DISCRETIZATION OF SINGULAR SYSTEMS AND ERROR ESTIMATION

NICHOLAS P. KARAMPETAKIS *, RALLIS KARAMICHALIS *

* Department of Mathematics
Aristotle University of Thessaloniki, Thessaloniki 54124, Greece
e-mail: karampet@math.auth.gr, rkaramic@math.auth.gr

This paper proposes a discretization technique of a descriptor differential system. The methodology used is both triangular first order hold (interpolating FOH) discretization and zero order hold (ZOH) for the input function. Upper bounds for the error between the continuous and discrete time solution are produced for both discretization methods and are shown to be better than any other existing method in the literature.

Keywords: Descriptor Systems, Discretization, Truncation Error, First Order Hold, Zero Order Hold

1. Introduction

In digital control, and in several areas of engineering, we need to discretize continuous-time state-space equations. The discretization process though, introduces an error between the continuous and the discretized solution. More specifically, we study Linear Time Invariant (LTI) differential systems of the form

\[ E \dot{x}(t) = Ax(t) + Bu(t) \]  

with \( E, A \in \mathbb{F}^{n \times n}, \) that is the set of all square matrices with elements in the field \( F = \mathbb{R} \) or \( \mathbb{C} \), \( \det E = 0 \) and \( B \in \mathbb{F}^{n \times l} \) are constant matrices. We also assume that state vector \( x(t) \in \mathbb{F}^{n \times 1} \), where each \( x_i(t) : F \mapsto F \), has consistent initial conditions and that input vector \( u(t) \in \mathbb{F}^{l \times 1} \) where also each \( u_i(t) : F \mapsto F \). In the special case where \( E \) is invertible and therefore the system is the known state-space system, a zero-order hold discretized model of (1) is given in (Levine, 2008). A first order hold (FOH) discretized model of (1) by extrapolation (resp. interpolation) of the first derivative of the input is given in (Toshiyuki and Mituhiko, 1993) (resp. (Franklin et al., 1997)). In case where \( E \) is singular, we may use the forward or backward Euler method, or even the Gears method proposed in (Sincovec et al., 1981) in order to get a discretized singular model of (1). From the literature of the discretization methods for descriptor differential systems, we mainly focus on two different interesting methods. The first one, (see (Karampetakis and Gregoriadou, 2011), (Karampetakis, 2004) and (López-Estrada et al., 2012)), which is also used is the latest version of Wolfram Mathematica 9, is based on the Laurent expansion of \((sE-A)^{-1}\). Both of them are somehow equivalent using Zero Order Hold (ZOH) approximation. This paper is an extension to the first method, using Triangular First Order Hold (interpolating FOH) approximation.

Consequently, in this paper, we provide the following interesting results: a) two new upper bounds for the norm of the difference between the continuous solution and the discretized solution \( \|x(kT) - x_k\| \) are given by extending the already known upper bound suggested in (Karageorgos et al., 2011) for the zero order hold approximation and providing a new upper bound for the first order hold approximation, b) the proposed bounds penalize our choice for the sampling period \( T \) and thus we can estimate a maximum period \( T \) if we demand the error to not exceed a given value. Finally, ZOH and interpolating FOH are compared via an example and advantages of interpolating FOH over ZOH are presented.

2. Problem Formulation and Preliminaries

Linear generalized differential systems of type \( E \dot{x}(t) = Ax(t), E, A \in \mathbb{R}^{n \times n} \) with \( \text{det} E = 0 \), where \( x \in \mathbb{R}^{n \times 1} \) and \( x_0 \) an initial value, are required in modelling of many physical, electrical and mechanical problems. Systems of this type are related to matrix pencil theory since the algebraic geometric and dynamic properties stem from the structure of the associated pencil \( sE - A \).
Given $E, A \in F^{m \times n}$ and an indeterminate $s$, the matrix pencil $sE - A$ is called regular when $m = n$ and $\text{det}(sE - A) \neq 0$. In any other case, the pencil will be called singular. The pencil $sE - A$ is said to be strictly equivalent to the pencil $s\tilde{E} - \tilde{A}$ if and only if there exist $P, Q \in C^{m \times n}$ such that $P(sE - A)Q = s\tilde{E} - \tilde{A}$ where $\text{det}P, \text{det}Q \neq 0$. It is known (Gantmacher, 1959) that $sE - A$ is strictly equivalent to its Weierstrass normal form $sE_w - A_w$ i.e. there exist nonsingular matrices $P, Q$ such that

$$P(sE - A)Q = \begin{pmatrix} sI_p - J_p & 0 \\ 0 & sH_q - I_q \end{pmatrix} = sE_w - A_w,$$

where $H_q \in \mathbb{R}^{q \times q}$ is nilpotent and $J_p \in \mathbb{R}^{p \times p}$ with $p + q = n$.

$$H_q = \text{blockdiag} \{H_{q_1}, H_{q_2}, \ldots, H_{q_k}\}$$

$$H_{q_i} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{\mu_i \times \mu_i}, i \in k$$

with $\sum_{i=1}^{k} \mu_i = q$ and

$$J_p = \text{blockdiag} \{J_{\sigma_1}(a_1), J_{\sigma_2}(a_2), \ldots, J_{\sigma_{\ell}}(a_{\ell})\}$$

$$J_{\sigma_i}(a_i) = \begin{bmatrix} a_i & 1 & 0 & \cdots & 0 \\ 0 & a_i & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & a_i \end{bmatrix} \in \mathbb{R}^{\sigma_i \times \sigma_i}, i \in \ell$$

with $\sum_{i=1}^{\ell} \sigma_i = p$, $\ell \geq 0$ is the number of the finite elementary divisors (f.e.d.) of $sE - A$ of the form $(s - a_i)^{\sigma_i}$, which uniquely characterize the block $sH_q - I_q$, $J_p$. The infinite elementary divisors (i.e.d.) of $sE - A$, that uniquely characterize the block $sI_p - J_p$, are given by

$$w^{\mu_1}, w^{\mu_2}, \ldots, w^{\mu_k}$$

where $\mu_i$ are the sizes of the Jordan blocks $H_{q_i}, i \in k$ of $H_q$ and they can be defined as the f.e.d.’s of the “dual” pencil $E - wA$ at $w = 0$. The relation between the i.e.d. and the infinite pole-zero structure of $sE - A$ is given in (Vardulakis and Karcanias, 1983). The matrices $P, Q$ used for transforming $sE - A$ to $sE_w - A_w$ are not unique. A numerical algorithm is given in (Duan, 2010) for the calculation of these matrices, whereas a theoretical algorithm based on the finite and infinite generalized eigenvectors of the matrix pencil $sE - A$ is given in (Vardulakis, 1991).

Now, we consider the transformation $x(t) = Qy(t)$ and we get the following results. As it has been already mentioned about the mathematical tools used during the discretization process, only Weierstrass canonical form (WCF) is required. As this paper extends (Karageorgos et al., 2010) using first order hold approximation instead of zero order hold in order to get better results, some commonly used lemmas are presented without their proofs, although full references are provided. We already know that system (1), has the following continuous time solution, see (Dai, 1989), (Karageorgos et al., 2010) and (Koumboulis and Mertzios, 1999):

$$x(t) = Q_{n,p} \left( e^{J_p(t-t_0)} y_p(t_0) + \int_{t_0}^{t} e^{J_p(t-s)} B_p u(s) ds \right) - Q_{n,q} \sum_{i=0}^{q-1} H_q B_q i_u(i)(t)$$

where $Q = \begin{bmatrix} Q_{n,p} & Q_{n,q} \\ Q_{n,p} & B_p q_i \\ Q_{n,q} & \end{bmatrix}$, $y(t_0) = y_p(t_0)$, $u(i)\text{the }i\text{th}-\text{derivative of the input function }u(t)$. However, (2) can be transformed in a more useful format. We have:

$$x(t) = Q_{n,p} e^{J_p(t-t_0)} y_p(t_0) + Q_{n,q} y_q(t_0) + Q_{n,p} \int_{t_0}^{t} e^{J_p(t-s)} B_p i u(s) ds - Q_{n,q} y_q(t_0)$$

$$- Q_{n,q} \sum_{i=0}^{q-1} H_q B_q i_u(i)(t)$$

$$= \begin{bmatrix} Q_{n,p} & Q_{n,q} \end{bmatrix} \begin{bmatrix} e^{J_p(t-t_0)} & O_{p,q} \\ O_{q,p} & I_q \end{bmatrix} \begin{bmatrix} y_p(t_0) \\ y_q(t_0) \end{bmatrix}$$

$$+ Q_{n,p} \int_{t_0}^{t} e^{J_p(t-s)} B_p i u(s) ds$$

$$+ Q_{n,q} \left( -y_q(t_0) - \sum_{i=0}^{q-1} H_q B_q i_u(i)(t) \right)$$

In order now for the system (1) to obtain consistent initial conditions, see (Karageorgos et al., 2010), we should consider that

$$y_p(t_0) = -y_q(t_0) = \sum_{i=0}^{q-1} H_q B_q i_u(i)(t_0)$$

and as a result we obtain

$$x(t) = Q \begin{bmatrix} O_{p,q} & I_q \end{bmatrix} Q_{n,p} \begin{bmatrix} e^{J_p(t-t_0)} \\ y_p(t) \end{bmatrix} - y_q(t)$$

$$+ Q_{n,p} \int_{t_0}^{t} e^{J_p(t-s)} B_p i u(s) ds$$

$$+ Q_{n,q} \sum_{i=0}^{q-1} H_q B_q i_u(i)(t) - y_q(t_0) - y_q(t)$$
Moreover, by definition, the state-transition matrix of the autonomous linear descriptor differential system, \(E \dot{x}(t) = Ax(t)\), is given by

\[
\Phi(t, t_0) = Q \left[ e^{J_p(t-t_0)} \begin{bmatrix} I_q & O_{p,q} \\ O_{q,p} & I_q \end{bmatrix} \right] Q^{-1},
\]

Finally, after noticing that

\[
\Phi(t, s)Q_{n,p} = \Phi(t, s) \left[ Q_{n,p} \begin{bmatrix} I_{p,p} \\ O_{q,p} \end{bmatrix} \right] Q_{n,q}
= \left[ Q_{n,p}e^{J_p(t-s)} \begin{bmatrix} I_{p,p} \\ O_{q,p} \end{bmatrix} \right] = Q_{n,p}e^{J_p(t-s)}
\]

we get

\[
x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, s)Q_{n,p}B_{p,l}u(s)ds + Q_{n,q} \sum_{i=0}^{q-1} H^i_q B_{q,l} \left( u^{(i)}(t_0) - u^{(i)}(t) \right) \tag{3}
\]

Now, let \(T > 0\) be the constant sampling period. We also assume that \(t_0 = 0\). We also consider two cases. In the first one, the input function \(u(\tau)\) is constant in the interval \([kT, (k+1)T)\) and we approximate it by using (ZOH) approximation,

\[
u(\tau) = u(kT) \quad \forall \tau \in [kT, (k+1)T)
\]

In the second case, the input function \(u(\tau)\) is not constant in the interval \([kT, (k+1)T)\) and we approximate it by using the Triangular First Order Hold (interpolating FOH) approximation,

\[
u(\tau) = u(kT) + \frac{u((k+1)T) - u(kT)}{T} (\tau - kT)
\]

\(\forall \tau \in [kT, (k+1)T)\) In order to combine these formulas into one formula we write

\[
u(\tau) = u(kT) + \chi_{tf} \frac{u((k+1)T) - u(kT)}{T} (\tau - kT)
\]

\(\forall \tau \in [kT, (k+1)T)\), where \(\chi_{tf} = 1\) or 0 whether we consider interpolating FOH or ZOH approximation, respectively. For use of simplicity, hereafter, we use the notation \(x_k := x(kT) \quad \forall k = 0, 1, 2, \ldots\) From equation (3) by setting \(t = kT\) and \(t = (k+1)T\), we get

\[
x_k = \Phi(kT, 0)x_0 + Q_{n,q} \sum_{i=0}^{q-1} H^i_q B_{q,l} (u^{(i)}_0 - u^{(i)}_{k+1})
+ \int_{0}^{kT} \Phi(kT, s)Q_{n,p}B_{p,l}u(s)ds \tag{4}
\]

\[
x_{k+1} = \Phi((k+1)T, 0)x_0
+ Q_{n,q} \sum_{i=0}^{q-1} H^i_q B_{q,l} (u^{(i)}_0 - u^{(i)}_{k+1})
+ \int_{0}^{(k+1)T} \Phi((k+1)T, s)Q_{n,p}B_{p,l}u(s)ds \tag{5}
\]

Based on the group property of the flow we conclude to the following Lemma.

**Lemma 1.** The following equalities hold:

\[
\Phi(T, 0)\Phi(kT, s) = \Phi((k+1)T, s)
\]

\[
\Phi(T, 0)Q_{n,q} = Q_{n,q}
\]

From equations (4) and (5) and using the above lemma, we multiply \(x_k\) by \(\Phi(T, 0)\) and then we subtract from \(x_{k+1}\), so we finally get

\[
x_{k+1} - \Phi(T, 0)x_k = Q_{n,q} \sum_{i=0}^{q-1} H^i_q B_{q,l} (u^{(i)}_0 - u^{(i)}_{k+1})
- \Phi(T, 0)Q_{n,q} \sum_{i=0}^{q-1} H^i_q B_{q,l} (u^{(i)}_0 - u^{(i)}_{k+1})
+ \int_{kT}^{(k+1)T} \Phi((k+1)T, s)Q_{n,p}B_{p,l}u(s)ds \tag{6}
\]

and therefore the following recursive formula derives

\[
x_{k+1} = \Phi(T, 0)x_k + Q_{n,q} \sum_{i=0}^{q-1} H^i_q B_{q,l} (u^{(i)}_{k+1} - u^{(i)}_{k})
+ \int_{kT}^{(k+1)T} \Phi((k+1)T, s)Q_{n,p}B_{p,l}u(s)ds \tag{6}
\]

But,

\[
\int_{kT}^{(k+1)T} \Phi((k+1)T, s)Q_{n,p}B_{p,l}u(s)ds = \int_{kT}^{(k+1)T+w} \Phi((k+1)T, s)Q_{n,p}B_{p,l}u(s)ds
\]

\[
\int_{0}^{T} \Phi((k+1)T, s)Q_{n,p}B_{p,l}u(s)ds = \int_{0}^{T} \Phi(T-w, 0)Q_{n,p}B_{p,l} \left( u_k + \chi_{tf} \frac{u_{k+1} - u_k}{T} \right) dw
\]

\[
\int_{0}^{T} \Phi(T-w, 0)Q_{n,p}B_{p,l}u_k dw
+ \chi_{tf} \int_{0}^{T} \Phi(T-w, 0)Q_{n,p}B_{p,l} \frac{u_{k+1} - u_k}{T} dw \tag{7}
\]
Finally by setting $\lambda = T - w$ in (7) and replacing in (6) we get the following recursive formula,

$$x_{k+1} = \Phi(T, 0)x_k + Q_{n,q} \sum_{i=0}^{q-1} H_q^i B_q,i (u_k^{(i)} - u_k^{(i) - 1})$$

$$+ \int_0^T \Phi(\lambda, 0)d\lambda Q_{n,p} B_{p,i} u_k$$

$$+ \chi tf \int_0^T \Phi(\lambda, 0) (T - \lambda) d\lambda Q_{n,p} B_{p,i} \frac{u_{k+1} - u_k}{T}$$

Relation (8) is the discretized model of (1) under the ZOH or the interpolating FOH approximation.

**Theorem 1.** The solution of (3) under interpolating FOH $(\chi_{tf} = 1)$ or ZOH $(\chi_{tf} = 0)$ approximation is given by the following analytic formula

$$x_k = \Phi(kT, 0)x_0 + Q_{n,q} \sum_{i=0}^{q-1} H_q^i B_q,i (u_k^{(i)} - u_k^{(i) - 1})$$

$$+ \int_0^T \Phi(jT + \lambda, 0) d\lambda Q_{n,p} B_{p,i} u_{k-j-1}$$

$$+ \chi tf \int_0^T \Phi(jT + \lambda, 0)(T - \lambda)d\lambda Q_{n,p} B_{p,i} \frac{u_{k-j-1} - u_{k-j-2}}{T}$$

$$+ \sum_{j=0}^{k-2} \int_0^T \Phi(jT + \lambda, 0) (T - \lambda) d\lambda Q_{n,p} B_{p,i} \frac{u_{k-j-1} - u_{k-j-2}}{T}$$

$$+ Q_{n,q} \sum_{i=0}^{q-1} H_q^i B_q,i (u_k^{(i)} - u_k^{(i) - 1})$$

(9)

**Proof.** First of all, for $k = 0$ in (8) we have the case $k = 1$ in (9). We assume that this is true for $k - 1$, that is

$$x_{k-1} = \Phi((k - 1)T, 0)x_0$$

$$+ Q_{n,q} \sum_{i=0}^{q-1} H_q^i B_q,i (u_{k-1}^{(i)} - u_{k-1}^{(i) - 1})$$

$$+ \int_0^{T} \Phi(jT + \lambda, 0) d\lambda Q_{n,p} B_{p,i} u_{k-j-2}$$

$$+ \sum_{j=0}^{k-2} \int_0^T \Phi(jT + \lambda, 0)(T - \lambda) d\lambda Q_{n,p} B_{p,i} \frac{u_{k-j-1} - u_{k-j-2}}{T}$$

and we prove it for $k$. By replacing $x_{k-1}$ in the recursive formula (8), we get

$$x_k = \Phi(T, 0) \left( \Phi((k - 1)T, 0)x_0 + Q_{n,q} \sum_{i=0}^{q-1} H_q^i B_q,i (u_{k-1}^{(i)} - u_{k-1}^{(i) - 1}) \right)$$

$$+ \sum_{j=0}^{k-2} \int_0^T \Phi(jT + \lambda, 0)(T - \lambda)d\lambda Q_{n,p} B_{p,i} \frac{u_{k-j-1} - u_{k-j-2}}{T}$$

or equivalently

$$x_k = \Phi(kT, 0)x_0 + \Phi(T, 0)Q_{n,q} \sum_{i=0}^{q-1} H_q^i B_q,i (u_k^{(i)} - u_k^{(i) - 1})$$

$$+ Q_{n,q} \sum_{i=0}^{q-1} H_q^i B_q,i (u_{k-1}^{(i)} - u_{k-1}^{(i) - 1})$$

$$+ \sum_{j=0}^{k-2} \int_0^T \Phi((j + 1)T + \lambda, 0)d\lambda Q_{n,p} B_{p,i} u_{k-j-2}$$

$$+ \sum_{j=0}^{k-2} \int_0^T \Phi(jT + \lambda, 0)(T - \lambda)d\lambda Q_{n,p} B_{p,i} \frac{u_{k-j-1} - u_{k-j-2}}{T}$$

Now, by setting $i = j + 1$ in order to group similar terms,
we have

\[ x_k = \Phi(kT,0)x_0 + Q_{n,q} \sum_{i=0}^{q-1} H_q B_q.t(u^{(i)}_0 - u^{(i)}_{k-1} + u^{(i)}_{k-1} - u^{(i)}_k) + \sum_{i=1}^{k-1} \int_0^T \Phi(iT + \lambda,0) d\lambda Q_{n,p} B_{p,t} u_{k-i-1} + \int_0^T \Phi(\lambda,0) d\lambda Q_{n,p} B_{p,t} u_{k-1} + \chi_{tf} \sum_{i=1}^{k-1} \int_0^T \Phi(iT + \lambda,0)(T - \lambda) d\lambda Q_{n,p} B_{p,t} \cdot \frac{u_k - u_{k-1}}{T} + \chi_{tf} \int_0^T \Phi(\lambda,0)(T - \lambda) d\lambda Q_{n,p} B_{p,t} \cdot \frac{u_k - u_{k-1}}{T} \]

which completes the induction. ■

3. Error Analysis and Upper Bound

Up to this point, having already found an analytic formula for the discretized solution \( x_k \), we provide an analytic expression for the norm of the difference between the continuous time solution at the moments \( t = kT \) and the discrete points \( x_k \) of the discretized solution. Moreover we bound this norm and we end with two upper bounds for ZOH and interpolating FOH respectively. From (3) and (9) we get

\[ x(kT) - x_k = \int_0^{kT} \Phi(kT,s)Q_{n,p} B_{p,t} u(s) ds - \sum_{j=0}^{k-1} \int_0^{T} \Phi(jT + \lambda,0) Q_{n,p} B_{p,t} \cdot \left( u_{k-j-1} + \chi_{tf}(T - \lambda) \frac{u_{k-j} - u_{k-j-1}}{T} \right) d\lambda \]

or by making the substitution \( T - \lambda = w \),

\[ x(kT) - x_k = \int_0^{kT} \Phi(kT,s)Q_{n,p} B_{p,t} u(s) ds - \sum_{j=0}^{k-1} \int_0^{T} \Phi(jT + \lambda,0) Q_{n,p} B_{p,t} \cdot \left( u_{k-j-1} + \chi_{tf}(T - \lambda) \frac{u_{k-j} - u_{k-j-1}}{T} \right) d\lambda = \sum_{j=0}^{k-1} \int_0^{r(j+1)T} \Phi(kT,s)Q_{n,p} B_{p,t} u(s) ds - \sum_{j=0}^{k-1} \int_0^{r(j+1)T} \Phi((j + 1)T - w,0) Q_{n,p} B_{p,t} \cdot \left( u_{k-j-1} + \chi_{tf}(T - \lambda) \frac{u_{k-j} - u_{k-j-1}}{T} \right) d\lambda \]

By setting \( i = k - j - 1 \) we get

\[ x(kT) - x_k = \sum_{j=0}^{k-1} \int_0^{r(j+1)T} \Phi(kT,s)Q_{n,p} B_{p,t} u(s) ds - \sum_{i=0}^{k-1} \int_0^{T} \Phi((i + 1)T - w,0) Q_{n,p} B_{p,t} \cdot \left( u_i + \chi_{tf}(\lambda - iT) \frac{u_{i+1} - u_i}{T} \right) d\lambda = \sum_{i=0}^{k-1} \int_0^{r(i+1)T} \Phi(kT,s)Q_{n,p} B_{p,t} u(s) ds - \sum_{i=0}^{k-1} \int_0^{r(i+1)T} \Phi((i + 1)T - w,0) Q_{n,p} B_{p,t} \cdot \left( u_i + \chi_{tf}(\lambda - iT) \frac{u_{i+1} - u_i}{T} \right) d\lambda \]

We now set \( \lambda = w + iT \) and we have,

\[ x(kT) - x_k = \sum_{i=0}^{k-1} \int_0^{r(i+1)T} \Phi(kT,s)Q_{n,p} B_{p,t} u(s) ds - \sum_{i=0}^{k-1} \int_0^{r(i+1)T} \Phi((i + 1)T - w,0) Q_{n,p} B_{p,t} \cdot \left( u_i + \chi_{tf}(s - iT) \frac{u_{i+1} - u_i}{T} \right) ds \]
Also, we have that
\[ x(kT) - x_k = \sum_{i=0}^{k-1} \int_{iT}^{(i+1)T} \Phi(kT, s)Q_{n,p}B_{p,i} \cdot \left( u(s) - u_i - \chi_{tf}(s - iT)\frac{u_{i+1} - u_i}{T} \right) ds \]
(10)

Having now in compact form the difference between the continuous and discretized solution we have the following interesting results.

**Theorem 2.** The upper bound of the error of (3) under ZOH (\(\chi_{tf} = 0\)) approximation is given by
\[
\| x(kT) - x_k \| \leq M_1 \| Q_{n,p} \| \| B_{p,i} \| \| Q \| \| Q^{-1} \|
\cdot \left\{ \left( e^{\|J_p\|T} - \|J_p\|T - 1 \right) \left( e^{\|J_p\|kT} - 1 \right) \|J_p\|^2 e^{\|J_p\|T - 1} + \sqrt{q}kT^2 \right\}
\]
while under interpolating FOH (\(\chi_{tf} = 1\)) approximation is given by
\[
\| x(kT) - x_k \| \leq M_1 \| Q_{n,p} \| \| B_{p,i} \| \| Q \| \| Q^{-1} \|
\cdot \left| \int_{iT}^{(i+1)T} \|\Phi(kT, s)\| \|u(s) - u_i\| ds \right|
\]
(11)

**Proof.** For ZOH approximation (\(\chi_{tf} = 0\)) we get,
\[
\| x(kT) - x_k \| \leq \| Q_{n,p} \| \| B_{p,i} \| \cdot \left| \sum_{i=0}^{k-1} \int_{iT}^{(i+1)T} \|\Phi(kT, s)\| |u(s) - u_i| ds \right|
\]
But from Theorem 12.2.3 in (Kenneth R. Davidson, 2010), we have that
\[
\| u(s) - u_i \| \leq |s - iT|\|u'(c)| \quad \text{with} \quad c \in (iT, iT + T)
\]
Also, we have that
\[
\|\Phi(kT, s)\| = \| Q \left[ e^{J_p(kT-s)} \right] \| \| \cdot \| B_{p,i} \|
\leq \left\| \int_{iT}^{(i+1)T} \|\Phi(kT, s)\| ds \right\| \|u(s) - u_i\| \| B_{p,i} \|
\]
and \(e^{\|J_p\|T} \leq e^{\|J_p\|kT}\) and so we finally get,
\[
\| x(kT) - x_k \| \leq M_1 \| Q_{n,p} \| \| B_{p,i} \| \cdot \left| \sum_{i=0}^{k-1} \int_{iT}^{(i+1)T} e^{\|J_p\|s} (s - iT) ds \right|
\]
and finally the upper bound formula for ZOH is
\[
\| x(kT) - x_k \| \leq M_1 \| Q_{n,p} \| \| B_{p,i} \| \| Q \| \| Q^{-1} \|
\cdot \left| \sum_{i=0}^{k-1} \int_{iT}^{(i+1)T} (s - iT) ds \right|
\]
(12)

The polynomial \( u_i + (s - iT)\frac{u_{i+1} - u_i}{T} \) interpolates the function \( u(s) \) and so,
\[
\| u(s) - u_i - (s - iT)\frac{u_{i+1} - u_i}{T} \| \leq \frac{1}{4(n+1)} M_2 \left( \frac{b - a}{n} \right)^{n+1} = \frac{1}{8} M_2 T^2
\]
because \( n = 1 \) and \( b - a = (iT + T) - iT = T \). At this point we have,
\[
\| x(kT) - x_k \| \leq M_1 \| Q_{n,p} \| \| B_{p,i} \| \cdot \left| \sum_{i=0}^{k-1} \int_{iT}^{(i+1)T} \|\Phi(kT, s)\| ds \right|
\]
Finally, because
\[
e^{\|J_p\|T} \sum_{i=0}^{k-1} \int_{iT}^{(i+1)T} e^{-\|J_p\|s} ds =
\]
and finally the upper bound formula for interpolating FOH is,
\[
\| x(kT) - x_k \| \leq \frac{1}{8} M_2 T^2 \| Q_{n,p} \| \| B_{p,i} \| \| Q \| \| Q^{-1} \|
\cdot \left\{ \left( e^{\|J_p\|T} - \|J_p\|T - 1 \right) \left( e^{\|J_p\|kT} - 1 \right) \|J_p\|^2 e^{\|J_p\|T - 1} + \sqrt{q}kT^2 \right\}
\]
Formulas (11) and (12), for ZOH and interpolating FOH respectively, are the upper bounds we wanted to prove.

The difference of these two formulas from the respective formulas of (Karageorgos et al., 2010) and (Karageorgos et al., 2011) is the result of two factors. Firstly, the discretization of the input function $u(t)$ used in this paper is not only zero order hold approximation but, in addition to this, we are also using triangular first order hold discretization. Secondly, a sharp upper bound for $\|\Phi(kT, s)\|$ which appears in both cases (ZOH and interpolating FOH), contributes to a general better result. Now, we can continue to the comparison throughout an example.

4. Illustrative Example

Let us now consider a system of the form $E\dot{x}(t) = Ax(t) + Bu(t)$, that is

$$
\begin{bmatrix}
-1.5 & 2 & 1.5 & 0.5 \\
0.5 & 0 & -0.5 & -0.5 \\
0.5 & -1 & -0.5 & 0.5 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t) \\
\dot{x}_3(t) \\
\dot{x}_4(t)
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
+ \begin{bmatrix}
0 \\
2 \\
1 \\
1
\end{bmatrix} u(t)
$$

Then, there exist nonsingular matrices

$$
P = \begin{bmatrix}
1 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix}
\quad \text{and} \quad
Q = \begin{bmatrix}
1 & 2 & 1 & 1 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1
\end{bmatrix}
$$

such that

$$
P EQ = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\quad \text{and} \quad
PAQ = \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

Since there exists not only one $Q$, $P$ that transforms $sE - A$ to $sE_w - A_w$ and the error depends on $Q$, we may select the one with the least norm. However, we have not proceeded to such details. For this system we have $p = q = 2$, $n = 4$. Assume also that $u(t) = t^3$ $k = 500$ and $T = 10^{-3}$. As a result $M_1 = \|u(\cdot)\|_\infty = \frac{3}{4}$ and $M_2 = \|u^{(2)}(\cdot)\|_\infty = 3$ with $t \in [0, kT]$. Moreover,

$$
Q_{4,2} = \begin{bmatrix}
1 & 2 \\
1 & 1 \\
0 & 0 \\
1 & 0
\end{bmatrix}, \quad B_{2,1} = \begin{bmatrix}
0 \\
2
\end{bmatrix}
$$

Therefore, $\|Q\| = 2\sqrt{3}$, $\|Q^{-1}\| = \frac{\sqrt{3}}{2}$, $\|Q_{4,2}\| = \sqrt{8}$ and $\|B_{2,1}\| = 2$. Applying these values to the formulas (11) and (12), we get that the upper bound for ZOH is $0.02529229$ while for interpolating FOH it is $2.529615 \times 10^{-3}$, about $10^{-3}$ times smaller.

Also, we can estimate the maximum allowed sampling period for which the error does not exceed a given value. For instance if we want the error to not exceed $10^{-2}$ for $k = 100$, for ZOH we get $T_{max} = 0.00153203$ while for interpolating FOH $T_{max} = 0.0110291$. This, proves the fact that due to the better approximation interpolating FOH offers instead of ZOH, we do not need to sample our system that much often in order to be under the maximum error allowed.

Last thing to do, is to compare these two upper bounds as steps $(k)$ increase. In the following table we see the values of the upper bounds for $T = 10^{-3}$. From this table we can see that, although for small $k$ ZOH is quite good, when $k$ increases interpolating FOH is significantly better.

<table>
<thead>
<tr>
<th>$k$</th>
<th>ZOH</th>
<th>FOH</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$4.0659 \times 10^{-6}$</td>
<td>$4.0664 \times 10^{-6}$</td>
</tr>
<tr>
<td>2</td>
<td>$8.1347 \times 10^{-6}$</td>
<td>$8.1357 \times 10^{-6}$</td>
</tr>
<tr>
<td>3</td>
<td>$1.2206 \times 10^{-6}$</td>
<td>$1.2208 \times 10^{-7}$</td>
</tr>
<tr>
<td>4</td>
<td>$1.6281 \times 10^{-6}$</td>
<td>$1.6283 \times 10^{-7}$</td>
</tr>
<tr>
<td>5</td>
<td>$2.0359 \times 10^{-6}$</td>
<td>$2.0361 \times 10^{-7}$</td>
</tr>
<tr>
<td>10</td>
<td>$4.0791 \times 10^{-4}$</td>
<td>$4.0796 \times 10^{-7}$</td>
</tr>
<tr>
<td>100</td>
<td>$0.0042190$</td>
<td>$4.2195 \times 10^{-6}$</td>
</tr>
<tr>
<td>500</td>
<td>$0.025292$</td>
<td>$2.5296 \times 10^{-6}$</td>
</tr>
<tr>
<td>750</td>
<td>$0.043766$</td>
<td>$4.3773 \times 10^{-7}$</td>
</tr>
<tr>
<td>1000</td>
<td>$0.069024$</td>
<td>$6.9037 \times 10^{-8}$</td>
</tr>
</tbody>
</table>

5. Conclusion

In this paper, new upper bound formulas regarding the discretization error of a singular descriptor system are considered. These two bounds differ on the way we approximate the input function, either zero order hold (ZOH) or triangular first order hold (interpolating FOH). In addition to this, the improvements of these sharper bounds stem from the upper bound of matrix $\|\Phi(kT, s)\|$ which conduces to an overall better result from what is already known in (Karageorgos et al., 2011). The whole theory is illustrated by an example. The results presented in this work and (Karageorgos et al., 2011), (Karageorgos et al., 2010) can be further extended to descriptor systems with delay (Jugo, 2002),(Chen and Wang, 1999) or even more to AutoRegressive Moving Average Representations. Alternatively, we can use the fundamental matrix sequence of the matrix pencil $sE-A$, in order to extend the results presented in (Karampetakis and Gregoriadou, 2011) to the Triangular First order hold method and compare with the existing results of this work. Other hold methods, can also be applied as like as the First order hold method (backward-Euler approximation of the derivative.
of the input) that can be combined with several hold methods for the approximation of the derivative of the inputs. Instead of studying the use of zero order hold devices, we can also study by using the same approach that we use in this work, the use of fractional order hold devices (or generalized first order (Jury, 1958)) that can improve, if properly tuned, the performance of hybrid control systems (Basterretxea et al., 2008).

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Nicholas P. Karampetakis

Nicholas P. Karampetakis was born in Drama, Greece in 1967. He received the bachelor degree in Mathematics and Ph.D. in Mathematics from the Department of Mathematics, Aristotle University of Thessaloniki, Greece in 1989 and 1993 respectively. From November 1994 to September 1995 he was a research associate with the Department of Mathematical Sciences, Loughborough University of Technology, England. From September 1995 to March 1999 he was a Research Associate with the Department of Mathematics, Aristotle University of Thessaloniki, Greece. From August 2000, to July 2009 he was an Assistant Professor to the same Department, whereas from July 2009, he has been Associate Professor. During the above periods he received many fellowships from the Greek Government, British Council and Engineering and Physical Sciences Research Council of England, and contributes in many research projects sponsored by the Greek Government and the European Union. His present research interests lie mainly in algebraic methods for computer aided design of control systems (CACSD), numerical and symbolic algorithms for CACSD, polynomial matrix theory and issues related to the mathematical systems theory. Dr. Karampetakis is a frequent contributor to the field in the form of journal articles, conference presentations and reviews. He is Senior Member of IEEE and a vice-Chair of the IEEE Action Group on Symbolic Methods for CACSD. He is also an Associate Editor a) of the Journal of Multidimensional Systems and Signal Processing and b) of the International Journal of Applied Mathematics and Computer Science.
Rallis Karamichalis was born in 1988 in Thessaloniki. He received his Bachelor degree in Mathematics (2010, "Excellent") and his M.Sc. in Theoretical Computer Science and Theory of Systems and Control (2012, "Excellent") from the Department of Mathematics, Aristotle University of Thessaloniki, Greece. He is currently pursuing his PhD in the Department of Computer Science, University of Western Ontario, Canada. He has numerous awards from mathematical competitions, including three medals from National Mathematical Olympiads and a bronze medal from International Mathematical Olympiad in 2006.