Modeling of discrete time auto-regressive systems with given forward and backward behavior

Lazaros Moysis
School of Mathematical Sciences
Aristotle University of Thessaloniki
Thessaloniki, Greece 54124
Email: lazarosm@math.auth.gr

Nicholas P. Karampetakis
School of Mathematical Sciences
Aristotle University of Thessaloniki
Thessaloniki, Greece 54124
Email: karampet@math.auth.gr

Abstract—We study the behavior of discrete time AR-representations. A theorem is provided connecting the backward behavior of a system, due to its infinite elementary divisors, with the forward behavior of its dual system. We first use this result to construct a system satisfying a given backward behavior. In addition to this, we propose a way to combine this result with previous ones to create an algorithm for computing a system satisfying a given forward and backward behavior.

I. INTRODUCTION

Let \( \mathbb{R} \) be the field of reals, \( \mathbb{R} [s] \) be the ring of polynomials, \( \mathbb{R}[s]^{m \times n} \) be the set of \( m \times n \) matrices with elements in \( \mathbb{R}[s] \). Consider the system of algebraic and difference equations, or otherwise a discrete time (Auto-Regressive) AR-representation, of the form

\[
A(\sigma) \beta(k) = 0
\]

(1)

where \( k = 0, 1, \ldots, N - q \), or equivalently

\[
A_q \beta(k + q) + A_{q-1} \beta(k + q - 1) + \ldots + A_0 \beta(k) = 0
\]

(2)

where \( \sigma \) denotes the forward shift operator i.e. \( \sigma \beta(k) = \beta(k + 1) \). \( A(\sigma) \in \mathbb{R}^{r \times r} \) is a regular polynomial matrix i.e. \( \det \{A(\sigma)\} \neq 0 \) with \( A_q \neq 0 \) and \( \beta(k) \in \mathbb{R}^r \), \( k = 0, 1, \ldots, N \). The number \( q \) is often called the lag of the matrix. The solution space of such systems consists of both forward and backward solutions defined over a finite time i.e.

\[
B := \{ \beta(k) \in \mathbb{R}^r, \text{satisfies (1), for } k=0,1,\ldots,N-q \}
\]

Forward solutions are defined in a sense that the initial conditions are given and \( \beta(k) \) is to be determined in a forward fashion from its previous values. Backward solutions are defined in a sense that the final conditions are given and \( \beta(k) \) is to be determined in a backward fashion from its future values.

This research has been co-financed by the European Union (European Social Fund ESF) and Greek national funds through the Operational Program “Education and Lifelong Learning” of the National Strategic Reference Framework (NSRF)—Research Funding Program:ARCHIMedes III. Investing in knowledge society through the European Social Fund.

The solution of such systems has been previously studied by various authors in [1], [2], [3] and their algebraic structure in [4], [5], [6], [7]. A method for the construction of the forward (resp. backward) solution space of (2), based on the finite (resp. infinite) elementary divisor structure of \( A(\sigma) \) has been presented in [1] (resp. [2]), whereas the results have been extended to the non-regular case by [8]. An interesting problem that we face in this paper is the inverse problem, that is: Given a certain forward/backward solution space, find a system of algebraic and difference equations or otherwise the polynomial matrix \( A(\sigma) \) given in (1), that has exactly the prescribed solution space. This problem has initially been presented and solved by [1], but only for the case of the forward solution space. More specifically, [1] proposed a formula that connects the form of \( A(\sigma) \) and a finite Jordan pair that can be constructed from the prescribed forward solution space. The main aim of this work is to extend the results presented in [1], for the case where except of the given forward behavior, a backward behavior is also given. In order to achieve this, we connect the backward solution space of (1) with the forward solution space of its dual system

\[
A_0 \beta(k + q) + A_1 \beta(k + q - 1) + \ldots + A_q \beta(k) = 0
\]

(4)

and use the results given in [1]. Otherwise, we use a methodology that was used in for the first time in [9] for the case of continuous time systems, where the finite and infinite elementary divisor structure of \( A(\sigma) \) is connected with the smooth and impulsive behavior of (1) respectively.

II. PRELIMINARIES

Theorem 1: [10] Let \( A(\sigma) \) as in (3). There exist unimodular matrices \( U_L(\sigma) \in \mathbb{R}[\sigma]^{r \times r} \), \( U_R(\sigma) \in \mathbb{R}[\sigma]^{r \times r} \) i.e. \( \det \{U_L(\sigma)\} \neq 0 \), \( \det \{U_R(\sigma)\} \neq 0 \), such that

\[
S_{A(\sigma)}^C = U_L(\sigma)A(\sigma)U_R(\sigma) = \text{blockdiag} [1, \ldots, 1, f_z(\sigma), f_{z+1}(\sigma), \ldots, f_r(\sigma)]
\]

(5)

with \( 1 \leq z \leq r \) and \( f_i(\sigma)/f_{i+1}(\sigma) i = z, z + 1, \ldots, r \). \( S_{A(\sigma)}^C \) is called the Smith form of \( A(\sigma) \) (in \( \mathbb{C} \)) where \( f_i(\sigma) \in \mathbb{R}[\sigma] \) are the invariant polynomials of \( A(\sigma) \). The zeros \( \lambda_j \in \mathbb{C} \) of \( f_j(\sigma) \in \mathbb{R}[\sigma], j = z, z + 1, \ldots, r \) are called finite zeros of \( A(\sigma) \). Assume that \( A(\sigma) \) has 1 distinct zeros
and the partial multiplicities of each zero \( \lambda_i \in \mathbb{C} \), \( i = 1, \ldots, l \) are \( 0 \leq n_{i,z} \leq n_{i,z+1} \leq \ldots \leq n_{i,u} \), i.e.

\[
f_j(\sigma) = (\sigma - \lambda_i)^{n_{i,j}} f_j(s)
\]

\( j = z, z+1, \ldots, r \) with \( f_j(\lambda_i) \neq 0 \) The terms \( (\sigma - \lambda_i)^{n_{i,j}} \) are called finite elementary divisors of \( A(\sigma) \) at \( \sigma = \lambda_i \). We also denote by \( n \) the sum of the degrees of the finite elementary divisors of \( A(\sigma) \), i.e. \( n := \deg \left[ \prod_{j=z}^{r} f_j(\sigma) \right] = \sum_{j=z}^{r} n_{i,j} \).

Similarly, we can find \( U_L(\sigma) \in \mathbb{R}[\sigma]^{r \times r}, U_R(\sigma) \in \mathbb{R}[\sigma]^{r \times r} \) having no poles and zeros at \( \sigma = \lambda_0 \) such that

\[
S_{A(\sigma)}^{0}(\sigma) = U_L(\sigma)A(\sigma)U_R(\sigma) = \text{blockdiag} \left[ 1, \ldots, 1, (\sigma - \lambda_0)^{n_1}, \ldots, (\sigma - \lambda_0)^{n_r} \right]
\]

\( S_{A(\sigma)}^{0}(\sigma) \) is called the Smith form of \( A(\sigma) \) at the local point \( \sigma = \lambda_0 \).

**Theorem 2:** [10] Let \( A(\sigma) \) defined in (3). There exist biproper matrices \( U_L(\sigma) \in \mathbb{R}^{r \times r}(\sigma), U_R(\sigma) \in \mathbb{R}^{r \times r}(\sigma) \) (i.e. having no poles or zeros at infinity) such that

\[
U_L(\sigma)A(\sigma)U_R(\sigma) = S_{\infty}(\sigma) = \text{blockdiag} \left( \frac{u}{s^q_1, s^q_2, \ldots, s^q_{u-k}}, \frac{1}{s^{q_{u+1}}, \ldots, s^{q_r}} \right)
\]

with \( 1 \leq u \leq r, q_1 \geq \ldots \geq q_k > 0 \) and \( q_r \geq q_{r-1} \geq \ldots \geq q_{u+1} > 0 \).

\( S_{\infty}(\sigma) \) is called the Smith form of \( A(\sigma) \) at infinity. The first \( k \) terms \( q_1, \ldots, q_k \) (resp. the latter \( r-u \) terms \( q_{u+1}, \ldots, q_r \)) are the orders of the poles (resp. zeros) at \( \sigma = \infty \) of \( A(\sigma) \). It is proved in [10] that \( q_1 = q \).

Define the dual polynomial matrix of \( A(\sigma) \) as

\[
\hat{A}(\sigma) := \sigma^q A \left( \frac{1}{\sigma} \right) = A_0 \sigma^q + A_1 \sigma^{q-1} + \ldots + A_q
\]

**Theorem 3:** [10] Let \( \hat{A}(\sigma) \) defined in (6). There exist matrices \( \hat{U}_L(\sigma) \in \mathbb{R}[\sigma]^{r \times r}, \hat{U}_R(\sigma) \in \mathbb{R}[\sigma]^{r \times r} \) having no poles or zeros at \( \sigma = 0 \), such that

\[
S^0_{\hat{A}(\sigma)}(\sigma) = \hat{U}_L(\sigma)\hat{A}(\sigma)\hat{U}_R(\sigma) = \text{blockdiag} \left[ \sigma^{\mu_1}, \ldots, \sigma^{\mu_r} \right]
\]

\( S^0_{\hat{A}(\sigma)}(\sigma) \) is the local Smith form of \( \hat{A}(\sigma) \) at \( \sigma = 0 \). The terms \( \sigma^{\mu_i} \) are the finite elementary divisors of \( \hat{A}(\sigma) \) at zero and are called the infinite elementary divisors of \( A(\sigma) \).

The connection between the Smith form at infinity of \( A(\sigma) \) and the Smith form at zero of the dual matrix is given in [11] and [10] : \( S^0_{\hat{A}(\sigma)}(\sigma) = \sigma^q S^\infty_{A(\sigma)} \left( \frac{1}{\sigma} \right) = \text{blockdiag} \left[ 1, \sigma^{q_1}, \ldots, \sigma^{q_k}, \sigma^{q_k+\hat{q}_k+1}, \ldots, \sigma^{q_r+\hat{q}_r} \right] \)

Therefore, the orders of the infinite elementary divisors are given by

\[
\mu_1 = q - q_1 \quad q = q_1 \quad 0
\]

\[
\mu_j = q - q_j \quad j = 2, 3, \ldots, u
\]

\[
\mu_j = q + \hat{q}_j \quad j = u + 1, \ldots, r
\]

We denote by \( \mu \), the sum of the degrees of the infinite elementary divisors of \( A(\sigma) \) i.e.

\[
\mu := \sum_{j=1}^{r} \mu_j
\]

**Lemma 4:** [12] Let \( A(\sigma) = A_q \sigma^q + A_{q-1} \sigma^{q-1} + \ldots + A_0 \in \mathbb{R}[\sigma]^{r \times r}[\sigma] \). Let also \( n, \mu \) be the sum of degrees of the finite and infinite elementary divisors of \( A(\sigma) \), as defined previously. Then

\[
n + \mu = r \times q
\]

### III. JORDAN PAIRS AND SOLUTIONS OF \( A(\sigma) \beta(k) = 0 \)

#### A. Finite Jordan Pairs and Forward Solution Space

Let \( (C_{\lambda_1}, J_{\lambda_1}) \in \mathbb{R}^{n_1 \times n_1}, (J_{\lambda_1}, J_{\lambda_1}) \in \mathbb{R}^{n_1 \times n_1} \) be a matrix pair, where \( J_{\lambda_1} \) is in Jordan form, corresponding to the zero of \( A(\sigma) \) of multiplicity \( n_1 \). This means that \( J_{\lambda_1} \) consists of Jordan blocks with sizes equal to the partial multiplicities of \( \lambda_1 \). This is called a finite eigenpair of \( A(\sigma) \) (or a finite Jordan pair) corresponding to \( \lambda_1 \) iff

\[
\text{rankcol} \left( C_{\lambda_1}, J_{\lambda_1} \right)^{n_1-1}_{k=0} = n_1 = \sum_{k=0}^{q} A_k C_{\lambda_1}, J_{\lambda_1} = 0
\]

Taking an eigenpair for each finite eigenvalue \( \lambda_i \) of \( A(\sigma) \), we can create the finite spectral pair of \( A(\sigma) \) i.e. \( C_F \in \mathbb{R}^{r \times n}, J_F \in \mathbb{R}^{n \times n} \), where

\[
C_F = [C_1, C_2, \ldots, C_r] \quad J_F = \text{blockdiag} \left[ J_1, J_2, \ldots, J_s \right]
\]

The forward solution space \( B_F \) of (1) is created by the column span of the matrix

\[
\Psi = C_F(J_F)^k
\]

and \( \dim B_F = n \). The finite spectral pair (7) satisfies similar properties i.e.

\[
\text{rankcol} \left( C_F J_F \right)^{n-1}_{k=0} = n = \sum_{k=0}^{q} A_k C_F J_F = 0
\]

#### B. Infinite Jordan Pairs and Backward Solution Space

An eigenpair of the dual matrix \( \hat{A}(\sigma) \) corresponding to the eigenvalue \( \hat{\lambda} = 0 \) is called an infinite eigenpair of \( A(\sigma) \) (or an infinite Jordan pair). Taking an eigenpair for each finite zero \( \hat{\lambda} = 0 \) of \( \hat{A}(\sigma) \) we create the infinite spectral pair of \( A(\sigma) \)

\[
C_\infty = [C_{\infty 1}, \ldots, C_{\infty r}] \quad J_\infty = \text{blockdiag} \left[ J_{\infty 1}, \ldots, J_{\infty r} \right]
\]

that satisfies the following

\[
\text{rankcol} \left( C_{\infty} J_\infty \right)^{n-1}_{k=0} = \mu = \sum_{k=0}^{q} A_k C_{\infty} J_\infty q^{-k} = 0
\]

The backward solution space \( B_B \) of (1) is created by the column span of the matrix

\[
\Psi_B = C_{\infty}(J_\infty)^{N-k}
\]

and \( \dim B_B = \mu \). Later we shall provide an analytic formula for the backward solution space.
IV. CONSTRUCTION OF A SYSTEM WITH GIVEN FORWARD BEHAVIOR

Suppose that a finite number of vector valued functions \( \beta_{i,q}(k) \in \mathbb{N} \rightarrow \mathbb{R}^r \), of the form

\[
\beta_i(k) = \lambda_i^k \beta_{i,q_i-1} + k \lambda_i^{k-1} \beta_{i,q_i-2} + \ldots + \left( \frac{k}{k-q_i} \right) \lambda_i^{k-q_i-1} \beta_{i,0}
\]

for \( \lambda_i \neq 0, i = 1, 2, \ldots, l \) or

\[
\beta_i(k) = \delta(k)x_{i,q_i-1} + \ldots + \delta(k-(q_i-1))x_{i,0}
\]

for \( \lambda_i = 0 \), are given. Create the matrices

\[
C_i := \begin{bmatrix} \beta_{i,0} & \beta_{i,1} & \cdots & \beta_{i,q_i-2} & \beta_{i,q_i-1} \end{bmatrix} \in \mathbb{R}^{r \times q_i}
\]

\[
J_i := \begin{bmatrix} \lambda_i & 1 & \cdots & 0 \\
0 & \lambda_i & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 \\
0 & \cdots & 0 & \lambda_i 
\end{bmatrix} \in \mathbb{R}^{q_i \times q_i}
\]

and combine these matrices into

\[
C := [C_1 \quad C_2 \quad \cdots \quad C_{l-1} \quad C_l] \in \mathbb{R}^{r \times n}
\]

\[
J = \text{blockdiag} \begin{bmatrix} J_1 & J_2 & \cdots & J_l \end{bmatrix} \in \mathbb{R}^{n \times n}
\]

where \( n := \sum_{i=1}^{l} q_i \). Then we have the following Theorem.

**Theorem 5:** [1] Let a be a complex number other than \( \lambda_i \) and define

\[
A(\sigma) = I_r - C(\sigma - A_J)^{-q} \times \left\{ (\sigma - a) V_q + (\sigma - a)^2 V_{q-1} + \ldots + (\sigma - a)^q V_1 \right\}
\]

where \( q = \text{ind}(C,J) \) i.e. the least integer such that the matrix \( S_q = \text{col}(CJ^{w+1}) \) has full column rank, and \( [V_1 \quad V_2 \quad \cdots \quad V_q] \) is the inverse of \( S_{1-q} = \text{col}(C(J - a I_n)^{-w})^{q-1} \). Then \( \beta_i(k) \) are solutions of (1). Furthermore, \( q \) is the minimal possible lag of any polynomial matrix with this property.

Note that \( A(\sigma) \) is not the only polynomial matrix satisfying the given behavior. Any other unimodal equivalent matrix \( \tilde{A}(\sigma) \) such that \( \tilde{A}(\sigma) = U(\sigma)A(\sigma) \) with \( \det U(\sigma) = c \in \mathbb{R} \) gives a solution to our problem. We need to mention here is that although we have created an Auto-Regressive representation for the given forward solution space, if equation \( n + \mu = r \times q \) is not satisfied for \( \mu = 0 \), then the above algorithm will give rise to an AR-representation which will include an extra forward/backward behavior due to the presence of a non-zero \( \mu \).

Now we will provide a theorem for the backward solution space.

**Theorem 6:** [8] Let \( \tilde{U}_R(\sigma) = (\tilde{u}_1(\sigma) \quad \cdots \quad \tilde{u}_r(\sigma)) \) and \( \tilde{u}_j^{(i)}(\sigma), \tilde{A}(\sigma)^{(i)} \) be the \( i \)-th derivatives of \( \tilde{u}_j(\sigma) \) and \( \tilde{A}(\sigma) \) for \( i = 0, 1, \ldots, q_j - 1 \) and \( j = 1, 2, \ldots, r \). Define

\[
x_{j,i} = \frac{1}{i!} \tilde{u}_j^{(i)}(\sigma)
\]

Then, the discrete time vector value functions

\[
\beta_j(k) = x_{j,0} \beta_j^{(0)} + \ldots + x_{j,q_j-1} \beta_j^{(q_j-1)}
\]

for \( j = 2, \ldots, r \) are linearly independent backward solutions of \( A(\sigma)\beta(k) = 0 \).

**Lemma 7:** [8] The vectors \( x_{j,i} \), defined in (8) for \( i = 0, 1, \ldots, q_j - 1 \) and \( j = k + 1, \ldots, r \) form Jordan Chains for \( \tilde{A}(\sigma) \) corresponding to the eigenvalue \( \lambda_i = 0 \) with lengths \( \mu_j \). Thus, they satisfy the following system of equations:

\[
\begin{pmatrix}
A_q & 0 & \cdots & 0 \\
\vdots & A_q & \cdots & 0 \\
A_0 & \vdots & \ddots & \vdots \\
0 & A_0 & \ddots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & A_q
\end{pmatrix}
\begin{pmatrix}
x_{j,0} \\
x_{j,1} \\
x_{j,q_j-1} \\
\vdots \\
x_{j,q_j}
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
\vdots \\
0
\end{pmatrix}
\]

(9)

According to [10], the above system (9) corresponds only to the infinite zero elementary divisors (I.Z.E.D.) i.e. \( \mu_j = q + \tilde{q}_j \), and not to the infinite pole elementary divisors (I.P.E.D.) i.e. \( \mu_j = q - q_j \) for \( j = 2, 3, \ldots, r \).

**Theorem 8:** The vector

\[
\beta_j(k) = x_{j,q_j-1} \delta(N-k) + \ldots + x_{j,0} \delta(N-k-q_j) \quad (q_j)
\]

is a solution of the AR-representation (1) iff the vector

\[
\tilde{\beta}_j(k) = x_{j,0} \delta(k-q_j) + \ldots + x_{j,q_j-1} \delta(k-q_j) \quad (q_j)
\]

is a solution of the dual system (4) i.e. \( \tilde{A}(\sigma)\tilde{\beta}(k) = 0 \).

**Proof:** (\( \Rightarrow \)) First we will prove that if \( \beta_j(k) \) is a solution of (1) then \( \tilde{\beta}_j(k) \) is a solution of (4). Define the following matrices

\[
A_j^T = \begin{pmatrix} A_{i+1} & A_i & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
A_j & A_{j-1} & \cdots & A_i \end{pmatrix}
\]

Taking Z-transforms on the dual system we have that

\[
Z \left\{ \tilde{A}(\sigma)\tilde{\beta}(z) \right\} = 0 \Rightarrow \tilde{A}(z)\tilde{\beta}(z) = \tilde{\beta}_m
\]

(10)

where \( \tilde{\beta}(z) := \{ \tilde{\beta}_j(k) \} \) and

\[
\tilde{\beta}_m := (z^qI_r \quad \cdots \quad zI_q) A_j^{-1} \left( \tilde{\beta}(0) \quad \tilde{\beta}(1) \quad \cdots \quad \tilde{\beta}(q_j-1) \right)^T
\]

By substituting

\[
\tilde{\beta}(z) = Z \left\{ \tilde{\beta}_j(k) \right\} = x_{j,0} z^{q_j-1} + x_{j,1} z^{q_j-2} + \ldots + x_{j,q_j-1} \quad (q_j)
\]

in place of \( \tilde{\beta}(z) \) in (10) we get

\[
\tilde{A}(z)\tilde{\beta}(z) = (z^qA_0 + z^{q-1}A_1 + \cdots + zA_{q_j-1} + A_q) \times
\]
\[\begin{align*}
\times (x_{j,0}z^{-q_0}q_1 + x_{j,1}z^{-q_0}q_1 + \cdots + x_{j,q}z^{-q_0}q_1 - 1) = \\
&= \left(z^{-q_0}I_r - z^{-q_0}I_r \cdots - z^{-q_0}I_r\right) A_0 \cdots 0 \cdots A_q \\
&\times A_q \cdots A_q ) x_{j, q} z^{-q_0}q_1 - 1 = \\
&= \left(z^{-q_0}I_r - z^{-q_0}I_r \cdots - z^{-q_0}I_r\right) A_0^{-1} x_{j, q} z^{-q_0}q_1 - 1 + \\
&\left(I_r \cdots - z^{-q_0}I_r \cdots - z^{-q_0}I_r\right) Q^T x_{j,0} z^{-q_0}q_1 - 1 = \\
&= \left(z^{-q_0}I_r - z^{-q_0}I_r \cdots - z^{-q_0}I_r\right) A_0^{-1} x_{j, q} z^{-q_0}q_1 - 1 + \\
&\left( z^{-q_0}I_r \cdots - z^{-q_0}I_r \cdots - z^{-q_0}I_r\right) Q^T x_{j, q} z^{-q_0}q_1 - 1 = \\
&= \left(z^{-q_0}I_r \cdots - z^{-q_0}I_r\right) A_0^{-1} x_{j, q} z^{-q_0}q_1 - 1 \\
\end{align*}\]

We can easily check that (10) and (11) coincide in case where:
\[
\begin{pmatrix}
\tilde{\beta}_j(0) \\
\vdots \\
\tilde{\beta}_j(q - 1)
\end{pmatrix} = \begin{pmatrix} x_{j, q} z^{-q_0}q_1 - 1 \\
\vdots \\
x_{j, q_1}
\end{pmatrix}
\]

Therefore \(\tilde{\beta}_j(k)\) is a solution of (4) for the above initial values.

\((\Leftarrow)\) Now we will show the opposite, that if \(\tilde{\beta}_j(k)\) is a solution of (4), then \(\beta_j(k)\) is a solution of (1).

Since \(\tilde{\beta}_j(k)\) is a solution of (4), we shall have that
\[
\tilde{A}(z)\tilde{\beta}_j(z) = \left(z^q A_0 + z^{q-1} A_1 + \cdots z A_{q-1} + A_q \right) \times \begin{pmatrix} x_{j,0} z^{-q_0}q_1 + x_{j,1}z^{-q_0}q_1 + \cdots + x_{j,q}z^{-q_0}q_1 - 1 \end{pmatrix} = \\
\begin{pmatrix} z^{-q_0}I_r \cdots - z^{-q_0}I_r \cdots - z^{-q_0}I_r\right) A_0^{-1} (\tilde{\beta}_j(0) \cdots \tilde{\beta}_j(q - 1))^T
\]

and therefore the strictly proper part in (11) i.e
\[
\left( z^{-q_0}I_r \cdots - z^{-q_0}I_r \cdots - z^{-q_0}I_r\right) Q x_{j, q} z^{-q_0}q_1 - 1
\]

must be equal to zero or otherwise the condition (9) must be satisfied. However, (9) guarantees that \(\beta_j(k)\) is a solution of (1).

The previous theorem proves that the problem of finding an AR-representation of the form (1) with the following backward behavior:
\[
\beta_j(k) = x_{j,q} - q_0q_1 - 1 \delta(N-k) + \cdots + x_{j,0} \delta(N-k-q_0q_1 + 1)
\]
is equivalent to finding an AR-representation of the form (4) satisfying the forward behavior
\[
\tilde{\beta}_j(k) = x_{j,q} \delta(k - q - q_0q_1 + 1) + \cdots + x_{j,q_1} \delta(k - q) + \\
\cdots + x_{j,q} \cdot q_0q_1 - 1 \delta(k)
\]

However this problem can easily be solved from the results that we already presented, using Gohberg’s formula from [1] and Theorem 5. This is showcased in the following theorem.

**Theorem 9:** Let the following \(l\) vector valued functions
\[
\beta_j(k) = \sum_{w=0}^{q_0q_1 - 1} x_{j,w} \delta(N - w - k)
\]
where \(x_{j,w} \in C^*, \ 1 \leq j \leq l\). Define
\[
\begin{pmatrix} 0 & 1 & \cdots & 0 \\
0 & 0 & \cdots & 1 \\
0 & 0 & \cdots & 0
\end{pmatrix} \in \mathbb{R}^{(q_0q_1) \times \mu_j}
\]

\(J_j = \begin{pmatrix} 0 & 1 & \cdots & 0 \\
0 & 0 & \cdots & 1 \\
0 & 0 & \cdots & 0
\end{pmatrix} \in \mathbb{R}^{(q_0q_1) \times \mu_j}
\]

where \(j = 1, 2, \ldots, l\) and \(\mu_j = q_0 + q_1\). Let
\[
C = (C_1 \ C_2 \ \cdots \ C_l) \in \mathbb{R}^{r \times \mu}
\]

\(J = \text{blockdiag} (J_1, J_2, \ldots, J_l) \in \mathbb{R}^{r \times \mu}
\]

with \(\mu = \sum_{j=1}^{l} \mu_j\). Let \(a \neq 0\) be a complex number and define
\[
A(\sigma) = I_r - C(J-aI_r)^{-q} \{(\sigma-a)V_q + \cdots + (\sigma-a)^q V_1\}
\]
where \(q = \text{ind}(C, J)\) is the lowest integer such that the matrix \(S_q - 1 = \text{col} \left(C J^q \right)_{i=0}^{q-1}\) has full column rank and \(V = (V_1 \ V_2 \ \cdots \ V_q)\) is the left inverse of \(S_{q-1} = \text{col} \left(C(J-aI_r)^{-q} V_q \right)_{i=0}^{q-1}\). Then \(\beta_j(k)\) is a solution of (1), where \(A(\sigma) = \sigma^q A(\frac{1}{\sigma})\). Furthermore, \(q\) is the minimal possible degree of any \(r \times r\) matrix polynomial with this property.

**V. CONSTRUCTION OF A SYSTEM WITH GIVEN FORWARD AND BACKWARD BEHAVIOR**

In this section, we will combine all the results presented in previous sections, in order to create an algorithm for constructing a system that satisfies both a given forward and a given backward behavior. We have already presented methods for constructing a system with given either a forward or a backward behavior. In the last case, we have constructed a dual polynomial matrix \(\tilde{A}(\sigma)\) with a forward behavior resulting from a given backward behavior. Our aim, is to work in a similar way for the forward behavior as well i.e. find a forward behavior of the dual system (4) that corresponds to a forward behavior of (1). If we succeed this, then the problem of finding a system with given forward-backward behavior is reduced to the problem of finding its dual system which exhibits a certain forward behavior.

**Theorem 10:** Let \(A(\sigma)\) defined in (3) with \(\text{rank}_{R(\sigma)} A(\sigma) = r\). If \(\beta_j(k) = C_j J_j^k x_0\) is a solution of (1), where \((C_j \in \mathbb{R}^{r \times n_j}; J_j \in \mathbb{R}^{n_j \times n_j})\) is a finite Jordan pair corresponding to the zero \(\lambda_j \neq 0\) of \(A(\sigma)\), then \(\tilde{\beta}_j(k) = C_j J_j^{-1} \left(J_j^k\right)^{-1} x_0\) is a solution of (4).

**Proof:** Since \(\beta_j(k) = C_j J_j^k x_0\) is a solution of (1), we have
\[
\begin{pmatrix}
\beta(0) \\
\vdots \\
\beta(q - 1)
\end{pmatrix} = \begin{pmatrix} C_j \\
\vdots \\
C_j J_j^{q - 1}
\end{pmatrix} x_0
\]
Taking $Z$-transform of $A(\sigma)\beta(k) = 0$ we have:

$$A(z)C_j(zI_n_j - J_j)^{-1} = (zq_{r_1} z^{q-1}{r_1} \cdots zI_r) \cdot \cdot A^1_j(C_j C_j J_j \cdots C_j J_j^{-1})^T x_0$$

Multiplying both sides by $z^q$ and replacing $z$ with $\frac{1}{z}$ we get

$$\tilde{A}(z)C_j\left(\frac{1}{z}I_n_j - J_j\right)^{-1} = (I_r \cdots z^{q-1}I_r) \cdot \cdot A^1_j(C_j C_j J_j \cdots C_j J_j^{-1})^T x_0$$

$$\Rightarrow \tilde{A}(z)C_j(I_n_j - zJ_j)^{-1} = (z^{q-1}I_r \cdots I_r) \cdot \cdot A^f_j(C_j C_j J_j \cdots C_j J_j^{-1})^T x_0$$

$$\Rightarrow \tilde{A}(z)C_j(J_j^{-1} - z) J_j^{-1} = (z^{q-1}I_r \cdots I_r) \cdot \cdot A^f_j(C_j C_j J_j \cdots C_j J_j^{-1})^T x_0$$

$$\Rightarrow -\tilde{A}(z)C_j J_j^{-1}(zI_n_j - J_j)^{-1} = (z^{q-1}I_r \cdots I_r) \cdot \cdot A^f_j(C_j C_j J_j \cdots C_j J_j^{-q})^T J_j^{-1} x_0 \quad \text{(11)}$$

where

$$A^f = \begin{pmatrix} A_1 & \cdots & A_q \\ \vdots & \ddots & \vdots \\ A_q & \cdots & 0 \end{pmatrix}$$

Since $(C_j, J_j)$ is a finite Jordan pair of $A(\sigma)$, it satisfies:

$$A_0 C_j + A_1 C_j J_j + \cdots + A_q C_j J_j^q = 0$$

Premultiplying (11) by $zI_n_j$ we get

$$-\tilde{A}(z)C_j J_j^{-1}(zI_n_j)(zI_n_j - J_j)^{-1} =$$

$$= - \left( z^{q}I_r \cdots zI_r \right) A^f_j^{-1} (C_j J_j^{-1} \cdots C_j J_j^{-q})^T x_0$$

Therefore, by taking the inverse $Z$-transform, we conclude that

$$\tilde{\beta}(k) = Z^{-1}\left\{ \tilde{\beta}(z) \right\} =$$

$$= Z^{-1}\left\{ C_j J_j^{-1}(zI_n_j)(zI_n_j - J_j)^{-1} x_0 \right\} =$$

$$= C_j J_j^{-1}(J_j^{-1})^k x_0$$

is a solution of (4) for initial conditions

$$\begin{pmatrix} \tilde{\beta}(0) \\ \vdots \\ \tilde{\beta}(q-1) \end{pmatrix} = \begin{pmatrix} C_j J_j^{-1} \\ \vdots \\ C_j J_j^{-q} \end{pmatrix} x_0$$

In the above theorem, we managed to match a given solution of (1) to a solution of (4), a result that we will utilise in the following.

Since the matrix $J_j^{-1}$ is not in Jordan form, we can find a nonsingular constant matrix $U \in \mathbb{R}^{n_j \times n_j}$ such that $J_j^{-1} = U J_j U^{-1}$ where $J_j$ is in Jordan form. With this change, the solution of $	ilde{A}(\sigma)\tilde{\beta}(k) = 0$ can also be written as follows:

$$\tilde{\beta}(k) = C_j J_j^{-1}(J_j^{-1})^k x_0 = C_j U J_j U^{-1}(J_j^{-1})^k x_0 =$$

$$= \tilde{C}_j(J_j)^k (U^{-1} x_0)$$

where

$$\tilde{C}_j = C_j U J_j$$

So we can see that instead of using the matrix pair $(C_j J_j^{-1} \in \mathbb{R}^{n_j \times n_j}, J_j^{-1} \in \mathbb{R}^{n_j \times n_j})$ where the matrix $J_j^{-1}$ is not in Jordan form, we can use the matrix pair

$$(\tilde{C}_j = C_j U J_j \in \mathbb{R}^{n_j \times n_j}; \tilde{J}_j = U^{-1} J_j^{-1} U \in \mathbb{R}^{n_j \times n_j})$$

Summarizing our results, in order to construct an AR-representation for a certain forward-backward behavior, one must follow the next algorithm.

**Algorithm 11:** Construction of an AR-Representation with given forward and backward behavior.

Step 1 Transform the finite Jordan pairs

$$(C_j \in \mathbb{R}^{r \times n_j}; J_j \in \mathbb{R}^{n_j \times n_j})$$

that correspond to solutions of the form $\tilde{\beta}(k) = C_j J_j^k x_0$ to the finite Jordan Pairs

$$(\tilde{C}_j = C_j U J_j \in \mathbb{R}^{n_j \times n_j}; \tilde{J}_j = U^{-1} J_j^{-1} U \in \mathbb{R}^{n_j \times n_j})$$

that correspond to solutions of the form $\tilde{\beta}(k) = \tilde{C}_j (J_j)^k (U^{-1} x_0)$ of the dual system that we are looking for.

Step 2 Transform the infinite Jordan pairs to finite Jordan pairs of the dual system using Theorem 10.

Step 3 Construct the polynomial matrix $\tilde{A}(\sigma)$ using the method presented in Theorem 5.

Step 4 Get the polynomial matrix $A(\sigma) = \sigma^q \tilde{A}(\frac{1}{\sigma})$ and thus the AR-representation (1) that we are looking for.

As we have already mentioned, $A(\sigma)$ is not the only polynomial matrix satisfying the given behavior. Any other unimodular equivalent matrix $\tilde{A}(s)$ such that $\tilde{A}(s) = U(\sigma)\tilde{A}(\sigma)$ with $\det U = c \in \mathbb{R}$ gives rise to the dual polynomial matrix of the solution that we are looking for.

**Definition 12:** [13], [14] $A_1(\sigma), A_2(\sigma) \in \mathbb{R}[\sigma]^{r \times m}$ are said to be divisor equivalent if there exist polynomial matrices $M(s), N(s)$ such that:

$$\begin{pmatrix} M(s) \\ A_2(s) \end{pmatrix} \begin{pmatrix} A_1(s) \\ -N(s) \end{pmatrix} = 0$$

where the matrices $(M(s) \ A_2(s))$ and $(A_1(s) \ -N(s))$:

1) have full rank over $\mathbb{R}[s]$ and no finite elementary divisors,

2) have no infinite elementary divisors.

Divisor Equivalence is a very important form of equivalence, since it has the property of preserving both the finite and the infinite elementary structure of a polynomial matrix. Note, that all the divisor equivalent polynomial matrices...
of $A(\sigma)$ that have been constructed in Algorithm 11 are solutions to our problem.

The above algorithm may display computational difficulties in the case where the forward behavior has polynomial vector valued functions (since these are connected with the finite elementary divisors of $A(\sigma)$ at zero). In that case, the Jordan matrix $J$, will have zero determinant and thus, will not be invertible. This problem can easily be surpassed by replacing $\sigma$ with $\sigma + b$ in $A(\sigma)$ i.e. moving all possible zeros of $A(\sigma)$ to non-zero places. The following lemma indicates this solution.

**Lemma 13:** [9] If $(C\ J)$ is a finite Jordan Pair of $A(\sigma)$ then $(C\ J + bI_n)$ is a finite Jordan Pair of $A(\sigma - b)$.

If $(C_\infty\ J_\infty)$ is an infinite Jordan Pair of $A(\sigma)$ then $(C_\infty\ J_\infty(I_n + bJ_\infty)^{-1})$ is an infinite Jordan Pair of $A(\sigma - b)$.

**Example 14:** We are interested to construct an AR-representation of the form (1) with the following forward and backward behavior:

$$\beta_1(k) = \begin{pmatrix} \frac{1}{\beta_{1,1}} \end{pmatrix} 2^k + \begin{pmatrix} \frac{2}{\beta_{1,0}} \end{pmatrix} k 2^{k-1}$$

$$\beta_2(k) = \begin{pmatrix} \frac{1}{\beta_{1,1}} \end{pmatrix} 2^k + \begin{pmatrix} \frac{2}{\beta_{1,0}} \end{pmatrix} k 2^{k-1}$$

First, define the matrix pairs

$$C_1 = \begin{pmatrix} \beta_{1,0} & \beta_{1,1} \end{pmatrix} = \begin{pmatrix} \frac{2}{\beta_{1,0}} & 1 \\ 0 & \frac{1}{\beta_{1,1}} \end{pmatrix};\ J_1 = \begin{pmatrix} \frac{2}{\beta_{1,0}} & 1 \\ 0 & \frac{1}{\beta_{1,1}} \end{pmatrix}$$

$$J_1^{-1} = \begin{pmatrix} \frac{1}{\beta_{1,0}} & \frac{1}{\beta_{1,1}} \\ 0 & \frac{1}{\beta_{1,0}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\beta_{1,0}} & \frac{1}{\beta_{1,1}} \\ 0 & \frac{1}{\beta_{1,0}} \end{pmatrix}$$

$$\tilde{C}_1 = C_1 U \tilde{J}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$C_2 = \begin{pmatrix} x_{1,0} & x_{1,1} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{\beta_{1,0}} \\ \frac{1}{\beta_{1,1}} & 0 \end{pmatrix}$$

$$J_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

The complete matrix pair is

$$C = \begin{pmatrix} \tilde{C}_1 & C_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 2 & 3 & -1 \end{pmatrix}$$

$$J = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\beta_{1,0}} & \frac{1}{\beta_{1,1}} & 0 & 0 \\ 0 & \frac{1}{\beta_{1,0}} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

For $q = 1$, the matrix $S_0 = C$ does not have full column rank. For $q = 2$, the matrix $S_1 = (C\ CJ)^T$ has full column rank. Let $a = 1$, so

$$\tilde{A}(\sigma) = I_2 - C(J - I_4)^{-2}\{ (s - 1)V_2 + (s - 1)^2V_1 \}$$

where

$$(V_1\ V_2) = \left( \begin{pmatrix} C & C(J - I_4)^{-1} \end{pmatrix} \right)^{-1}\tilde{A}(\sigma)$$

Finishing the computations, we conclude to

$$\tilde{A}(\sigma) = \begin{pmatrix} \frac{1}{3} (6 - 23\sigma + 22\sigma^2) \\ \frac{2}{3} (1 - 3\sigma + 2\sigma^2) \end{pmatrix}$$

Therefore, the matrix we are looking for is

$$A(\sigma) = \sigma^2 \tilde{A}(\sigma) = \begin{pmatrix} \frac{1}{3} (6\sigma^2 - 23\sigma + 22) \\ \frac{2}{3} (\sigma - 1) \end{pmatrix}$$

As a matter of fact, the above matrix pairs constitute finite and infinite Jordan Pairs for $A(\sigma)$ since they satisfy

$$A_2C_1J_1^2 + A_1C_1J_1 + A_0C_1 = 0 \quad \text{rank}(C_1C_1J_1) = 2$$

$$A_0C_2J_2^2 + A_1C_2J_2 + A_2C_2 = 0 \quad \text{rank}(C_2C_2J_2) = 2$$

VI. CONCLUSIONS

We have proposed an algorithm for constructing a regular AR-representation which satisfies a given forward and backward behavior. Our aim is to extend this theory for the case non-regular AR-representations, since we have noticed that for the given forward-backward behavior we can always achieve a non-regular AR-representation which satisfy this behavior.

**REFERENCES**


