# On the Modeling of Discrete Time Auto-Regressive Representations

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***Abstract*—It is well known [2], [6], that given the discrete-time AutoRegressive representation $A(\sigma)\beta(k) = 0$, where $\sigma$ denotes the shift forward operator and $A(\sigma)$ a polynomial matrix, we can always construct the forward-backward behavior of this system, by using the finite and infinite elementary divisor structure of $A(\sigma)$. The main theme of this work is to study the inverse problem: given a specific forward-backward behavior, find a family of polynomial matrices $A(\sigma)$, such that the system $A(\sigma)\beta(k) = 0$ has exactly the prescribed behavior. As we shall see, the problem can be reduced either to a linear system equation problem or to an interpolation problem.**

## I. INTRODUCTION

LET $\mathbb{R}$ be the field of reals, $\mathbb{R}[s]$ the ring of polynomials with coefficients from $\mathbb{R}$ and $\mathbb{R}(s)$ the field of rational functions. By $\mathbb{R}[s]^{p\times m}, \mathbb{R}(s)^{p\times m}, \mathbb{R}_{pr}(s)^{p\times m}$ we denote the sets of $p\times m$ polynomial, rational and proper rational matrices with real coefficients. We are going to study the behavior of systems of algebraic difference equations that are described by the form of an (Auto-Regressive) AR-representation, that is

$$A(\sigma)\beta(k) = 0 \tag{1}$$

with k=0,1,...,N-q, or equivalently

$$A_q\beta(k+q) + A_{q-1}\beta(k+q-1) + ... + A_0\beta(k) = 0$$

where $A(\sigma) = A_q\sigma^q + A_{q-1}\sigma^{q-1} + ... + A_0 \in \mathbb{R}^{r\times r}[\sigma]$ is a regular polynomial matrix, i.e. $\det[A(\sigma)] \neq 0$, $\beta(k) \in \mathbb{R}^r$ and denotes the forward shift operator $\sigma\beta(k) = \beta(k+1)$. The number $q$ is often called the lag of the matrix. The solution space of such systems consists of both forward and backward solutions and is denoted as

$$B := \{\beta(k) : [0, N] \to \mathbb{R}^r \,|(1) \text{ is satisfied } \forall k \in [0, N-q]\,\}$$

The forward (resp. backward) solution space is connected to the finite (resp. infinite) elementary divisors of $A(\sigma)$. Methods for constructing the solution space of (1) have been initially presented in [2], whereas an extension of this method to non regular systems is given in [6]. In this paper, we shall study the inverse problem, that is: Given a certain forward/backward solution space, find a system of algebraic/difference equations with the prescribed solution space. A partial solution to this problem has been described in [2], where only the forward behavior was studied. We shall extend these results for the case where both a forward and backward behavior is under consideration by using a different methodology. The core of our proposed method is the fact that the vectors that consist a solution of the system (forward or backward), actually satisfy a certain system of equations, which we are going to solve in terms of the unknown coefficients of A($\sigma$), in order to receive the original system. This method has a significant advantage over the one proposed in [2]. It can be easily modified to give a solution for non-regular systems, or systems, whereas [2] worked only for the case of regular matrices. The whole theory is implemented via an example in the last section.

This research has been co-financed by the European Union (European Social Fund ESF) and Greek national funds through the Operational Program "Education and Lifelong Learning" of the National StrategicReference Framework (NSRF)—Research Funding Program:ARCHIMEDES III. Investing in knowledge society through the European Social Fund.

978-1-4799-6773-5/14/$31.00 2014 IEEE

## II. PRELIMINARIES

Consider the matrix

$$A(\sigma) = A_q\sigma^q + A_{q-1}\sigma^{q-1} + ... + A_1\sigma + A_0 \tag{2}$$

where $A_i \in \mathbb{R}^{r\times r}$, $A_q \neq 0$ and $\det(A(\sigma)) \neq 0$.

*Definition 1:* [8] Let A($\sigma$) as in (2). There exist unimodular matrices $U_L(\sigma) \in \mathbb{R}[\sigma]^{r\times r}$, $U_R(\sigma) \in \mathbb{R}[\sigma]^{r\times r}$i.e. $\det(U_L(\sigma))$ and $\det(U_R(\sigma))$ are nonzero constants, such that

$$\begin{aligned} S^{\mathbb{C}}_{A(\sigma)}(\sigma) &= U_L(\sigma)A(\sigma)U_R(\sigma) = \\ &= diag\,[1, ..., 1, f_z(\sigma), f_{z+1}(\sigma), ..., f_r(\sigma)] \end{aligned} \tag{3}$$

with $1 \leq z \leq r$ and $f_i(\sigma)/f_{i+1}(\sigma)$. $S^{\mathbb{C}}_{A(\sigma)}(\sigma)$ is called the *Smith form* of $A(\sigma)$. $f_i(\sigma)$ are called the *invariant polynomials* of $A(\sigma)$. The zeros $s_i \in \mathbb{C}$ of $f_j(\sigma)$, $j = z, z+1, \ldots, r$ are called *finite zeros* of $A(\sigma)$. Assume that the partial multiplicities of each zero $\sigma_i \in \mathbb{C}$, $i \in k$ are $0 \leq n_{i,z} \leq n_{i,z+1} \leq ... \leq n_{i,r}$ i.e.

$$f_j(\sigma) = (\sigma - \lambda_i)^{n_{i,j}}\hat{f}_j(s)$$

j=z,z+1,...,r ; $\hat{f}_j(\sigma_i) \neq 0$ The terms $(\sigma - \lambda_i)^{n_{i,j}}$ are called finite elementary divisors of A($\sigma$) at $\sigma = \lambda_i$. We also denote by n the sum of the degrees of the finite elementary divisors of A(s), i.e. $n := \deg\left[\prod_{j=z}^{r} f_j(\sigma)\right] = \sum_{i=1}^{k}\sum_{j=z}^{r} n_{i,j}$.

Similarly, we can find $U_L(\sigma) \in \mathbb{R}(\sigma)^{r\times r}$, $U_R(\sigma) \in \mathbb{R}(\sigma)^{r\times r}$ having no poles and zeros at $\sigma = \lambda_0$ such that

$$S^{\lambda_0}_{A(\sigma)}(\sigma) = U_L(\sigma)A(\sigma)U_R(\sigma) = \\ = diag\left[1, ..., 1, (\sigma-\lambda_0)^{n_z}, ..., (\sigma-\lambda_0)^{n_r}\right]$$

$S^{\lambda_0}_{A(\sigma)}(\sigma)$ is called the Smith form at the local point $\sigma = \lambda_0$.

*Definition 2:* [8] Let A($\sigma$) as in (2). There exist biproper matrices $U_L(\sigma) \in \mathbb{R}^{r\times r}_{pr}(s)$, $U_R(\sigma) \in \mathbb{R}^{r\times r}_{pr}(s)$ (i.e. having no poles or zeros at infinity) such that

$$\mathrm{U}_L(\sigma)\mathrm{A}(\sigma)\mathrm{U}_R(\sigma) = S^{\infty}_{\mathrm{A}(\sigma)}(\sigma) = \\ = diag\left(\sigma^{q_1}, \dots, \sigma^{q_k}, \frac{1}{\sigma^{\tilde{q}_{k+1}}}, \dots, \frac{1}{\sigma^{\tilde{q}_r}}\right)$$

where
$q_1 \geq q_2 \geq \ldots \geq q_k \geq 0$ and $\tilde{q}_r \geq \tilde{q}_{r-1} \geq \ldots \tilde{q}_{k+1} \geq 0$.

$S^{\infty}_{\mathrm{A}(\sigma)}(\sigma)$ is called the *Smith form at infinity* of A($\sigma$). The first k terms $q_1, ..., q_k$ are the poles and the latter (r-k) terms $\tilde{q}_{k+1}, ..., \tilde{q}_r$ the zeros at $\sigma = \infty$ of A($\sigma$). It is proved in [8] that $q_1 = q$. We denote by $\mu$ the sum of the degrees of the infinite elementary divisors i.e. $\mu := \sum_{j=1}^{k} q_j + \sum_{j=k+1}^{r} \tilde{q}_j$.

*Definition 3:* [2], [8] The dual polynomial matrix of A($\sigma$) is defined as

$$\tilde{\mathrm{A}}(\sigma) := \sigma^q A(\frac{1}{\sigma}) = A_0\sigma^q + A_1\sigma^{q-1} + ... + A_q \tag{4}$$

*Definition 4:* [2], [8] Let $\tilde{A}(\sigma)$ as in (4). There exist matrices $\tilde{U}_L(\sigma) \in \mathbb{R}(\sigma)^{r\times r}$, $\tilde{U}_R(\sigma) \in \mathbb{R}(\sigma)^{r\times r}$ having no poles or zeros at $\sigma = 0$, such that

$$S^0_{\tilde{A}(\sigma)}(\sigma) = \tilde{U}_L(\sigma)\tilde{A}(\sigma)\tilde{U}_R(\sigma) = diag\left[\sigma^{\mu_1}, ..., \sigma^{\mu_r}\right]$$

$S^0_{\tilde{A}(\sigma)}(\sigma)$ is the Smith form of $\tilde{A}(\sigma)$ at zero. The terms $\sigma^{\mu_j}$ are the finite elementary divisors of $\tilde{A}(\sigma)$ at zero and are called the infinite elementary divisors of $\tilde{A}(\sigma)$. The connection between the Smith form at infinity of A($\sigma$) and the Smith form at zero of the dual matrix is

$$S^0_{\tilde{A}(\sigma)}(\sigma) = \sigma^q S^{\infty}_{A(\sigma)}(\tfrac{1}{\sigma}) = \\ = diag\left[1, \sigma^{q-q_2}, \dots, \sigma^{q-q_k}, \sigma^{q+\tilde{q}_{k+1}}, \dots, \sigma^{q+\tilde{q}_r}\right]$$

So the orders of the infinite elementary divisors are given by $\mu_1 = 0$, $\mu_j = q - q_j$, $j = 2, 3, ..., k$, $\mu_j = q + \tilde{q}_j$, $j = k+1, ..., r$

*Lemma 5:* [2], [8] Let $A(\sigma) = A_q\sigma^q + A_{q-1}\sigma^{q-1} + ... + A_0 \in \mathbb{R}^{r\times r}[\sigma]$. Let also n, $\mu$ be the sum of degrees of the finite and infinite elementary divisors of $A(\sigma)$, as defined previously. Then

$$n + \mu = r \times q \tag{5}$$

## III. SOLUTION AND MODELLING OF THE FORWARD BEHAVIOR OF $A(\sigma)\beta(k) = 0$

### *A. Finite Elementary Divisors and Forward Solution Space*

Let us assume that $A(\sigma)$ has $l$ distinct zeros $\lambda_1, \lambda_2, ..., \lambda_l$ where for simplicity of notation we assume that $\lambda_i \in \mathbb{C}$, $i = 1, 2, ..., l$ and let $S^C_{A(\sigma)}(\sigma) = U_L(\sigma)A(\sigma)U_R(\sigma) = blockdiag\left[1, ..., 1, f_z(\sigma), f_{z+1}(\sigma), ..., f_r(\sigma)\right]$. Assume that the partial multiplicities of the zeros $\lambda_\iota \in \mathbb{C}$ are $0 \leq n_{i,z} \leq n_{i,z+1} \leq ... \leq n_{i,r}$ i.e. $f_j(\sigma) = (\sigma-\lambda_i)^{n_{i,j}}\hat{f}_j(\sigma)$ j=z,z+1,,r with $\hat{f}_j(\lambda_i) \neq 0$. Let $u_j(\sigma) \in \mathbb{R}[\sigma]^{r\times 1}$ ,$j \in \mathbb{R}$ be the columns of $U_R(\sigma)$ and $u^{(\varphi)}_j(\sigma) := (\partial^\varphi/\partial\sigma^\varphi)u_j(\sigma)$, . Let also

$$\beta^i_{j,\varphi} := \tfrac{1}{\varphi!}u^{(\varphi)}_j(\lambda_i) \quad i = 1, 2, ..., l \quad j = z, z+1, \dots, r$$

Define the vector valued functions

$$\xi^i_{j,\varphi}(k) := \lambda^k_i\beta^i_{j,\varphi} + k\lambda^{k-1}_i\beta^i_{j,\varphi-1} + ... + \begin{pmatrix} k \\ \varphi \end{pmatrix}\lambda^{k-\varphi}_i\beta^i_{j,0} \\ for \lambda_i \neq 0 \tag{6}$$

$$\xi^i_{j,\varphi}(k) := \delta(k)x^i_{j,\varphi} + \delta(k-1)x^i_{j,\varphi-1} + ... + \delta(k-\varphi)x^i_{j,0} \\ for \lambda_i = 0 \tag{7}$$

$$i = 1, 2, ..., l \quad j = z, z+1, \dots, r \quad \varphi = 0, 1, ..., n_{i,j-1}$$

where by $\delta(k)$ or $\delta_k$ we denote the known Kronecker delta function.

*Theorem 6:* [6], [8] The time sequences $\xi^i_{j,q}(k)$, as defined above, are solutions of (1). In addition, let

$$C_{i,j} := \left[\begin{array}{ccccc} \beta^i_{j,0} & \beta^i_{j,1} & ... & \beta^i_{j,n_{i,j}-2} & \beta^i_{j,n_{i,j}-1} \end{array}\right]$$

$$J_{i,j} := \begin{bmatrix} \lambda_i & 1 & ... & 0 \\ 0 & \lambda_i & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & ... & 0 & \lambda_i \end{bmatrix} \in \mathbb{R}^{n_{i,j}\times n_{i,j}}$$

where , $j = z, z+1, \dots, r$ and

$$C^F_i := \left[\begin{array}{cccc} C_{i,z} & C_{i,z+1} & \cdots & C_{i,r} \end{array}\right] \\ J^F_i := blockdiag\left[\begin{array}{cccc} J_{i,z} & J_{i,z+1} & \cdots & J_{i,r} \end{array}\right]$$

Finally let

$$C^D_F := \left[C^F_1 \quad \cdots \quad C^F_l\right] \quad J^D_F := blockdiag\left[J^F_1 \quad \cdots \quad J^F_l\right]$$

The forward solution space of the system (1) is spanned by:

$$B^D_F = \left\langle C^D_F\left(J^D_F\right)^k\right\rangle$$

*Lemma 7:* [8] The vectors $\beta^i_{j,\varphi}$, previously defined as:

$$\beta^i_{j,\varphi} := \tfrac{1}{\varphi!}u^{(\varphi)}_j(\lambda_i) \quad i = 1, 2, ..., l \quad j = z, z+1, \dots, r$$

satisfy the system of equations:

$$A(\lambda_i)\beta^i_{j,0} = 0 \\ A^{(1)}(\lambda_i)\beta^i_{j,0} + A(\lambda_i)\beta^i_{j,1} = 0 \\ \vdots \\ \frac{1}{(n_{ij}-1)!}A^{(n_{ij}-1)}(\lambda_i)\beta^i_{j,0} + \cdots + A(\lambda_i)\beta^i_{j,(n_{ij}-1)} = 0$$

The above system of equations can be rewritten as

$$\begin{pmatrix} \frac{A^{(n_{ij}-1)}(\lambda_i)}{(n_{ij}-1)!} & \cdots & A(\lambda_i) \end{pmatrix} \underbrace{\begin{pmatrix} \beta^i_{j,0} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \beta^i_{j,(n_{ij}-1)} & \cdots & \beta^i_{j,0} \end{pmatrix}}_{W_i} \\ = \begin{pmatrix} 0 & \cdots & 0 \end{pmatrix} \tag{8}$$

By solving the above system of equations, we can obtain the matrices, $A^{(n_{ij}-1)}(\lambda_i), ..., A'(\lambda_i), A(\lambda_i)$, that represent the values of $A(\sigma)$ and its derivatives at $\lambda_i$. Thus the evaluation of $A(\sigma)$ is reduced to a Hermite interpolation problem. Alternatively, using the relation

$$\frac{A^{(\varepsilon)}(\lambda_i)}{\varepsilon!} = \binom{q}{\varepsilon} A_q {\lambda_i}^{q-\varepsilon} + \cdots + \binom{\varepsilon+1}{\varepsilon} A_{\varepsilon+1}\lambda_i + \binom{\varepsilon}{\varepsilon} A_\varepsilon$$

we rewrite (8) as follows:

$$\begin{pmatrix} A_q & \cdots & A_0 \end{pmatrix} Q_i W_i = \begin{pmatrix} 0 & \cdots & 0 \end{pmatrix} \quad (9)$$

where

$$Q_i = \begin{pmatrix} \binom{q}{n_{ij}-1}{\lambda_i}^{q-(n_{ij}-1)} I & \cdots & {\lambda_i}^q I \\ \vdots & \ddots & \vdots \\ \binom{n_{ij}}{n_{ij}-1}\lambda_i I & \binom{r_{idx}}{c_{idx}}{\lambda_i}^{r_{idx}-c_{idx}} & \vdots \\ I & \ddots & {\lambda_i}^2 I \\ \vdots & \ddots & \lambda_i \\ 0 & \cdots & I \end{pmatrix}$$

with $r_{idx} = q, q-1, ..., 0$ and $c_{idx} = (n_{ij}-1), ..., 0$ the row and column indexes, counting from left to right and top to bottom.

### B. Construction of a system with forward behavior

The previous lemma is important, because it states that in order for a time sequence (6) or (7) to be a solution of $A(\sigma)\beta(k) = 0$, the vectors $\beta^i_{j,\varphi}$ need to satisfy (9). Thus, our problem has been reduced to a linear system equation problem. That is, given a time sequence in the form of $\xi^i_{j,\varphi}(k)$, we can solve (9) in terms of the unknowns $A_0, A_1, ..., A_q$ in order to construct $A(\sigma)$. These results give rise to the following algorithm for construction of a system that satisfies a desired forward behavior.

*Algorithm 8:* Suppose that a finite number of functions of the form

$$\beta^i_j(k) := \lambda_i^k \beta^i_{j,q_{ij}-1} + \cdots + \binom{k}{q_{ij}-1} \lambda_i^{k-(q_{ij}-1)} \beta^i_{j,0}$$
$$\beta_{j,q_j}(k) = \delta(k) x_{j,q_j-1} + ... + \delta(k-(q_j-1)) x_{j,0}$$

are given.

**Step 1** Using the equation $n + \mu = r \times q$, where $\mu = 0$, find q. In case this equation is not satisfied, then let q=1, q=2,... until matrix dimensions agree.

**Step 2** Create the matrices $Q_i, W_i$.

**Step 3** Create the combined matrices

$$Q = \begin{pmatrix} Q_1 & \cdots & Q_l \end{pmatrix} \in \mathbb{R}^{(q+1)r \times n_{ij} r l}$$
$$W = blockdiag\begin{pmatrix} W_1 & \cdots & W_l \end{pmatrix} \in \mathbb{R}^{n_{ij} r l \times n_{ij} l}$$

**Step 4** Solve the system of equations

$$\begin{pmatrix} A_q & \cdots & A_0 \end{pmatrix} QW = \begin{pmatrix} 0 & \cdots & 0 \end{pmatrix}$$

in terms of the unknown matrices $A_i$. Choose the free entries $a_{ij}$ of each matrix $A_i$ so that $\det[A(\sigma)] \neq 0$.

In case where in step1, there exists $q$ such that $n + 0 > rq$ and $n < r(q+1)$, we can continue with the rest of the steps for $(q+1)$. The resulting matrix $A(\sigma)$ will describe a system of algebraic/difference equations with $\beta_i(k)$ as part of its solution space.

*Example 9:* Let the following time sequences

$$\beta_1(k) = \underbrace{\begin{pmatrix} 2 \\ 3 \end{pmatrix}}_{\beta_{12}} 2^k + \underbrace{\begin{pmatrix} 4 \\ 1 \end{pmatrix}}_{\beta_{11}} k2^{k-1} + \underbrace{\begin{pmatrix} 4 \\ 1 \end{pmatrix}}_{\beta_{10}} \frac{k(k-1)}{2} 2^{k-2}$$
$$\beta_2(k) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} 3^k$$

**Step 1** We want to create a system $A(\sigma)\beta(k) = 0$ that has the above sequences in its solution space. Since $n = q_1 + q_2 = 3 + 1 = 4$ and r=2, we have $n + 0 = 2q \Rightarrow q = 2$. So

$$A(\sigma) = A_2\sigma^2 + A_1\sigma + A_0$$

**Step 2** Create the matrices

$$Q_1 = \begin{pmatrix} I & 2\cdot 2I_2 & 2^2 I_2 \\ 0 & I_2 & 2I_2 \\ 0 & 0 & I_2 \end{pmatrix} \quad W_1 = \begin{pmatrix} \beta_{10} & 0 & 0 \\ \beta_{11} & \beta_{10} & 0 \\ \beta_{12} & \beta_{11} & \beta_{10} \end{pmatrix}$$
$$Q_2 = \begin{pmatrix} 3^2 I_2 \\ 3I_2 \\ I_2 \end{pmatrix} \quad W_2 = \beta_{20}$$

**Step 3** Create the combined matrices

$$Q = \begin{pmatrix} Q_1 & Q_2 \end{pmatrix} \quad W = \begin{pmatrix} W_1 & 0 \\ 0 & W_2 \end{pmatrix}$$

**Step 4** Solve the system

$$\begin{pmatrix} A_2 & A_1 & A_0 \end{pmatrix} QW = 0_{2\times 4}$$

where

$$A_i = \begin{pmatrix} a_{i1} & a_{i2} \\ a_{i3} & a_{i4} \end{pmatrix}$$

By solving this system in Mathematica and choosing $a_{12} = 1$, $a_{14} = a_{22} = a_{24} = -1$ we end up with

$$A(\sigma) = \begin{pmatrix} \frac{13}{4} - \frac{15s}{8} + \frac{s^2}{8} & \frac{19}{4} + s - s^2 \\ \frac{23}{4} - \frac{29s}{8} + \frac{3s^2}{8} & \frac{41}{4} - s - s^2 \end{pmatrix}$$

with $\det[A(\sigma)] = \frac{1}{4}(s-3)(s-2)^3$.

## IV. SOLUTION AND MODELLING OF THE BACKWARD BEHAVIOR OF $A(\sigma)\beta(k) = 0$

### A. Infinite Elementary Divisors and Backward Solution Space

Let

$$S^0_{\tilde{A}(\sigma)}(\sigma) = \tilde{U}_L(\sigma)\tilde{A}(\sigma)\tilde{U}_R(\sigma) = blockdiag[\sigma^{\mu_1}, ..., \sigma^{\mu_r}]$$

be the Smith form of $\tilde{A}(\sigma)$ at zero. Let also $\tilde{U}_R(\sigma) = \begin{bmatrix} \tilde{u}_1(\sigma) & \cdots & \tilde{u}_r(\sigma) \end{bmatrix}$ where $\tilde{u}_j(\sigma) \in R(\sigma)^{r\times 1}$ and $\tilde{u}_j^{(i)}(\sigma)$, $\tilde{A}^{(i)}(\sigma)$ be the i-th derivatives of $\tilde{u}_j(\sigma)$ and $\tilde{A}(\sigma)$ respectively, for i=0,1,,$\mu_j$ and $j = 2, ..., r$. Let

$$x_{j,i} := \frac{1}{i!}\tilde{u}_j^{(i)}(0) \quad i = 0, 1, ..., \mu_j - 1 \quad j = 2, ..., r$$

and define the vector valued functions

$$\xi_j^B(k) := x_{j,\mu_j-1}\delta(N-k) + ... + x_{j,0}\delta(N-(k+\mu_j-1))$$
$$i = 0,1,\ldots,\mu_j-1; j = 2,...,r$$

*Theorem 10:* The time sequences $\xi_j^B(k)$ are solutions of (1) for final conditions

$$\begin{bmatrix} \xi(\mathrm{N}) \\ \vdots \\ \xi(\mathrm{N}-q+1) \end{bmatrix} = \begin{bmatrix} x_{j,i} \\ \vdots \\ x_{q_j-1} \end{bmatrix}$$

In addition, let

$$C_j^B = \begin{bmatrix} x_{j,0} & x_{j,1} & \cdots & x_{j,\mu_j-1} \end{bmatrix} \in \mathbb{R}^{r\times\mu_j}$$

$$J_j^B := \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 0 \end{pmatrix} \in \mathbb{R}^{\mu_j\times\mu_j}$$

and

$$C_B^D := \begin{bmatrix} C_2^D & \cdots & C_r^D \end{bmatrix} \quad J_B^D := blockdiag\begin{bmatrix} J_2^D & \cdots & J_r^D \end{bmatrix}$$

The backward solution space of (1) is spanned by

$$B_B^D = \left\langle C_B^D \left(J_B^D\right)^{N-k} \right\rangle$$

*Lemma 11:* [8] The vectors $x_{j,i}$, satisfy the system of equations:

$$\begin{pmatrix} A_q & 0 & 0 & \cdots & \cdots & 0 \\ \vdots & A_q & 0 & \cdots & \cdots & 0 \\ A_0 & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & A_0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_0 & \cdots & A_q \end{pmatrix} \begin{pmatrix} x_{j,0} \\ x_{j,1} \\ \vdots \\ x_{j,q-1} \\ x_{j,q} \\ \vdots \\ x_{j,q+q_j-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \quad (10)$$

Although the system of equations in [8] refer to the infinite zero elementary divisors (i.z.e.d.), we can easily check that they are also satisfied for the infinite pole elementary divisors (i.p.e.d).

$$S^0_{\tilde{A}(\sigma)}(\sigma) = diag\left[1, \underbrace{\sigma^{q-q_2}, \ldots, \sigma^{q-q_k}}_{i.p.e.d.}, \underbrace{\sigma^{q+\tilde{q}_{k+1}}, \ldots, \sigma^{q+\tilde{q}_r}}_{i.z.e.d.}\right]$$

Equations (10) can be rewritten as

$$\begin{pmatrix} A_q \\ \vdots \\ A_0 \end{pmatrix}^T \underbrace{\begin{pmatrix} x_{j,0} & x_{j,1} & \cdots & x_{j,q} & \cdots & x_{j,q+q_j-1} \\ 0 & x_{j,0} & \cdots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \cdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & x_{j,0} & \cdots & x_{j,q_j-1} \end{pmatrix}}_{Q^{B_z}{}_j} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}^T \quad (11)$$

for the case of i.z.e.d. and

$$\begin{pmatrix} A_q \\ \vdots \\ A_{q_k+1} \end{pmatrix}^T \underbrace{\begin{pmatrix} x_{j,0} & x_{j,1} & \cdots & x_{j,q-q_k-1} \\ 0 & x_{j,0} & \cdots & \vdots \\ \vdots & \ddots & \cdots & \vdots \\ 0 & \cdots & 0 & x_{j,0} \end{pmatrix}}_{Q^{B_p}{}_j} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}^T \quad (12)$$

for the case of i.p.e.d.

### *B. Construction of a system with backward behavior*

The previous lemma is important, because it states that in order for a time function

$$\xi_{j,i}^B(k) := x_{j,\mu_j-1}\delta(N-k) + ... + x_{j,0}\delta(N-(k+\mu_j-1))$$

to be a solution of $A(\sigma)\beta(k) = 0$, the vectors $x_{j,i}$ need to satisfy (11) (or (12)). But since this constitutes a linear system of equations, it can be used to solve the inverse problem. That is, given a time sequence in the form of $x_{j,i}$, we can use relation (11) (or (12)) and solve this linear system for the unknowns $A_0, A_1, ..., A_q$ in order to construct $A(\sigma)$. These results give rise to the following algorithm for construction of a system that satisfies a desired backward behavior.

*Algorithm 12:* Suppose that a finite number of functions of the form

$$\xi_{j,i}^B(k) := x_{j,\mu_j-1}\delta(N-k) + ... + x_{j,0}\delta(N-(k+\mu_j-1))$$

are given.

**Step 1** Using the equation $n + \mu = r \times q$, where n=0, find q. In case this equation is not satisfied, then let q=1, q=2,... until matrix dimensions agree.

**Step 2** Create the matrices $Q^{B_z}{}_j$ and/or $Q^{B_p}{}_j$.

**Step 3** Create the combined matrix

$$Q^B = \begin{pmatrix} Q^B{}_1 & \cdots & Q^B{}_r \end{pmatrix}$$

**Step 4** Solve the system of equations

$$\begin{pmatrix} A_q & \cdots & A_0 \end{pmatrix} Q^B = \begin{pmatrix} 0 & \cdots & 0 \end{pmatrix} \quad (13)$$

or

$$\begin{pmatrix} A_q & \cdots & A_{q_k+1} \end{pmatrix} Q^B = \begin{pmatrix} 0 & \cdots & 0 \end{pmatrix} \quad (14)$$

in terms of the unknown matrices $A_i$. Choose the free entries $a_{ij}$ of each matrix $A_i$ so that $\det[A(\sigma)] \neq 0$.

*Example 13:* Let the time sequence

$$\beta_1(k) = \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}_{x_{1,2}} \delta_{\mathrm{N}-k} + \underbrace{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}_{x_{1,1}} \delta_{N-k-1} + \underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}_{x_{1,0}} \delta_{N-k-2}$$

**Step 1** We want to create a system $A(\sigma)\beta(k) = 0$ that has the above sequences as its solution. Since $\mu = 3$ and $r = 3$, we have $\mu + 0 = 3q \Rightarrow q = 1$. So

$$A(\sigma) = A_1\sigma + A_0$$

Since q=1, we have either $\mu_1 = q - q_j = 3 \Rightarrow 1 - q_j = 3 \Rightarrow q_j = -2$ which is rejected, or $\mu_1 = q + q_j = 3 \Rightarrow 1 + q_j = 3 \Rightarrow q_j = 2$ which is accepted.

**Step 2 & 3** By using equation (14). Create the matrix

$$Q^B = \left( \begin{array}{c|c|c} x_{1,0} & x_{1,1} & x_{1,2} \\ 0 & x_{1,0} & x_{1,1} \end{array} \right)$$

**Step 4** Solve the system

$$\left( \begin{array}{cc} A_1 & A_0 \end{array} \right) Q^B = 0_{3\times 3}$$

where $A_i \in \mathbb{R}^{3\times 3}$. $A(\sigma)$ is given by

$$A(\sigma) = \left( \begin{array}{ccc} 1+\frac{\sigma}{2} & 4 & 2+\frac{\sigma}{2} \\ 1+\frac{\sigma}{4} & 2+\sigma & \frac{1}{2}+\frac{\sigma}{4} \\ \frac{1}{4}+\frac{3\sigma}{8} & 1+\sigma & \frac{3\sigma}{8} \end{array} \right)$$

with $\det[A(\sigma)] = 1$, $\det\left[\tilde{A}(\sigma)\right] = \det[A_0\sigma + A_1] = -3\sigma^3$.

## V. MODELLING OF A SYSTEM WITH FORWARD AND BACKWARD BEHAVIOUR

So far we have provided methods for constructing a system that satisfies a desired forward or backward behavior. Since these methods are functional, we can combine them to give a solution to the general inverse problem. Construct a system of algebraic/difference equations that satisfies a desired forward and a backward behavior. The answer is simple; we can just solve both systems (9) and (11) and find a solution that satisfies both. As a result, the system produced will have a solution space spanned by the given time sequences. These results give rise to the following algorithm.

*Algorithm 14:* Suppose that a finite number of functions of the form

$$\begin{array}{c} \beta_j^i(k) := \lambda_i^k \beta_{j,q_{ij}-1}^i + ... + \binom{k}{q_{ij}-1} \lambda_i^{k-(q_{ij}-1)} \beta_{j,0}^i \\ \beta_{j,q_j}(k) = \delta(k) x_{j,q_j-1} + ... + \delta(k-(q_j-1)) x_{j,0} \\ \xi_j^B(k) := x_{j,\mu_j-1}\delta(N-k) + ... + x_{j,0}\delta(N-(k+i)) \end{array}$$

are given.

**Step 1** Using the equation $n + \mu = r \times q$, where $n = \sum q_{ij}$ and $\mu = \sum \mu_j$, find q. . In case this equation is not satisfied, then let q=1, q=2,... until matrix dimensions agree.

**Step 2** Create the matrices Q and W according to Algorithm 8 and $Q^B$ according to Algorithm 12.

**Step 3** Solve the system of equations

$$\begin{array}{c} \left( \begin{array}{ccc} A_q & \cdots & A_0 \end{array} \right) QW = \left( \begin{array}{ccc} 0 & \cdots & 0 \end{array} \right) \\ AND \\ \left( \begin{array}{ccc} A_q & \cdots & A_0 \end{array} \right) Q^B = \left( \begin{array}{ccc} 0 & \cdots & 0 \end{array} \right) \end{array}$$

in terms of the unknown matrices $A_i$. Choose the free entries $a_{ij}$ of each matrix $A_i$ so that $\det[A(\sigma)] \neq 0$.

We have constructed an algorithm of finding the polynomial matrix A($\sigma$) corresponding to certain behavior. As we have already mentioned, A($\sigma$) is not the only polynomial matrix satisfying the given behavior. More specifically, all the *divisor equivalent* [7] polynomial matrices are solutions to our problem.

*Example 15:* Let the following time sequences

$$\beta_1(k) = \underbrace{\left( \begin{array}{c} 1 \\ 1 \end{array} \right)}_{\beta_{11}} + \underbrace{\left( \begin{array}{c} 3 \\ 1 \end{array} \right)}_{\beta_{10}} k \quad \beta_2(k) = \underbrace{\left( \begin{array}{c} 2 \\ 0 \end{array} \right)}_{\beta_{21}} 2^k + \underbrace{\left( \begin{array}{c} 4 \\ 1 \end{array} \right)}_{\beta_{20}} k 2^{k-1}$$

$$\beta_3(k) = \underbrace{\left( \begin{array}{c} -1 \\ 0 \end{array} \right)}_{\beta_{33}} \delta(\mathrm{N}-k) + \underbrace{\left( \begin{array}{c} -1 \\ -1 \end{array} \right)}_{\beta_{32}} \delta(\mathrm{N}-k-1) +$$

$$+ \underbrace{\left( \begin{array}{c} 1 \\ 1 \end{array} \right)}_{\beta_{31}} \delta(\mathrm{N}-k-2) + \underbrace{\left( \begin{array}{c} 1 \\ 0 \end{array} \right)}_{\beta_{30}} \delta(\mathrm{N}-k-3)$$

**Step 1** We want to create a system $A(\sigma)\beta(k) = 0$ that has the above functions as its solution. Since $\mu = 4$ and r=2, we have $8 = 2q \Rightarrow q = 4$. So

$$A(\sigma) = A_4\sigma^4 + A_3\sigma^3 + A_2\sigma^2 + A_1\sigma + A_0$$

**Step 2** For the first and second time sequences, we create the matrices

$$Q_i = \left( \begin{array}{cc} 4{\lambda_i}^3 \mathrm{I}_2 & {\lambda_i}^4 \mathrm{I}_2 \\ 3{\lambda_i}^2 \mathrm{I}_2 & {\lambda_i}^3 \mathrm{I}_2 \\ 2\lambda_i \mathrm{I}_2 & {\lambda_i}^2 \mathrm{I}_2 \\ \mathrm{I}_2 & \lambda_i \mathrm{I}_2 \\ 0_2 & \mathrm{I}_2 \end{array} \right) \quad W_i = \left( \begin{array}{cc} \beta_{i0} & 0 \\ \beta_{i1} & \beta_{i0} \end{array} \right)$$

$$Q = \left( \begin{array}{cc} Q_1 & Q_2 \end{array} \right) \quad W = \left( \begin{array}{cc} W_1 & 0 \\ 0 & W_2 \end{array} \right)$$

For the third time sequence, since q=4, we have either
$\mu = q - q_j = 4 \Rightarrow 4 - q_j = 4 \Rightarrow q_j = 0$ which is accepted or
$\mu = q + q_j = 4 \Rightarrow 4 + q_j = 4 \Rightarrow q_j = 0$ which is accepted too. But in order for the matrix dimensions to agree, we will use (14).

$$Q^B = \left( \begin{array}{cccc} \beta_{30} & \beta_{31} & \beta_{32} & \beta_{33} \\ 0 & \beta_{30} & \beta_{31} & \beta_{32} \\ 0 & 0 & \beta_{30} & \beta_{31} \\ 0 & 0 & 0 & \beta_{30} \end{array} \right)$$

**Step 3** solve the system

$$\begin{array}{c} \left( \begin{array}{ccccc} A_4 & A_3 & A_2 & A_1 & A_0 \end{array} \right) QW = 0_{2\times 4} \\ \left( \begin{array}{cccc} A_4 & A_3 & A_2 & A_1 \end{array} \right) Q^B = 0_{2\times 4} \end{array}$$

by using Mathematica and get

$$A(\sigma) =$$

$$= \left( \begin{array}{cc} \frac{3}{5} - \frac{11\sigma}{5} + 2\sigma^2 - \frac{3\sigma^3}{5} & 2\sigma - \frac{6\sigma^2}{5} - \frac{4\sigma^3}{5} + \frac{3\sigma^4}{5} \\ \frac{1}{10} - \frac{29\sigma}{20} + \frac{7\sigma^2}{4} - \frac{3\sigma^3}{5} & 1 + \sigma - \frac{29\sigma^2}{20} - \frac{11\sigma^3}{20} + \frac{3\sigma^4}{5} \end{array} \right)$$

withdet $[A(\sigma)] = (\sigma-2)^2(\sigma-1)^2$ and $\det\left[\tilde{A}(\sigma)\right] = \frac{3}{20}(\sigma-1)^2\sigma^4(2\sigma-1)^2$.

## VI. MODELLING OF NON-REGULAR SYSTEMS

In the previous sections, we have provided methods for the construction of a regular system with prescribed forward and backward behavior. This is of course not the only case one should consider. Similar results, can be extended for the case of non-square or non-regular systems $A(\sigma)\beta(k) = 0$ with $A(\sigma) \in \mathbb{R}[\sigma]^{p\times m}$ and $rankA(\sigma) = r < min\{p, m\}$ or with $A(\sigma) \in \mathbb{R}[\sigma]^{p\times p}$ and $det[A(\sigma)] = 0$. In such systems, the complete solution space depends not only on the finite and infinite elementary divisors of $A(\sigma)$, but also on the right null space structure. Results regarding such systems have been presented in [1], [5], [6]. We shall utilise this knowledge regarding the right null space structure in order to extend our modeling method to non-regular systems.

*Definition 16:* [6] Let $A(\sigma) \in \mathbb{R}[\sigma]^{p\times m}$ with $rankA(\sigma) = r < min\{p, m\}$. Define by

$$V_R := \left\{ u(\sigma) \in \mathbb{R}[\sigma]^{m\times 1} : A(\sigma)u(\sigma) = 0_{q\times 1} \right\}$$

the m-r dimensional vector space of the right kernel of $A(\sigma)$. Let $\{\varepsilon_{r+1}(\sigma), \dots, \varepsilon_m(\sigma)\}$ be a minimal polynomial basis of $V_R$. Then, define by $\varepsilon_i$ the degree of the m-tuples $\varepsilon_i(\sigma)$, i.e. $\varepsilon_i := \deg[\varepsilon_i(\sigma)]$ where degree of an m-tuple is the greatest degree among its components. The indices $\{\varepsilon_{r+1}, \dots, \varepsilon_m\}$ are called right minimal indices of $A(\sigma)$ and constitute invariant elements of the polynomial matrix. We also define by $\varepsilon$ the sum of the right minimal indices, i.e. $\varepsilon = \sum_{i=r+1}^{m-r} \varepsilon_i$.

In the same fashion, we can define the left minimal polynomial base of the left kernel of $A(\sigma)$ as $\{\eta_{r+1}(\sigma), \dots, \eta_p(\sigma)\}$ and the left minimal indices of $A(\sigma)$ $\{\eta_{r+1}, \dots, \eta_p\}$. These indices constitute invariant elements ot the polynomial matrix. Their sum is denoted by $\eta = \sum_{i=r+1}^{p} \eta_i$.

*Lemma 17:* [6] Let $A(\sigma) = A_q\sigma^q + ... + A_0 \in \mathbb{R}[\sigma]^{p\times m}$ with $rankA(\sigma) = r < min\{p, m\}$ Let also $n, \mu, \varepsilon, \eta$ be the sum of degrees of the finite and infinite elementary divisors and the sum of the sum of the right and left minimal indices of $A(\sigma)$ respectively. Then

$$n + \mu + \varepsilon + \eta = r \times q \tag{15}$$

The right kernel of $A(\sigma)$ is connected to the behavior of the system [6]. It should be mentioned that the right null space structure gives rise to either forward or backward solutions of the system.

Considering these results, we can extend the previous methods, bearing in mind that in case equation (5) is not satisfied, an exact solution may still exist for a non-regular system. In this case, the lag q of the system can be computed by (15).

*Example 18:* Let the three vector time sequences

$$\beta_1(k) = \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\beta_{11}} + \underbrace{\begin{pmatrix} 3 \\ 1 \end{pmatrix}}_{\beta_{10}} k \quad \beta_2(k) = \underbrace{\begin{pmatrix} 2 \\ 0 \end{pmatrix}}_{\beta_{21}} 2^k + \underbrace{\begin{pmatrix} 4 \\ 1 \end{pmatrix}}_{\beta_{20}} k2^{k-1}$$

$$\beta_3(k) = \underbrace{\begin{pmatrix} -1 \\ -1 \end{pmatrix}}_{\beta_{32}} \delta_{N-k} + \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\beta_{31}} \delta_{\mathrm{N}-k-1} + \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\beta_{30}} \delta_{\mathrm{N}-k-2}$$

We want to construct a system that has these time sequences as its solution.

**Step 1** Assume that $\mu = 3, n = 4, r = 2$ the equality $n+\mu = rq$ is not satisfied for any q. If however we assume that the system is not of full row rank, i.e. $r < 2$ we can find that for $\mu = 0, \varepsilon = 3, n = 4, r = 1$ we get $n + \mu + \varepsilon = rq \Rightarrow q = 7$. So the system constructed will be of the form

$$A(\sigma) = A_7\sigma^7 + \cdots + A_1\sigma + A_0$$

with $A_i = \begin{pmatrix} a_{i1} & a_{i2} \end{pmatrix}$, since r=1.

**Step 2 & 3** Following the same procedure as in the previous sections, we solve the system

$$\begin{pmatrix} A_7 & A_6 & A_5 & A_4 & A_3 & A_2 & A_1 & A_0 \end{pmatrix} QW = 0_{1\times 4}$$
$$\begin{pmatrix} A_7 & A_6 & A_5 \end{pmatrix} Q^B = 0_{1\times 3}$$

and get

$$A(\sigma) = \Big( \frac{609}{4294} - s^3 - s^4 + \frac{5521s^5}{8588} - \frac{2243s^6}{8588} \quad \frac{620}{2147} + s + s^2 + s^4 + s^5 - \frac{1035s^6}{8588} + \frac{2243s^7}{8588} \Big)$$

## VII. CONCLUSIONS

We have studied the backward behavior of systems of algebraic/difference equations. We exploited already known results in order to construct a system satisfying a given solution space in the form of time sequences. We also extended these results to non-regular systems. It should be mentioned that the methods presented here are subject to computation errors. For small scale systems of low order q, the commonly used Mathematica Commands are appropriate for providing solutions to the linear system equations. Nevertheless, for problems of higher dimension, one should consider using more appropriate methods, in order to avoid computational difficulties. These results can surely be implemented for the case of continuous systems, where smooth and impulsive behaviors are of interest.

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