Modeling of discrete time auto-regressive systems with given forward and backward behavior

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***Abstract*—We study the behavior of discrete time AR-representations. A theorem is provided connecting the backward behavior of a system, due to its infinite elementary divisors, with the forward behavior of its dual system. We first use this result to construct a system satisfying a certain backward behavior. In addition to this, we propose a way to combine this result with previous ones to create an algorithm for computing a system satisfying a given forward and backward behavior.**

I. Introduction

Let $\mathbb{R}$ be the field of reals, $\mathbb{R}[s]$ be the ring of polynomials, $\mathbb{R}[s]^{m\times n}$ be the set of $m\times n$ matrices with elements in $\mathbb{R}[s]$. Consider the system of algebraic and difference equations, or otherwise a discrete time (Auto-Regressive) AR-representation, of the form

$$A(\sigma)\beta(k)=0 \tag{1}$$

where $k=0,1,...,N-q$, or equivalently

$$A_q\beta(k+q)+A_{q-1}\beta(k+q-1)+...+A_0\beta(k)=0 \tag{2}$$

where σ denotes the forward shift operator i.e. $\sigma\beta(k)=\beta(k+1)$

$$A(\sigma)=A_q\sigma^q+A_{q-1}\sigma^{q-1}+...+A_1\sigma+A_0 \tag{3}$$

where $A(\sigma)\in\mathbb{R}^{r\times r}[\sigma]$ is a regular polynomial matrix i.e. $\det[A(\sigma)]\neq 0$ with $A_q\neq 0$ and $\beta(k)\in\mathbb{R}^r, k=0,1,...N$. The number q is often called the *lag* of the matrix. The solution space of such systems consists of both forward and backward solutions defined over a finite time i.e.

$$B:=\{\beta(k)\in\mathbb{R}^r, \text{satisfies (1), for k=0,1,...,N-q }\}$$

Forward solutions are defined in a sense that the initial conditions are given and $\beta(k)$ is to be determined in a forward fashion from its previous values. Backward solutions are defined in a sense that the final conditions are given and $\beta(k)$ is to be determined in a backward fashion from its future values.

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The solution of such systems has been previously studied by various authors in [1], [2], [3] and their algebraic structure in [4], [5], [6], [7]. A method for the construction of the forward (resp. backward) solution space of (2), based on the finite (resp. infinite) elementary divisor structure of $A(\sigma)$ has been presented in [1] (resp. [2]), whereas the results have been extended to the non-regular case by [8]. An interesting problem that we face in this paper is the inverse problem, that is: Given a certain forward/backward solution space, find a system of algebraic and difference equations or otherwise the polynomial matrix $A(\sigma)$ given in (1), that has exactly the prescribed solution space. This problem has initiallly been presented and solved by [1], but only for the case of the forward solution space. More specifically, [1] proposed a formula that connects the form of $A(\sigma)$ and a finite Jordan pair that can be constructed from the prescribed forward solution space. The main aim of this work is to extend the results presented in [1], for the case where except of the given forward behavior, a backward behavior is also given. In order to achieve this, we connect the backward solution space of (1) with the forward solution space of its dual system

$$A_0\beta(k+q)+A_1\beta(k+q-1)+...+A_q\beta(k)=0 \tag{4}$$

and use the results given in [1]. Otherwise, we use a methodology that was used in for the first time in [9] for the case of continuous time systems, where the finite and infinite elementary divisor structure of $A(\sigma)$ is connected with the smooth and impulsive behavior of (1) respectively.

II. Preliminaries

Theorem 1: [10] Let $A(\sigma)$ as in (3). There exist unimodular matrices $U_L(\sigma)\in\mathbb{R}[\sigma]^{r\times r}$,$U_R(\sigma)\in\mathbb{R}[\sigma]^{r\times r}$ i.e. $\det(U_L(\sigma)), \det(U_R(\sigma))\neq 0$, such that

$$\begin{aligned}S^{\mathbb{C}}_{A(\sigma)}(\sigma)&=U_L(\sigma)A(\sigma)U_R(\sigma)=\\&=blockdiag\,[1,...,1,f_z(\sigma),f_{z+1}(\sigma),...,f_r(\sigma)]\end{aligned} \tag{5}$$

with $1\le z\le r$ and $f_i(\sigma)/f_{i+1}(\sigma)$ $i=z,z+1,...,r$. $S^{\mathbb{C}}_{A(\sigma)}(\sigma)$ is called the *Smith form* of $A(\sigma)$ (in $\mathbb{C}$) where $f_i(\sigma)\in\mathbb{R}[\sigma]$ are the *invariant polynomials* of $A(\sigma)$. The zeros $\lambda_i\in\mathbb{C}$ of $f_j(\sigma)\in\mathbb{R}[\sigma]$, $j=z,z+1,...,r$ are called *finite zeros* of $A(\sigma)$. Assume that $A(\sigma)$ has l distinct zeros and the partial multiplicities of each zero $\lambda_i \in \mathbb{C}$, $i = 1, ..., l$ are $0 \le n_{i,z} \le n_{i,z+1} \le ... \le n_{i,r}$ i.e.

$$f_j(\sigma) = (\sigma - \lambda_i)^{n_{i,j}} \hat{f}_j(s)$$

$j = z, z+1, ..., r$ with $\hat{f}_j(\lambda_i) \neq 0$ The terms $(\sigma - \lambda_i)^{n_{i,j}}$ are called *finite elementary divisors* of $A(\sigma)$ at $\sigma = \lambda_i$. We also denote by n the sum of the degrees of the finite elementary divisors of $A(\sigma)$, i.e. $n := \deg \left[\prod_{j=z}^{r} f_j(\sigma) \right] = \sum_{i=1}^{k} \sum_{j=z}^{r} n_{i,j}$. Similarly, we can find $U_L(\sigma) \in \mathbb{R}(\sigma)^{r \times r}$, $U_R(\sigma) \in \mathbb{R}(\sigma)^{r \times r}$ having no poles and zeros at $\sigma = \lambda_0$ such that

$$S_{A(\sigma)}^{\lambda_0}(\sigma) = U_L(\sigma) A(\sigma) U_R(\sigma) = \\ = blockdiag\left[1, ..., 1, (\sigma - \lambda_0)^{n_z}, ..., (\sigma - \lambda_0)^{n_r}\right]$$

$S_{A(\sigma)}^{\lambda_0}(\sigma)$ is called the Smith form of $A(\sigma)$ at the local point $\sigma = \lambda_0$.

Theorem 2: [10] Let $A(\sigma)$ defined in (3). There exist biproper matrices $U_L(\sigma) \in \mathbb{R}_{pr}^{r \times r}(s)$, $U_R(\sigma) \in \mathbb{R}_{pr}^{r \times r}(s)$ (i.e. having no poles or zeros at infinity) such that

$$U_L(\sigma) A(\sigma) U_R(\sigma) = S_{\mathrm{A}(\sigma)}^{\infty}(\sigma) =$$

$$= blockdiag \left(\overbrace{\underbrace{s^{q_1}, s^{q_2}, ..., s^{q_k}}_{k}, I_{u-k}}^{u}, \overbrace{\frac{1}{s^{\hat{q}_{u+1}}}, \frac{1}{s^{\hat{q}_{u+2}}}, ..., \frac{1}{s^{\hat{q}_r}}}^{r-u} \right)$$

with $1 \le u \le r$, $q_1 \ge \ldots \ge q_k > 0$ and $\hat{q}_r \ge \hat{q}_{r-1} \ge \ldots \hat{q}_{u+1} > 0$.

$S_{\mathrm{A}(\sigma)}^{\infty}(\sigma)$ is called the *Smith form of* $A(\sigma)$ *at infinity*. The first k terms $q_1, ..., q_k$ (resp. the latter $(r - u)$ terms $\hat{q}_{u+1}, ..., \hat{q}_r$) are the orders of the *poles* (resp. *zeros*) at $\sigma = \infty$ of $A(\sigma)$. It is proved in [10] that $q_1 = q$.

Define the dual polynomial matrix of $A(\sigma)$ as

$$\tilde{\mathrm{A}}(\sigma) := \sigma^q A(\frac{1}{\sigma}) = A_0 \sigma^q + A_1 \sigma^{q-1} + ... + A_q \tag{6}$$

Theorem 3: [10] Let $\tilde{A}(\sigma)$ defined in (6). There exist matrices $\tilde{U}_L(\sigma) \in \mathbb{R}[\sigma]^{r \times r}$, $\tilde{U}_R(\sigma) \in \mathbb{R}[\sigma]^{r \times r}$ having no poles or zeros at $s = 0$, such that

$$S_{\tilde{A}(\sigma)}^{0}(\sigma) = \tilde{U}_L(\sigma) \tilde{A}(\sigma) \tilde{U}_R(\sigma) = blockdiag\left[\sigma^{\mu_1}, \ldots, \sigma^{\mu_r}\right]$$

$S_{\tilde{A}(\sigma)}^{0}(\sigma)$ is the local Smith form of $\tilde{A}(\sigma)$ at $s = 0$. The terms σ^{μ_j} are the finite elementary divisors of $\tilde{A}(\sigma)$ at zero and are called the *infinite elementary divisors of* $A(\sigma)$.

The connection between the Smith form at infinity of $A(\sigma)$ and the Smith form at zero of the dual matrix is given in [11] and [10] :

$$S_{\tilde{A}(\sigma)}^{0}(\sigma) = \sigma^q S_{A(\sigma)}^{\infty}(\tfrac{1}{\sigma}) = \\ = blockdiag\left[1, \sigma^{q-q_2}, \ldots, \sigma^{q-q_k}, \sigma^{q+\hat{q}_{k+1}}, \ldots, \sigma^{q+\hat{q}_r}\right]$$

Therefore, the orders of the infinite elementary divisors are given by

$$\begin{aligned} \mu_1 &= q - q_1 \overset{q=q_1}{=} 0 \\ \mu_j &= q - q_j \qquad j = 2, 3, ..., u \\ \mu_j &= q + \hat{q}_j \qquad j = u+1, ..., r \end{aligned}$$

We denote by μ the sum of the degrees of the infinite elementary divisors of $A(\sigma)$ i.e.

$$\mu := \sum_{j=1}^{r} \mu_j$$

Lemma 4: [12] Let $A(\sigma) = A_q \sigma^q + A_{q-1} \sigma^{q-1} + ... + A_0 \in \mathbb{R}^{r \times r}[\sigma]$. Let also n, μ be the sum of degrees of the finite and infinite elementary divisors of $A(\sigma)$, as defined previously. Then

$$n + \mu = r \times q$$

III. Jordan Pairs and Solutions of $A(\sigma)\beta(k) = 0$

A. Finite Jordan Pairs and Forward Solution Space

Let $(C_{\lambda_i} \in \mathbb{R}^{r \times n_i}, J_{\lambda_i} \in \mathbb{R}^{n_i \times n_i})$ be a matrix pair, where J_{λ_i} is in Jordan form, corresponding to the zero of $A(\sigma)$ of multiplicity n_i. This means that J_{λ_i}consists of Jordan blocks with sizes equal to the partial multiplicities of λ_i. This is called a *finite eigenpair* of $A(\sigma)$ (or a finite Jordan pair) corresponding to λ_i iff

$$rankcol \left(C_{\lambda_i} {J_{\lambda_i}}^k \right)_{k=0}^{n_i - 1} = n_i \qquad \sum_{k=0}^{q} A_k C_{\lambda_i} {J_{\lambda_i}}^k = 0$$

Taking an eigenpair for each finite eigenvalue λ_i of $A(\sigma)$, we can create the *finite spectral pair* of $A(\sigma)$ i.e. $C_F \in \mathbb{R}^{r \times n}$, $J_F \in \mathbb{R}^{n \times n}$, where

$$C_F = [C_1, C_2, ..., C_l] \qquad J_F = blockdiag\left[J_1, J_2, ..., J_l\right]$$

The forward solution space B_F of (1) is created by the column span of the matrix

$$\Psi = C_F (J_F)^k \tag{7}$$

and $\dim B_F = n$. The finite spectral pair (7) satisfies similar properties i.e.

$$rankcol \left(C_F {J_F}^k \right)_{k=0}^{n-1} = n \qquad \sum_{k=0}^{q} A_k C_F {J_F}^k = 0$$

B. Infinite Jordan Pairs and Backward Solution Space

An eigenpair of the dual matrix $\tilde{A}(\sigma)$ corresponding to the eigenvalue $\tilde{\lambda} = 0$ is called an *infinite eigenpair* of $A(\sigma)$ (or an infinite Jordan pair). Taking an eigenpair for each finite zero $\tilde{\lambda} = 0$ of $\tilde{A}(\sigma)$ we create the *infinite spectral pair* of $A(\sigma)$

$$C_\infty = [C_{\infty 1}, ..., C_{\infty r}] \qquad J_\infty = blockdiag\left[J_{\infty 1}, ..., J_{\infty r}\right]$$

that satisfies the following

$$rankcol \left(C_\infty {J_\infty}^k \right)_{k=0}^{\mu - 1} = \mu \qquad \sum_{k=0}^{q} A_k C_\infty {J_\infty}^{q-k} = 0$$

The backward solution space B_B of (1) is created by the column span of the matrix

$$\Psi_B = C_\infty (J_\infty)^{N-k}$$

and $\dim B_B = \mu$. Later we shall provide an analytic formula for the backward solution space.

IV. Construction of a System with Given Forward Behavior

Suppose that a finite number of vector valued functions $\beta_{i,q_i}(k)$: $\mathbb{N} \to \mathbb{R}^r$, of the form

$$\beta_i(k) := \lambda_i^k \beta_{i,q_i-1} + k\lambda_i^{k-1}\beta_{i,q_i-2} + \ldots + \binom{k}{k-q_i}\lambda_i^{k-q_i-1}\beta_{i,0}$$

for $\lambda_i \neq 0$, $i = 1, 2, ..., l$ or

$$\beta_i(k) = \delta(k)x_{i,q_i-1} + \ldots + \delta(k-(q_i-1))\,x_{i,0}$$

for $\lambda_i = 0$, are given. Create the matrices

$$C_i := \begin{bmatrix}\beta_{i,0} & \beta_{i,1} & \cdots & \beta_{i,q_i-2} & \beta_{i,q_i-1}\end{bmatrix} \in \mathbb{R}^{r\times q_i}$$

$$J_i := \begin{bmatrix}\lambda_i & 1 & \cdots & 0 \\ 0 & \lambda_i & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \lambda_i\end{bmatrix} \in \mathbb{R}^{q_i\times q_i}$$

and combine these matrices into

$$C := \begin{bmatrix}C_1 & C_2 & \cdots & C_{l-1} & C_l\end{bmatrix} \in \mathbb{R}^{r\times n}$$

$$J = blockdiag\begin{bmatrix}J_1 & J_2 & \cdots & J_l\end{bmatrix} \in \mathbb{R}^{n\times n}$$

where $n := \sum_{i=1}^{l} q_i$. Then we have the following Theorem.

Theorem 5: [1] Let a be a complex number other than λ_i and define

$$A(\sigma) = I_r - C(J - aI_n)^{-q} \times$$
$$\times \left\{(\sigma-a)V_q + (\sigma-a)^2 V_{q-1} + \ldots + (\sigma-a)^q V_1\right\}$$

where $q = ind(C, J)$ i.e. the least integer such that the matrix $S_{q-1} = col\,(CJ^w)_{w=0}^{q-1}$ has full column rank, and $\begin{bmatrix}V_1 & V_2 & \cdots & V_q\end{bmatrix}$ is the inverse of $S_{1-q} = col\left[C(J-aI_n)^{-w}\right]_{w=0}^{q-1}$. Then $\beta_i(k)$ are solutions of (1). Furthermore, q is the minimal possible lag of any polynomial matrix with this property.

Note that $A(\sigma)$ is not the only polynomial matrix satisfying the given behavior. Any other unimodular equivalent matrix $\hat{A}(\sigma)$ such that $\hat{A}(\sigma) = U(\sigma)A(\sigma)$ with $\det U(\sigma) = c \in \mathbb{R}$ gives a solution to our problem. What we need to mention here is that although we have created an Auto-Regressive representation for the given forward solution space, if equation $n + \mu = r \times q$ is not satisfied for $\mu = 0$, then the above algorithm will give rise to an AR-representation which will include an extra forward/backward behavior due to the presence of a non-zero μ.

Now we will provide a theorem for the backward solution space.

Theorem 6: [8] Let $\tilde{U}_R(\sigma) = \begin{pmatrix}\tilde{u}_1(\sigma) & \cdots & \tilde{u}_r(\sigma)\end{pmatrix}$ and $\tilde{u}_j^{(i)}(\sigma)$, $\tilde{A}(\sigma)^{(i)}$ be the i-th derivatives of $\tilde{u}_j(\sigma)$ and $\tilde{A}(\sigma)$ for $i = 0, 1, ..., \mu_j - 1$ and $j = 1, 2, ...r$. Define

$$x_{j,i} = \frac{1}{i!}\tilde{u}_j^{(i)}(\sigma) \tag{8}$$

Then, the discrete time vector value functions

$$\beta_j(k) = x_{j,\mu_j-1}\delta(N-k) + \ldots + x_{j,0}\delta(N-k-\mu_j+1)$$

for $j = 2, ...r$ are linearly independent backward solutions of $A(\sigma)\beta(k) = 0$.

Lemma 7: [8] The vectors $x_{j,i}$, defined in (8) for $i = 0, 1, \ldots, \mu_j - 1$ and $j = k+1, ..., r$ form Jordan Chains for $\tilde{A}(\sigma)$ corresponding to the eigenvalue $\lambda_i = 0$ with lengths μ_j. Thus, they satisfy the following system of equations:

$$\underbrace{\begin{pmatrix}A_q & 0 & 0 & \cdots & \cdots & 0 \\ \vdots & A_q & 0 & \cdots & \cdots & 0 \\ A_0 & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & A_0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_0 & \cdots & A_q\end{pmatrix}}_{Q}\begin{pmatrix}x_{j,0} \\ x_{j,1} \\ \vdots \\ x_{j,q-1} \\ x_{j,q} \\ \vdots \\ x_{j,q+\hat{q}_j-1}\end{pmatrix} = \begin{pmatrix}0 \\ 0 \\ \vdots \\ 0\end{pmatrix} \tag{9}$$

According to [10], the above system (9) corresponds only to the *infinite zero elementary divisors* (I.Z.E.D.) i.e. $\mu_j = q + \hat{q}_j$, and not to the *infinite pole elementary divisors* (I.P.E.D.) i.e. $\mu_j = q - q_j$ for $j = 2, 3, ..., r$.

Theorem 8: The vector

$$\beta_j(k) = x_{j,q+\hat{q}_j-1}\delta(N-k) + \ldots + x_{j,0}\delta(N-k-q-\hat{q}_j+1)$$

is a solution of the AR-representation (1) iff the vector

$$\tilde{\beta}_j(k) = x_{j,0}\delta(k-q-\hat{q}_j+1) + \cdots + x_{j,q_j-1}\delta(k-q) +$$
$$+ \cdots + x_{j,q+\hat{q}_j-1}\delta(k)$$

is a solution of the dual system (4) i.e. $\tilde{A}(\sigma)\tilde{\beta}_j(k) = 0$.

Proof: $(\Rightarrow)$ First we will prove that if $\beta_j(k)$ is a solution of (1) then $\tilde{\beta}_j(k)$ is a solution of (4). Define the following matrices

$$A_i^j = \begin{pmatrix}A_i & 0 & \cdots & 0 \\ A_{i+1} & A_i & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ A_j & A_{j-1} & \cdots & A_i\end{pmatrix} \qquad x_i^j = \begin{pmatrix}x_j \\ \vdots \\ x_i\end{pmatrix}$$

Taking Z-transforms on the dual system we have that

$$\mathrm{Z}\left\{\tilde{A}(\sigma)\tilde{\beta}(k)\right\} = 0 \Rightarrow \tilde{A}(z)\tilde{\beta}(z) = \tilde{\beta}_{in} \tag{10}$$

where $\tilde{\beta}(z) := \mathrm{Z}\left\{\tilde{\beta}(k)\right\}$ and

$$\tilde{\beta}_{in} := \begin{pmatrix}z^q I_r & \cdots & zI_r\end{pmatrix} A_0^{q-1} \begin{pmatrix}\tilde{\beta}(0) & \tilde{\beta}(1) & \cdots & \tilde{\beta}(q-1)\end{pmatrix}^T$$

By substituting

$$\tilde{\beta}_j(z) = \mathrm{Z}\left\{\tilde{\beta}_j(k)\right\} =$$
$$= x_{j,0}z^{-q-\hat{q}_j+1} + x_{j,1}z^{-q-\hat{q}_j+2} + \cdots + x_{j,q+\hat{q}_j-1}$$

in place of $\tilde{\beta}(z)$ in (10) we get

$$\tilde{A}(z)\tilde{\beta}_j(z) = \left(z^q A_0 + z^{q-1}A_1 + \cdots zA_{q-1} + A_q\right) \times$$

$$\times\left(x_{j,0}z^{-q-\hat{q}_j+1}+x_{j,1}z^{-q-\hat{q}_j+2}+\cdots+x_{j,q+\hat{q}_j-1}\right)=$$
$$=\begin{pmatrix} z^qI_r & z^{q-1}I_r & \cdots & I_r & z^{-1}I_r & \cdots & z^{-q-\hat{q}_j+1}I_r\end{pmatrix}\times$$
$$\times\begin{pmatrix} A_0 & \cdots & 0\\ \vdots & \ddots & \vdots\\ A_q & \ddots & A_0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & A_q\end{pmatrix} x_{j,0}^{j,q+\hat{q}_j-1}=$$
$$=\begin{pmatrix} z^qI_r & z^{q-1}I_r & \cdots & zI_r\end{pmatrix}A_0^{q-1}x_{j,q_j}^{j,q+\hat{q}_j-1}+$$
$$+\begin{pmatrix} I_r & \cdots & z^{-q}I_r & \cdots & z^{-q-q_j-1}I_r\end{pmatrix}Q^Tx_{j,0}^{j,q+\hat{q}_j-1}=$$
$$=\begin{pmatrix} z^qI_r & z^{q-1}I_r & \cdots & zI_r\end{pmatrix}A_0^{q-1}x_{j,q_j}^{j,q+\hat{q}_j-1}+$$
$$+\begin{pmatrix} z^{-q-\hat{q}_j+1}I_r & \cdots & z^{-q}I_r & \cdots & I_r\end{pmatrix}\underbrace{Qx_{j,q+\hat{q}_j-1}^{j,0}}_{=0}=$$
$$=\begin{pmatrix} z^qI_r & \cdots & zI_r\end{pmatrix}A_0^{q-1}x_{j,q_j}^{j,q+\hat{q}_j-1}$$

We can easily check that (10) and (11) coincide in case where:

$$\begin{pmatrix}\tilde{\beta}_j(0)\\ \vdots\\ \tilde{\beta}_j(q-1)\end{pmatrix}=\begin{pmatrix} x_{j,q+\hat{q}_j-1}\\ \vdots\\ x_{j,q_j}\end{pmatrix}$$

Therefore $\tilde{\beta}_j(k)$ is a solution of (4) for the above initial values.

($\Leftarrow$) Now we will show the opposite, that if $\tilde{\beta}_j(k)$ is a solution of (4), then $\beta_j(k)$ is a solution of (1).

Since $\tilde{\beta}_j(k)$ is a solution of (4), we shall have that

$$\tilde{A}(z)\tilde{\beta}_j(z)=\left(z^qA_0+z^{q-1}A_1+\cdots zA_{q-1}+A_q\right)\times$$
$$\times\left(x_{j,0}z^{-q-\hat{q}_j+1}+x_{j,1}z^{-q-\hat{q}_j+2}+\cdots+x_{j,q+\hat{q}_j-1}\right)=$$
$$=\begin{pmatrix} z^qI_r & \cdots & zI_r\end{pmatrix}A_0^{q-1}\begin{pmatrix}\tilde{\beta}_j(0) & \cdots & \tilde{\beta}_j(q-1)\end{pmatrix}^T$$

and therefore the strictly proper part in (11) i.e

$$\begin{pmatrix} z^{-q-\hat{q}_j+1}I_r & \cdots & z^{-q}I_r & \cdots & I_r\end{pmatrix}Qx_{j,q+\hat{q}_j-1}^{j,0}$$

must be equal to zero or otherwise the condition (9) must be satisfied. However, (9) guarantees that $\beta_j(k)$ is a solution of (1). ∎

The previous theorem proves that the problem of finding an AR-representation of the form (1) with the following backward behavior :

$$\beta_j(k)=x_{j,q+\hat{q}_j-1}\delta(N-k)+...+x_{j,0}\delta(N-k-q-\hat{q}_j+1)$$

is equivalent to the problem of finding an AR-representation of the form (4) satisfying the forward behavior

$$\tilde{\beta}_j(k)=x_{j,0}\delta(k-q-\hat{q}_j+1)+\cdots+x_{j,q_j-1}\delta(k-q)+$$
$$+\cdots+x_{j,q+\hat{q}_j-1}\delta(k)$$

However this problem can easily be solved from the results that we already presented, using Gohberg's formula from [1] and Theorem 5. This is showcased in the following theorem.

Theorem 9: Let the following l vector valued functions

$$\beta_j(k)=\sum_{w=0}^{q+\hat{q}_j-1}x_{j,w}\delta(N-w-k)$$

where $x_{j,w}\in\mathbb{C}^r$, $1\le j\le l$. Define

$$C_j=\begin{pmatrix} x_{j,0} & x_{j,1} & \cdots & x_{j,\hat{q}_j} & \cdots & x_{j,q+\hat{q}_j-1}\end{pmatrix}$$
$$J_j=\begin{pmatrix} 0 & 1 & \cdots & 0\\ 0 & 0 & \ddots & \vdots\\ \vdots & \ddots & \ddots & 1\\ 0 & \cdots & 0 & 0\end{pmatrix}\in\mathbb{R}^{\mu_j\times\mu_j}$$

where $j=1,2,...,l$ and $\mu_j=q+\hat{q}_j$. Let

$$C=\begin{pmatrix} C_1 & C_2 & \cdots & C_l\end{pmatrix}\in\mathbb{R}^{r\times\mu}$$
$$J=blockdiag\left(J_1,J_2,\ldots,J_l\right)\in\mathbb{R}^{\mu\times\mu}$$

with $\mu=\sum_{j=1}^{l}\mu_j$. Let $a\neq 0$ be a complex number and define

$$\tilde{A}(\sigma)=I_r-C(J-aI_r)^{-q}\left\{(\sigma-a)V_q+\cdots+(\sigma-a)^qV_1\right\}$$

where $q=ind(C,J)$ is the least integer such that the matrix $S_{q-1}=col\left(CJ^i\right)_{i=0}^{q-1}$ has full column rank and $V=\begin{pmatrix} V_1 & V_2 & \cdots & V_q\end{pmatrix}$ is the left inverse of $S_{1-q}=col\left[C(J-aI_n)^{-w}\right]_{w=0}^{q-1}$. Then $\beta_j(k)$ is a solution of (1), where $A(\sigma)=\sigma^q\tilde{A}(\frac{1}{\sigma})$. Furthermore, q is the minimal possible degree of any $r\times r$ matrix polynomial with this property.

V. Construction of a System with Given Forward and Backward Behavior

In this section, we will combine all the results presented in previous sections, in order to create an algorithm for constructing a system that satisfies both a given forward and a given backward behavior. We have already presented methods for constructing a system with given either a forward or a backward behavior. In the last case, we have constructed a dual polynomial matrix $\tilde{A}(\sigma)$ with a forward behavior resulting from a given backward behavior. Our aim, is to work in a similar way for the forward behavior as well i.e. find a forward behavior of the dual system (4) that corresponds to a forward behavior of (1). If we succeed this, then the problem of finding a system with given forward-backward behavior is reduced to the problem of finding its dual system which exhbits a certain forward behavior.

Theorem 10: Let $A(\sigma)$ defined in (3) with $rank_{R(\sigma)}A(\sigma)=r$. If $\beta_j(k)=C_jJ_j^kx_0$ is a solution of (1), where $\left(C_j\in\mathbb{R}^{r\times n_j},J_j\in\mathbb{R}^{n_j\times n_j}\right)$ is a finite Jordan pair corresponding to the zero $\lambda_j\neq 0$ of $A(\sigma)$, then $\tilde{\beta}_j(k)=C_jJ_j^{-1}\left(J_j^{-1}\right)^kx_0$ is a solution of (4).

Proof: Since $\beta(k)=C_jJ_j^kx_0$ is a solution of (1), we have

$$\begin{pmatrix}\beta(0)\\ \vdots\\ \beta(q-1)\end{pmatrix}=\begin{pmatrix} C_j\\ \vdots\\ C_jJ_j^{q-1}\end{pmatrix}x_0$$

Taking Z-transform of $A(\sigma)\beta(k)=0$ we have:

$$A(z)C_j(zI_{n_j})\left(zI_{n_j}-J_j\right)^{-1}=\begin{pmatrix} z^qI_r & z^{q-1}I_r & \cdots & zI_r \end{pmatrix}\cdot$$
$$\cdot A_q^1\begin{pmatrix} C_j & C_jJ_j & \cdots & C_jJ_j^{q-1}\end{pmatrix}^T x_0$$

Multiplying both sides by z^q and replacing z with $\frac{1}{z}$ we get

$$\tilde{A}(z)C_j(\frac{1}{z}I_{n_j})\left(\frac{1}{z}I_{n_j}-J_j\right)^{-1}=\begin{pmatrix} I_r & \cdots & z^{q-1}I_r\end{pmatrix}\cdot$$
$$\cdot A_q^1\begin{pmatrix} C_j & C_jJ_j & \cdots & C_jJ_j^{q-1}\end{pmatrix}^T x_0$$
$$\Rightarrow \tilde{A}(z)C_j\left(I_{n_j}-zJ_j\right)^{-1}=\begin{pmatrix} z^{q-1}I_r & \cdots & I_r\end{pmatrix}\cdot$$
$$\cdot A^f\begin{pmatrix} C_j & C_jJ_j & \cdots & C_jJ_j^{q-1}\end{pmatrix}^T x_0$$
$$\Rightarrow \tilde{A}(z)C_j\left(\left(J_j^{-1}-z\right)J_j\right)^{-1}=\begin{pmatrix} z^{q-1}I_r & \cdots & I_r\end{pmatrix}\cdot$$
$$\cdot A^f\begin{pmatrix} C_j & C_jJ_j & \cdots & C_jJ_j^{q-1}\end{pmatrix}^T x_0$$
$$\Rightarrow -\tilde{A}(z)C_jJ_j^{-1}\left(zI_{n_j}-J_j^{-1}\right)^{-1}=\begin{pmatrix} z^{q-1}I_r & \cdots & I_r\end{pmatrix}\cdot$$
$$\cdot A^f\begin{pmatrix} C_jJ_j & C_jJ_j^2 & \cdots & C_jJ_j^{q}\end{pmatrix}^T J_j^{-1}x_0 \qquad (11)$$

where

$$A^f=\begin{pmatrix} A_1 & \cdots & A_q \\ \vdots & \iddots & \vdots \\ A_q & \cdots & 0\end{pmatrix}$$

Since (C_j,J_j) is a finite Jordan pair of $A(\sigma)$, it satisfies:

$$A_0C_j+A_1C_jJ_j+\cdots+A_qC_jJ_j^q=0$$

Premultiplying (11) by zI_{n_j} we get

$$-\tilde{A}(z)C_jJ_j^{-1}\left(zI_{n_j}\right)\left(zI_{n_j}-J_j^{-1}\right)^{-1}=$$
$$=-\begin{pmatrix} z^qI_r & \cdots & zI_r\end{pmatrix}A_0^{q-1}\begin{pmatrix} C_jJ_j^{-1} & \cdots & C_jJ_j^{-q}\end{pmatrix}^T x_0$$

Therefore, by taking the inverse Z-transform, we conclude that

$$\tilde{\beta}(k)=\mathrm{Z}^{-1}\left\{\tilde{\beta}(z)\right\}=$$
$$=\mathrm{Z}^{-1}\left\{C_jJ_j^{-1}\left(zI_{n_j}\right)\left(zI_{n_j}-J_j^{-1}\right)^{-1}x_0\right\}=$$
$$=C_jJ_j^{-1}\left(J_j^{-1}\right)^kx_0$$

is a solution of (4) for initial conditions

$$\begin{pmatrix}\tilde{\beta}(0)\\ \vdots \\ \tilde{\beta}(q-1)\end{pmatrix}=\begin{pmatrix} C_jJ_j^{-1} \\ \vdots \\ C_jJ_j^{-q}\end{pmatrix}x_0$$

∎

In the above theorem, we managed to match a given solution of (1) to a soltuion of (4), a result that we will utilise in the following.

Since the matrix J_j^{-1} is not in Jordan form, we can find a nonsingular constant matrix $U\in\mathbb{R}^{n_j\times n_j}$ such that $J_j^{-1}=U\tilde{J}_jU^{-1}$ where $\tilde{J}_j$ is in Jordan form. With this change, the solution of $\tilde{A}(\sigma)\tilde{\beta}(k)=0$ can also be written as follows:

$$\tilde{\beta}(k) = C_jJ_j^{-1}\left(J_j^{-1}\right)^kx_0=$$
$$= C_jU\tilde{J}_jU^{-1}U\left(\tilde{J}_j\right)^kU^{-1}x_0=$$
$$= \tilde{C}_j\left(\tilde{J}_j\right)^k\left(U^{-1}x_0\right)$$

where

$$\tilde{C}_j=C_jU\tilde{J}_j$$

So we can see that instead of using the matrix pair $\left(C_jJ_j^{-1}\in\mathbb{R}^{r\times n_j}, J_j^{-1}\in\mathbb{R}^{n_j\times n_j}\right)$ where the matrix J_j^{-1} is not in Jordan form, we can use the matrix pair

$$\left(\tilde{C}_j=C_jU\tilde{J}_j\in\mathbb{R}^{r\times n_j};\tilde{J}_j=U^{-1}J_j^{-1}U\in\mathbb{R}^{n_j\times n_j}\right)$$

Summarizing our results, in order to construct an AR-representation for a certain forward-backward behavior, one must follow the next algorithm.

Algorithm 11: Construction of an AR-Representation with given forward and backward behavior.

Step 1 Transform the finite Jordan pairs

$$\left(C_j\in\mathbb{R}^{r\times n_j};\ J_j\in\mathbb{R}^{n_j\times n_j}\right)$$

that correspond to solutions of the form $\beta(k)=C_jJ_j^kx_0$ to the finite Jordan Pairs

$$\left(\tilde{C}_j=C_jU\tilde{J}_j\in\mathbb{R}^{r\times n_j};\tilde{J}_j=U^{-1}J_j^{-1}U\in\mathbb{R}^{n_j\times n_j}\right)$$

that correspond to solutions of the form $\tilde{\beta}(k)=\tilde{C}_j\left(\tilde{J}_j\right)^k\left(U^{-1}x_0\right)$ of the dual system that we are looking for.

Step 2 Transform the infinite Jordan pairs to finite Jordan pairs of the dual system using Theorem 10.

Step 3 Construct the polynomial matrix $\tilde{A}(\sigma)$ using the method presented in Theorem 5.

Step 4 Get the polynomial matrix $A(\sigma)=\sigma^q\tilde{A}(\frac{1}{\sigma})$ and thus the AR-representation (1) that we are looking for.

As we have already mentioned, $A(\sigma)$ is not the only polynomial matrix satisfying the given behavior. Any other unimodular equivalent matrix $\hat{A}(s)$ such that $\hat{A}(s)=U(\sigma)\tilde{A}(\sigma)$ with $\det U=c\in\mathbb{R}$ gives rise to the dual polynomial matrix of the solution that we are looking for.

Definition 12: [13], [14] $A_1(\sigma),A_2(\sigma)\in\mathbb{R}[\sigma]^{r\times m}$ are said to be *divisor equivalent* if there exist polynomial matrices $M(s)$, $N(s)$ such that:

$$\begin{pmatrix} M(s) & A_2(s)\end{pmatrix}\begin{pmatrix} A_1(s) \\ -N(s)\end{pmatrix}=0$$

where the matrices $\begin{pmatrix} M(s) & A_2(s)\end{pmatrix}$ and $\begin{pmatrix} A_1(s) \\ -N(s)\end{pmatrix}$:

1) have full rank over $\mathbb{R}[s]$ and no finite elementary divisors,
2) have no infinite elementary divisors.

Divisor Equivalence is a very important form of equivalence, since it has the property of preserving both the finite and the infinite elementary structure of a polynomial matrix. Note, that all the divisor equivalent polynomial matrices of $A(\sigma)$ that have been constructed in Algorithm 11 are solutions to our problem.

The above algorithm may display computational difficulties in the case where the forward behavior has polynomial vector valued functions (since these are connected with the finite elementary divisors of $A(\sigma)$ at zero). In that case, the Jordan matrix J, will have zero determinant and thus, will not be invertible. This problem can easily be surpassed by replacing σ with $\sigma + b$ in $A(\sigma)$ i.e. moving all possible zeros of $A(\sigma)$ to non-zero places. The following lemma indicates this solution.

Lemma 13: [9] If $\begin{pmatrix} C & J \end{pmatrix}$ is a finite Jordan Pair of $A(\sigma)$ then $\begin{pmatrix} C & J + bI_n \end{pmatrix}$ is a finite Jordan Pair of $A(\sigma - b)$.

If $\begin{pmatrix} C_\infty & J_\infty \end{pmatrix}$ is an infinite Jordan Pair of $A(\sigma)$ then $\begin{pmatrix} C_\infty & J_\infty (I_\mu + bJ_\infty)^{-1} \end{pmatrix}$ is an infinite Jordan Pair of $A(\sigma - b)$.

Example 14: We are interested to construct an AR-representation of the form (1) with the following forward and backward behavior:

$$\beta_1(k) = \underbrace{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{\beta_{1,1}} 2^k + \underbrace{\begin{pmatrix} 2 \\ 0 \end{pmatrix}}_{\beta_{1,0}} k2^{k-1}$$

$$\beta_2(k) = \underbrace{\begin{pmatrix} -1 \\ -1 \end{pmatrix}}_{x_{1,1}} \delta(N-k) + \underbrace{\begin{pmatrix} 0 \\ 3 \end{pmatrix}}_{x_{1,0}} \delta(N-k-1)$$

First, define the matrix pairs

$$C_1 = \begin{pmatrix} \beta_{1,0} & \beta_{1,1} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}; J_1 = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

$$J_1^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} \\ 0 & \frac{1}{2} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix}}_{U\tilde{J}_1 U^{-1}}$$

$$\tilde{C}_1 = C_1 U \tilde{J}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$C_2 = \begin{pmatrix} x_{1,0} & x_{1,1} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 3 & -1 \end{pmatrix} \quad J_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

The complete matrix pair is

$$C = \begin{pmatrix} \tilde{C}_1 & C_2 \end{pmatrix} = \left(\begin{array}{cc|cc} 1 & 0 & 0 & -1 \\ 0 & 2 & 3 & -1 \end{array}\right)$$

$$J = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix} = \left(\begin{array}{cc|cc} \frac{1}{2} & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

For $q = 1$, the matrix $S_0 = C$ does not have full column rank. For $q = 2$, the matrix $S_1 = \begin{pmatrix} C & CJ \end{pmatrix}^T$ has full column rank. Let $a = 1$, so

$$\tilde{A}(\sigma) = I_2 - C(J - I_4)^{-2}\left\{(s-1)V_2 + (s-1)^2 V_1\right\}$$

where

$$\begin{pmatrix} V_1 & V_2 \end{pmatrix} = \left[\begin{pmatrix} C & C(J - I_4)^{-1} \end{pmatrix}^T\right]^{-1}$$

Finishing the computations, we conclude to

$$\tilde{A}(\sigma) = \begin{pmatrix} \frac{1}{5}\left(6 - 23\sigma + 22\sigma^2\right) & -\frac{2}{5}\left(-1\sigma + \sigma^2\right) \\ \frac{3}{5}\left(1 - 3\sigma + 2\sigma^2\right) & \frac{1}{5}\left(\sigma + 4\sigma^2\right) \end{pmatrix}$$

Therefore, the matrix we are looking for is

$$A(\sigma) = \sigma^2 \tilde{A}\left(\frac{1}{\sigma}\right) = \begin{pmatrix} \frac{1}{5}\left(6\sigma^2 - 23\sigma + 22\right) & \frac{2}{5}\left(\sigma - 1\right) \\ \frac{3}{5}\left(\sigma^2 - 3\sigma + 2\right) & \frac{1}{5}\left(\sigma + 4\right) \end{pmatrix}$$

As a matter of fact, the above matrix pairs constitute finite and infinite Jordan Pairs for $A(\sigma)$ since they satisfy

$$A_2 C_1 J_1^2 + A_1 C_1 J_1 + A_0 C_1 = 0_2 \quad rank\begin{pmatrix} C_1 \\ C_1 J_1 \end{pmatrix} = 2$$

$$A_0 C_2 J_2^2 + A_1 C_2 J_2 + A_2 C_2 = 0_2 \quad rank\begin{pmatrix} C_2 \\ C_2 J_2 \end{pmatrix} = 2$$

VI. CONCLUSIONS

We have proposed an algorithm for constructing a regular AR-representation which satisfies a given forward and backward behavior. Our aim is to extend this theory for the case non-regular AR-representations, since we have noticed that for the given forward-backward behavior we can always achieve a non-regular AR-representation which satisfy this behavior.

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