T]$.

Proof: By setting $t = kT, i = 1, 2, \ldots$ in (10) we get

$$x_c(kT) = e^{\Phi_0 AkT} \Phi_0 Ex(0-) + \\ \sum_{i=0}^{k-1} \int_{iT}^{iT+T} e^{\Phi_0 A(kT-\tau)} \Phi_0 Bu(\tau) d\tau + \sum_{i=0}^{\mu-1} \Phi_{-i-1} Bu^{(i)}(kT) \tag{23}$$

and therefore the local truncation error will be

$$serror_{zoh} = \|x_c(kT) - x_d(kT)\| =$$

$$= \left\| \begin{array}{c} \left(\begin{array}{c} e^{\Phi_0 AkT} \Phi_0 Ex(0-) + \\ + \sum_{i=0}^{k-1} \int_{iT}^{iT+T} e^{\Phi_0 A(kT-\tau)} \Phi_0 Bu(\tau) d\tau + \\ + \sum_{i=0}^{\mu-1} \Phi_{-i-1} Bu^{(i)}(kT) \end{array} \right) - \\ - \left(\begin{array}{c} e^{\Phi_0 AkT} \Phi_0 Ex(0-) + \\ + \sum_{i=0}^{k-1} \int_{iT}^{iT+T} e^{\Phi_0 A(kT-\tau)} \Phi_0 Bu(iT) d\tau + \\ + \sum_{i=0}^{\mu-1} \Phi_{-i-1} Bu_{appr}^{(i)}(kT) \end{array} \right) \end{array} \right\| \leq$$

$$\leq \left\| \sum_{i=0}^{k-1} \int_{iT}^{iT+T} e^{\Phi_0 A(kT-\tau)} \Phi_0 B [u(\tau) - u(iT)] d\tau \right\| + \\ + \left\| \sum_{i=1}^{\mu-1} \Phi_{-i-1} B \left[u^{(i)}(kT) - u_{appr}^{(i)}(kT)\right] \right\| \leq$$

$$\leq \sum_{i=0}^{k-1} \int_{iT}^{iT+T} e^{\|\Phi_0 A(kT-\tau)\|} \|\Phi_0 B\| \|[u(\tau) - u(iT)]\| d\tau + \\ + \sum_{i=1}^{\mu-1} \|\Phi_{-i-1} B\| \left\| \left[u^{(i)}(kT) - u_{appr}^{(i)}(kT)\right] \right\| \leq$$

$$\leq e^{\|\Phi_0 A\|kT} \|\Phi_0 B\| \times \\ \times \sum_{i=0}^{k-1} \int_{iT}^{iT+T} e^{-\|\Phi_0 A\|\tau} \|[u(\tau) - u(iT)]\| d\tau + \\ + \sum_{i=1}^{\mu-1} \|\Phi_{-i-1} B\| \left\| \left[u^{(i)}(kT) - u_{appr}^{(i)}(kT)\right] \right\|$$

According to the Mean Value Theorem 2, $\exists c \in [iT, \tau)$ where $\tau \leq iT + T$ such that

$$\|(u(\tau) - u(iT))\| \leq (\tau - iT) \left\|u^{(1)}(c)\right\| \leq (\tau - iT) M_1$$

where $M_1 = \left\|u^{(1)}(t)\right\|_\infty$. Therefore,

$$serror_{zoh} \leq \\ \leq e^{\|\Phi_0 A\|kT} \|\Phi_0 B\| \sum_{i=0}^{k-1} \int_{iT}^{iT+T} e^{-\|\Phi_0 A\|\tau} (\tau - iT) M_1 d\tau + \\ + \sum_{i=1}^{\mu-1} \|\Phi_{-i-1} B\| \left\|u^{(i)}(kT) - u_{appr}^{(i)}(kT)\right\|$$

In order to estimate an upper bound for the norm of the difference $u^{(i)}(kT) - u_{appr}^{(i)}(kT)$ we use the Taylor approximation

$$u(kT+T) = u(kT) + \frac{u^{(1)}(kT)}{1!}(kT + T - kT) + \\ + \frac{u^{(2)}(\xi_1)}{2!}(kT + T - kT)^2, \xi_1 \in (kT, kT + T)$$

where

$$u^{(1)}(kT) - u_{appr}^{(1)}(kT) = \frac{u^{(2)}(\xi_1)}{2!} T$$

If $\tilde{M}_i = \max \left\|u^{(i)}(t)\right\|_\infty$ for $t \in [kT, kT + (\mu-1)T]$ then

$$\left\|u^{(1)}(kT) - u_{appr}^{(1)}(kT)\right\| = \left\|\frac{u^{(2)}(\xi_1)}{2!} T\right\| \leq \frac{\tilde{M}_2 T}{2!}$$

Following similar lines, using Taylor approximation with 4 terms, we get

$$\left\|u^{(2)}(kT) - \frac{u(kT+2T) - 2u(kT+T) + u(kT)}{T^2}\right\| = \\ = \left\|\frac{2^3 u^{(3)}(\xi_3)}{3!} T\right\| + \left\|\frac{2u^{(3)}(\xi_2)}{3!} T\right\|$$

where $\xi_3 \in (kT, kT + 2T)$ and $\xi_2 \in (kT, kT + T)$. Therefore,

$$\left\|u^{(2)}(kT) - u_{appr}^{(2)}(kT)\right\| \leq \sum_{l=0}^{2-1} \frac{\begin{pmatrix} 2 \\ l \end{pmatrix} (2-l)^3}{(3)!} \tilde{M}_3 T.$$

In the same way, we can show that

$$\left\|u^{(i)}(kT) - u_{appr}^{(i)}(kT)\right\| \leq \sum_{l=0}^{i-1} \frac{\begin{pmatrix} i \\ l \end{pmatrix} (i-l)^{i+1}}{(i+1)!} \tilde{M}_{i+1} T \tag{24}$$

By using the above relations, we get

$$serror_{zoh} \leq \\ \leq e^{\|\Phi_0 A\|kT} \|\Phi_0 B\| \sum_{i=0}^{k-1} \int_{iT}^{iT+T} e^{-\|\Phi_0 A\|\tau} (\tau - iT) M_1 d\tau + \\ + \sum_{i=1}^{\mu-1} \|\Phi_{-i-1} B\| \sum_{l=0}^{i-1} \frac{\begin{pmatrix} i \\ l \end{pmatrix} (i-l)^{i+1}}{(i+1)!} \tilde{M}_{i+1} T =$$

$$= \frac{\left(e^{\|\Phi_0 A\|T} - \|\Phi_0 A\|T - 1\right)\left(e^{\|\Phi_0 A\|kT} - 1\right)}{\|\Phi_0 A\|\left(e^{\|\Phi_0 A\|T} - 1\right)} \|\Phi_0 B\| M_1 +$$
$$+ \sum_{i=1}^{\mu-1} \|\Phi_{-i-1} B\| \sum_{l=0}^{i-1} \frac{\begin{pmatrix} i \\ l \end{pmatrix} (i-l)^{i+1}}{(i+1)!} \tilde{M}_{i+1} T$$

which verifies the proof. ■

Theorem 10: An upper bound for the local error $serror_{toh}$ between the continuous time response $x_c(t)$ of the singular system (8) and the discrete time response $x_d(kT)$ of its *triangle-order hold* discretization given in (19) at the discrete time instants $t = kT$ with $k = 0, 1, ...$ is given by

$$serror_{toh} \leq \left(\frac{\|\Phi_0 B\| \left(e^{\|\Phi_0 A\|kT} - 1\right)}{\|\Phi_0 A\|} \right) \frac{M_2 T^2}{8} +$$
$$+ \sum_{i=1}^{\mu-1} \|\Phi_{-i-1} B\| \sum_{l=0}^{i-1} \frac{\begin{pmatrix} i \\ l \end{pmatrix} (i-l)^{i+1}}{(i+1)!} \tilde{M}_{i+1} T$$

where $M_2 = \left\|u^{(2)}(t)\right\|_\infty$, $t \in [0, kT]$ and $\tilde{M}_i = \max \left\|u^{(i)}(t)\right\|_\infty, t \in [kT, kT + (\mu - 1) T]$.

Proof: The local truncation error will be, according to (18) and (23), the following

$$serror_{toh} = \|x_c(kT) - x_d(kT)\| =$$
$$= \left\| \begin{array}{c} \left(\begin{array}{c} e^{\Phi_0 AkT} \Phi_0 Ex(0-) + \\ + \sum_{i=0}^{k-1} \int_{iT}^{iT+T} e^{\Phi_0 A(kT-\tau)} \Phi_0 Bu(\tau) d\tau + \\ + \sum_{i=0}^{\mu-1} \Phi_{-i-1} Bu^{(i)}(kT) \end{array} \right) - \\ - \left(\begin{array}{c} e^{\Phi_0 AkT} \Phi_0 Ex(0-) + \\ + \sum_{i=0}^{k-1} \int_{iT}^{iT+T} e^{\Phi_0 A(kT-\tau)} \Phi_0 Bq_i(\tau) d\tau + \\ + \sum_{i=0}^{\mu-1} \Phi_{-i-1} Bu_{appr}^{(i)}(kT) \end{array} \right) \end{array} \right\| \leq$$
$$\leq e^{\|\Phi_0 A\|kT} \|\Phi_0 B\| \times$$
$$\times \sum_{i=0}^{k-1} \int_{iT}^{iT+T} e^{-\|\Phi_0 A\|\tau} \|[u(\tau) - q_i(\tau)]\| d\tau +$$
$$+ \sum_{i=1}^{\mu-1} \|\Phi_{-i-1} B\| \left\|u^{(i)}(kT) - u_{appr}^{(i)}(kT)\right\|$$

From Theorem 4, we have that

$$\|u(\tau) - q_i(\tau)\| \leq \frac{1}{4(1+1)} M_2 \left(\frac{iT + T - iT}{1} \right)^{1+1} = \frac{M_2 T^2}{8}$$

where $M_2 = \left\|u^{(2)}(t)\right\|_\infty$, $t \in [0, kT]$ and therefore

$$serror_{toh} \leq e^{\|\Phi_0 A\|kT} \|\Phi_0 B\| \sum_{i=0}^{k-1} \int_{iT}^{iT+T} e^{-\|\Phi_0 A\|\tau} \frac{M_2 T^2}{8} d\tau +$$
$$+ \sum_{i=1}^{\mu-1} \|\Phi_{-i-1} B\| \left\| \left[u^{(i)}(kT) - u_{appr}^{(i)}(kT)\right] \right\| =$$
$$= e^{\|\Phi_0 A\|kT} \|\Phi_0 B\| \frac{M_2 T^2}{8} \sum_{i=0}^{k-1} e^{-\|\Phi_0 A\|iT} \int_0^T e^{-\|\Phi_0 A\|w} dw +$$
$$+ \sum_{i=1}^{\mu-1} \|\Phi_{-i-1} B\| \left\| \left[u^{(i)}(kT) - u_{appr}^{(i)}(kT)\right] \right\|$$

where from (24)

$$\left\|u^{(i)}(kT) - u_{appr}^{(i)}(kT)\right\| \leq \sum_{l=0}^{i-1} \frac{\begin{pmatrix} i \\ l \end{pmatrix} (i-l)^{i+1}}{(i+1)!} \tilde{M}_{i+1} T$$

and

$$e^{\|\Phi_0 A\|kT} \sum_{i=0}^{k-1} e^{-\|\Phi_0 A\|iT} \int_0^T e^{-\|\Phi_0 A\|w} dw = \frac{\left(e^{\|\Phi_0 A\|kT} - 1\right)}{\|\Phi_0 A\|}$$

which verifies the proof. ■

VI. Conclusions

An error analysis for the discretization process of the zero-order and triangle-order hold discretization for state space and singular systems has been provided. Based on these results we can always estimate the sampling period T that will not permit the errors during the discretization process exceed a given value. The same approach provides bounds for the intersample error as well (by replacing T with the intesample value t in $[0, T]$). The results presented in this work can also be extended to singular systems with delay.

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