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Linearization of bivariate polynomial matrices expressed in non monomial basis

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Abstract The paper proposes a two step algorithm that reduces a bivariate polynomial matrix $T(s,z)$ expressed in Newton or Lagrange base to a bivariate matrix pencil $A+E_1s+E_2z$ with the same invariant polynomials and zero structure.

Keywords : bivariate polynomial matrix, matrix pencil, companion form, unimodular equivalence, zero coprime equivalence, Newton basis, Lagrange basis.

MSC Codes : *Primary* 93B18, *Secondary* 93B17, 93B25, 93B11, 93B40, 93C05, 93C35.

1 Introduction

An interesting problem in linear system analysis, is how to reduce a polynomial matrix to a simpler form as for example a matrix pencil by keeping invariant its zero structure, since this structure is usually connected with several physical system properties like controllability, observability, stability etc..

In the case of one variable polynomial matrices, a number of transformations have been proposed that preserve either the finite zero structure of polynomial matrices i.e. strict equivalence (Gantmacher, 1959), unimodular equivalence (Rosenbrock, 1970), extended unimodular equivalence (Pugh and Shelton, 1978), (Smith, 1981) or both the finite and infinite zero structure of polynomial matrices i.e. complete equivalence (Hayton and Fretwell, 1987), full equivalence (Walker and Pugh, 1990) and full unimodular equivalence (Karampetakis and Vardulakis, 1992). Based on these transformations, several works have been written for the reduction procedure of a polynomial (or system) matrix to a matrix pencil (Walker and Pugh, 1989), (Karampetakis and Vardulakis, 1993), (Vardulakis and Pugh, 1995), (Rosenbrock, 1970), (Wolovich, 1974). The main difference between

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all these approaches is based on the size of the reduced matrix pencil and the computational effort needed for this construction (Karampetakis and Vardulakis, 1993).

In the case of two variable polynomial matrices, transformations have been proposed that preserve either the invariant polynomials i.e. elementary-operation equivalence and unimodular equivalence (Pugh et al, 2005), (Lin et al, 2006), or both the invariant polynomials and the invariant zeros i.e. zero coprime equivalence (Johnson, 1993), (Lévy, 1981), (Boudellioua and Hayton, 1998) and (Johnson and Hayton, 1998). The reduction of an arbitrary bivariate polynomial matrix to a matrix pencil form by such kind of transformations was initially studied in (Johnson and Hayton, 1998), where an algorithm given by (Bosgra and Van Der Weiden, 1981) was extended to the bivariate case. A more direct approach, was used later by (Boudellioua, 2006) and (Pugh et al, 2005), with the proposal of a certain kind of companion forms which are connected with the original polynomial matrix either with the zero coprime equivalence transformation (Boudellioua, 2006) or with the elementary operation equivalence (Pugh et al, 2005). Other attractive approaches that are based on elementary operation equivalence someone can find in (Gałkowski, 1997b), (Gałkowski, 1997a). A new matrix pencil that preserves both the symmetric structure and the structural invariants, of the original 2-D polynomial matrix was later given in (Karampetakis, 2010). The most of the above reduction procedures are two-stage algorithms. Firstly, the polynomial matrix $A(s,z)$ is written as a polynomial in s with coefficient matrices whose elements are in the ring $R[z]$. Then by constructing certain block matrices from these coefficient matrices, a system matrix $T_s(s,z)$ is constructed in a various ways which is linear in s. The second stage of the reduction then follows in an analogous manner to the first and yields a matrix pencil $T_{sz}(s,z)$ which is linear both in s, z. This final matrix pencil is in a form which arises in the context of singular general models of 2-D linear discrete systems as studied by (Kaczorek, 1988).

A problem occurred in case where the polynomial matrix is not expressed in the known polynomial base, but in another base as for example the Newton or the Lagrange base (Amiraslani et al, 2009). In order to overcome this problem, we have first to express the polynomial matrix in the monomial base and then use the known techniques described above. However, we avoid this conversion since it is usually ill conditioned and may decrease the numerical stability of our algorithm (Corless, 2007), (Corless, 2004). Apart of this, we cannot take advantage of the sparsity of the polynomial matrix expressed in one of these two bases. Thus, direct methods have been proposed in (Amiraslani et al, 2009), (Corless and Litt, 2001) in order to solve the matrix pencil reduction problem when our one variable polynomial matrix is expressed either in a Newton or Lagrange base. These methods are especially efficient in cases where a polynomial matrix is a sparse representation in one of these bases. Bases other than the monomials find many applications i.e. the Bernstein-Bezier basis and the Lagrange basis found applications in Computer-Aided Geometric Design, the Legendre polynomials found applications in partial differential equations with symmetries in the boundary conditions etc. (Amiraslani et al, 2009), the Hermit polynomials found applications in polynomial solutions (Aruliah et al, 2007), (Shakoori, 2008).

The main aim of this work is to extend the results presented in (Amiraslani et al, 2009), (Corless and Litt, 2001) for the two-variable polynomial matrix case. More specifically we present in Section 3 (resp. Section 4), a two step reduction algorithm that transforms any two-variable polynomial matrix expressed in a Newton base (resp. Lagrange base), to a matrix pencil of the form $A + E_1 s + E_2 z$. The reduced pencil preserves both the determinantal zeros and the invariant zeros of the original polynomial matrix since it is connected with it, through the zero coprime equivalence transformation.

2 Preliminaries

In this section we give some necessary definitions that we use in the rest of the paper.

Definition 1 (Boudellioua, 2006) (zero left / right coprime) Two polynomial matrices $P_1(s,z)$ and $S_1(s,z)$ of appropriate dimensions are said to be *zero left coprime* if the matrix

$$\left[P_1(s,z)\ S_1(s,z)\right]$$

has a full rank for all $(s,z) \in \mathbb{C}^2$. Similarly, two polynomial matrices $P_2(s,z)$ and $S_2(s,z)$ of appropriate dimensions are said to be *zero right coprime* if the matrix

$$\left[P_2^T(s,z)\ S_2^T(s,z)\right]^T$$

has a full rank for all $(s,z) \in \mathbb{C}^2$.

Some known equivalence transformations that occur between bivariate polynomial matrices are the unimodular equivalence and the zero coprime equivalence transformations.

Definition 2 (Pugh et al, 2005) (unimodular equivalence, (U-E))
If $L(s,z) \in \mathbb{R}[s,z]^{m\times m}$ and $R(s,z) \in \mathbb{R}[s,z]^{n\times n}$ are unimodular matrices i.e. they have non-zero constant determinant, and

$$P_2(s,z) = L(s,z)P_1(s,z)R(s,z)$$

then $P_1(s,z)$, $P_2(s,z) \in \mathbb{R}[s,z]^{m\times n}$ are said to be **unimodular equivalent** (ue).

Definition 3 (Pugh et al, 2005) (zero coprime equivalence, (ZC-E)) Let $\mathcal{P}(m,n)$ be the set of $(r+m)\times(r+n)$ two-variable polynomial matrices with $r < \min(m,n)$. $P_1(s,z)$, $P_2(s,z) \in \mathcal{P}(m,n)$ are **zero coprime equivalent** (zce) if there exist polynomial matrices $L(s,z)$, $R(s,z)$ of appropriate dimensions with $L(s,z)$, $P_2(s,z)$ zero left coprime and $P_1(s,z)$, $R(s,z)$ zero right coprime, such that

$$L(s,z)P_1(s,z) = P_2(s,z)R(s,z)$$

Both *ue* and *zce* are equivalence relations and *ue* is a special case of *zce*. Whereas *ue* connect bivariate polynomial matrices of the same dimension, *zce* is applied to a larger set of polynomial matrices i.e. the set $\mathcal{P}(m,n)$ of matrices with the same difference between the number of rows and columns. However, *ue* and *zce* define the same equivalence class on the set of polynomial matrices with the same dimension (Pugh et al, 2005).

The zero structure of bivariate polynomial matrices is characterized by its invariant polynomials and zeros as defined below.

Definition 4 (Boudellioua, 2006), (Invariant polynomials and zeros) Given a $m\times n$ polynomial matrix $T(s,z)$, the $i-$th order **invariant polynomial** $\Phi_i(s,z)$ of $T(s,z)$ is defined by

$$\Phi_i(s,z)=\begin{cases}\frac{D_i(s,z)}{D_{i-1}(s,z)} & \Big| \; if\ 1\le i\le t,\\ 0 & \Big| \; if\ t\le i\le min(m,n)\end{cases}$$

where t is the normal rank of $T(s,z)$, $D_0(s,z)=1$, and $D_i(s,z)$ is the greatest common divisors of all the $i\times i$ minors of the given matrix $T(s,z)$ ($i-$th determinantal divisor). The zeros of $D_i(s,z)$ are called the $i-$th **determinantal zeros** of $T(s,z)$.The $i-$th ordered **invariant zeros** of $T(s,z)$ are the elements of the variety $\mathcal{V}_{\mathbb{R}}(\mathcal{I}_i^{[P]})$, defined by the ideal $\mathcal{I}_i^{[P]}$ generated by the $i\times i$ minors of $T(s,z)$.

The invariant zeros of two-variable polynomial matrices have no counter part in one-variable case and play a crucial role in control problems of 2-D systems, like stability (Jury, 1978), controllability and observability (Rogers and Owens, 2000), (Wood and Rogers, 2001), (Zerz, 1996). As we see below, *zce* preserves both the determinantal and invariant zero structure of polynomial matrices.

Lemma 1 *(Pugh et al, 2005) Suppose that two $m\times n$ bivariate polynomial matrices $T_1(s,z)$ and $T_2(s,z)$ are related by zero coprime equivalence and let $\Phi_1^{[T_1]}$, $\Phi_2^{[T_1]}$, ..., $\Phi_h^{[T_1]}$, where $h=min(m,n)$, denote the invariant polynomials of $T_1(s,z)$ and $\Phi_1^{[T_2]}$, $\Phi_2^{[T_2]}$, ...,$\Phi_k^{[T_2]}$, where $k=min(m,n)$, denote the invariant polynomials of $T_2(s,z)$. Then*

$$\Phi_{h-i}^{[T_1]}=c_i\Phi_{k-i}^{[T_2]},\ \ for\ i=0,1,\ldots,max(k-1,h-1)$$

where

$$\Phi_j^{[T_1]}=1=\Phi_j^{[T_2]},\ \ for\ any\ j<1,\ c_i\in\mathbb{R}\backslash\{0\}.$$

Lemma 2 *(Pugh et al, 2005) Suppose that two bivariate $m\times n$ polynomial matrices $T_1(s,z)$ and $T_2(s,z)$ are related by zero coprime equivalence and let $\mathcal{I}_j^{[T_1]}$ for $j=1,\ldots,h=min(m,n)$ denote the ideal generated by $j\times j$ minors of $T_1(s,z)$ and $\mathcal{I}_i^{[T_2]}$, for $i=1,\ldots,k=min(m,n)$, denoted the ideal generated by the $i\times i$ minors of $T_2(s,z)$. Then,*

$$\mathcal{I}_{h-i}^{[T_1]}=\mathcal{I}_{k-i}^{[T_2]},\ i=0,\ldots,\overline{h}$$

where

$$\overline{h}=min(h-1,k-1)\quad and\ for\ any\ i>h,$$

$$\mathcal{I}_{h-i}^{[T_1]}=\langle 1\rangle\ or\ \mathcal{I}_{k-i}^{[T_2]}=\langle 1\rangle\quad in\ case\ i<h\ or\ i<k.$$

3 On the reduction of a bivariate polynomial matrix expressed in Newton base to a bivariate matrix pencil

In this section we present a two step algorithm that reduces a bivariate polynomial matrix which is expressed in Newton base to a zero coprime equivalent bivariate matrix pencil.

Let a polynomial matrix $T(s,z) \in \mathbb{R}[s,z]^{m\times n}$ expressed in the Newton base i.e.

$$T(s,z) = \sum_{i=0}^{p}\sum_{j=0}^{q} H_{i,j} N_{i,j}(s,z),$$

where

$$N_{i,j}(s,z) = \prod_{\lambda=0}^{i-1}\prod_{\nu=0}^{j-1}(s-r_\lambda)(z-\varrho_\nu) \tag{1}$$
$$i=0,1,..,p, \quad j=0,1,...,q$$

with r_λ and ϱ_ν distinct interpolation points arbitrary selected and $N_{0,0}(s,z) \equiv 1$. Now, we use a two step reduction algorithm in order to reduce $T(s,z)$ to a *zce* bivariate matrix pencil.

○ **First Step**

Write $T(s,z)$ as a polynomial in the form

$$T(s,z) = A_p(z)(s-r_1)(s-r_2)\dots(s-r_p) + \cdots + A_1(z)(s-r_1) + A_0(z)$$

where $r_i, i=1,...,p$ are distinct interpolation points arbitrary selected.
Consider the matrix

$$T_s(s,z) = \begin{bmatrix} I_n & (r_p-s)I_n & \cdots & 0 & 0 \\ 0 & I_n & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_n & (r_1-s)I_n \\ A_p(z) & A_{p-1}(z) & \cdots & A_1(z) & A_0(z) \end{bmatrix}$$

Lemma 3 $T_E(s,z) = \begin{bmatrix} I_{np} & 0 \\ 0 & T(s,z) \end{bmatrix}$ *is unimodular equivalent to* $T_s(s,z)$.

Proof
There exists two matrices $P^{(1)}(s,z)$ and $Q^{(1)}(s,z)$ such that

$$T_s(s,z) = \underbrace{\begin{bmatrix} I_n & 0 & \cdots & 0 & 0 \\ 0 & I_n & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_n & 0 \\ V_1(z) & V_2(z) & \cdots & V_p(z) & I_m \end{bmatrix}}_{P^{(1)}(s,z)} T_E(s,z) \underbrace{\begin{bmatrix} I_n & (r_p-s)I_n & \cdots & 0 & 0 \\ 0 & I_n & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_n & (r_1-s)I_n \\ 0 & 0 & \cdots & 0 & I_n \end{bmatrix}}_{Q^{(1)}(s,z)}$$

where

$$\begin{aligned} V_1(z) &= A_p(z) \\ V_2(z) &= A_p(z)(s-r_p) + A_{p-1}(z) \\ &\vdots \\ V_{p-1}(z) &= A_p(z)(s-r_3)\dots(s-r_p) + \dots + A_2(z) \\ V_p(z) &= A_p(z)(s-r_2)\dots(s-r_p) + \dots + A_2(z)(s-r_2) + A_1(z). \end{aligned}$$

$P^{(1)}(s,z)$ (resp. $Q^{(1)}(s,z)$) is a $(pn+m)\times(pn+m)$ (resp. $(p+1)n\times(p+1)n$) unimodular matrix. Therefore $T_E(s,z)$ is unimodular equivalent to $T_s(s,z)$.

We conclude from the form of the unimodular transformation used in Lemma 3, that we can always reduce $T_E(s,z)$ to $T_s(s,z)$ by using row-column operations. Now we proceed to the second step of the reduction algorithm.

○ **Second Step**

$T_s(s,z)\in\mathbb{R}\left[s,z\right]^{[pn+m]\times[(p+1)n]}$ is written as a polynomial in z as follows

$$T_s(s,z) = B_q(s)(z-\varrho_1)\dots(z-\varrho_q) + \dots + B_1(s)(z-\varrho_1) + B_0(s).$$

where ϱ_j , $j=1,...,q$ are distinct interpolation points arbitrary selected. Let $T_{sz}(s,z)\in\mathbb{R}\left[s,z\right]^{[pn+m+qn(p+1)]\times[(p+1)(q+1)n]}$

$$T_{sz}(s,z) = \begin{bmatrix} I_{n(p+1)} & (\varrho_q - z)I_{n(p+1)} & \cdots & 0 & 0 \\ 0 & I_{n(p+1)} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_{n(p+1)} & (\varrho_1 - z)I_{n(p+1)} \\ B_q(s) & B_{q-1}(s) & \cdots & B_1(s) & B_0(s) \end{bmatrix}$$

Lemma 4 $T_E'(s,z) = \begin{bmatrix} I_{qn(p+1)} & 0 \\ 0 & T_s(s,z) \end{bmatrix}$ *is unimodular equivalent to* $T_{sz}(s,z)$.

Proof There exists two matrices $P^{(2)}(s,z)$ and $Q^{(2)}(s,z)$ such that

$$T_{sz}(s,z) = \underbrace{\begin{bmatrix} I_{n(p+1)} & 0 & \cdots & 0 & 0 \\ 0 & I_{n(p+1)} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_{n(p+1)} & 0 \\ V_1'(s,z) & V_2'(s,z) & \cdots & V_q'(s,z) & I_{(np+m)} \end{bmatrix}}_{P^{(2)}(s,z)} \times$$

$$\times T_E'(s,z) \underbrace{\begin{bmatrix} I_{n(p+1)} & (\varrho_q - z)I_{n(p+1)} & \cdots & 0 & 0 \\ 0 & I_{n(p+1)} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_{n(p+1)} & (\varrho_1 - z)I_{n(p+1)} \\ 0 & 0 & \cdots & 0 & I_{n(p+1)} \end{bmatrix}}_{Q^{(2)}(s,z)}$$

where

$$\begin{aligned} V_1'(s,z) &= B_q(s) \\ V_2'(s,z) &= B_q(s)(z-\varrho_q)+B_{q-1}(s) \\ &\vdots \\ V_{q-1}'(s,z) &= B_q(s)(z-\varrho_3)\dots(z-\varrho_q)+\dots+B_2(s) \\ V_q'(s,z) &= B_q(s)(z-\varrho_2)\dots(z-\varrho_q)+\dots+B_2(s)(z-\varrho_2)+B_1(s). \end{aligned}$$

$P^{(2)}(s,z)$ (resp. $Q^{(2)}(s,z)$) is a $[qn(p+1)+np+m]\times[qn(p+1)+np+m]$ (resp. $[n(p+1)(q+1)]\times[n(p+1)(q+1)]$) unimodular matrix. So, $T_E'(s,z)$ is unimodular equivalent to $T_{sz}(s,z)$.

From Lemma 3, Lemma 4 and the transitivity property of unimodular-equivalence we can conclude that we can always transform by using row-column operations and trivial inflations, a bivariate polynomial matrix to a unimodular equivalent matrix pencil which possess the same zero structure. However, a direct connection of $T(s,z)$ with the matrix $T_{sz}(s,z)$ is always possible with zero coprime equivalence as we can see in the following Theorem.

Theorem 1 *$T(s,z)$ is zero coprime equivalent to the matrix $T_{sz}(s,z)$.*

Proof There exists two matrices $M(s,z)$ and $N(s,z)$

$$\underbrace{\begin{bmatrix} 0_{(pq+p+q)n\times m} \\ I_m \end{bmatrix}}_{M(s,z)} T(s,z) = T_{sz}(s,z) \underbrace{\begin{bmatrix} (z-\varrho_1)\dots(z-\varrho_q)X(s) \\ (z-\varrho_1)\dots(z-\varrho_{q-1})X(s) \\ \vdots \\ (z-\varrho_1)X(s) \\ X(s) \end{bmatrix}}_{N(s,z)}$$

where

$$X(s) = \begin{bmatrix} (s-r_1)\dots(s-r_p)I_n & (s-r_1)\dots(s-r_{p-1})I_n & \cdots & (s-r_1)I_n & I_n \end{bmatrix}^T.$$

Moreover, the block matrices $\begin{bmatrix} M(s,z) & T_{sz}(s,z) \end{bmatrix}$ and $\begin{bmatrix} T(s,z) \\ N(s,z) \end{bmatrix}$ have full rank for all $(s,z)\in\mathbb{C}^2$. Thus, $M(s,z)$ and $T_{sz}(s,z)$ are zero left coprime and $N(s,z)$ and $T(s,z)$ are zero right coprime. Therefore $T(s,z)$ and $T_{sz}(s,z)$ are zero coprime equivalent.

We note that the matrix pencil $T_{sz}(s,z)$ is of the general form

$$T_{sz}(s,z) = A + E_1 s + E_2 z$$

where A, E_1 and E_2 are constant matrices, the 2-D pencil on which the generalized version of the 2-D singular Roesser system matrix is based (Lewis, 1992), (Uetake and Okubo, 1988).

The zero coprime equivalence of $T(s,z)$ and $T_{sz}(s,z)$ implies that $T(s,z)$ and $T_{sz}(s,z)$ possess the same zero structure i.e. the same invariant polynomials (and thus determinantal zeros) and invariant zeros in the sense of Lemmas 1 and 2 respectively.

Example 1 Consider the bivariate polynomial matrix $T(s,z)$ expressed in the Newton basis i.e.

$$T(s,z) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}(s-1)(s-2)(z-1)(z-2) + \begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix}(s-1)(z-1)(z-2) +$$

$$+ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}(z-1)(z-2) + \begin{bmatrix} 5 & 3 \\ 0 & 0 \end{bmatrix}(s-1)(s-2)(z-1) + \begin{bmatrix} 15 & 9 \\ 1 & 0 \end{bmatrix}(s-1)(z-1) +$$
$$+ \begin{bmatrix} 6 & 3 \\ 1 & 1 \end{bmatrix}(z-1) + \begin{bmatrix} 3 & 2 \\ 0 & 0 \end{bmatrix}(s-1)(s-2) + \begin{bmatrix} 11 & 6 \\ 1 & 1 \end{bmatrix}(s-1) + \begin{bmatrix} 6 & 7 \\ 2 & 3 \end{bmatrix},$$

where $s_1 = 1$, $s_2 = 2$ and $z_1 = 1$, $z_2 = 2$ are distinct interpolation points.

◦ **First Step**

We write $T(s,z)$ as a polynomial in s

$$T(s,z) = \underbrace{\begin{bmatrix} (z-1)(z-2) + 5(z-1) + 3 & (z-1)(z-2) + 3(z-1) + 2 \\ 0 & 0 \end{bmatrix}}_{A_2(z)}(s-1)(s-2) +$$

$$+ \underbrace{\begin{bmatrix} 3(z-1)(z-2) + 15(z-1) + 11 & 3(z-1)(z-2) + 9(z-1) + 6 \\ (z-1) + 1 & 1 \end{bmatrix}}_{A_1(z)}(s-1) +$$
$$+ \underbrace{\begin{bmatrix} (z-1)(z-2) + 6(z-1) + 6 & (z-1)(z-2) + 3(z-1) + 7 \\ (z-1) + 2 & (z-1) + 3 \end{bmatrix}}_{A_0(z)}$$
$$= A_2(z)(s-1)(s-2) + A_1(z)(s-1) + A_0(z).$$

Thus, we take the matrix pencil

$$T_s(s,z) = \begin{bmatrix} I_2 & -(s-2)I_2 & 0 \\ 0 & I_2 & -(s-1)I_2 \\ A_2(z) & A_1(z) & A_0(z) \end{bmatrix}$$

$$= \left[\begin{array}{cc|cc|cc} 1 & 0 & 2-s & 0 & 0 & 0 \\ 0 & 1 & 0 & 2-s & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 1-s & 0 \\ 0 & 0 & 0 & 1 & 0 & 1-s \\ \hline 2z+z^2 & 1+z^2 & 2+6z+3z^2 & 3+3z^2 & 2+3z+z^2 & 6+z^2 \\ 0 & 0 & z & 1 & z+1 & z+2 \end{array}\right].$$

◦ **Second Step**

$T_s(s,z)$ is written as a polynomial in z.

$$T_s(s,z) = \underbrace{\begin{bmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 1&1&3&3&1&1 \\ 0&0&0&0&0&0 \end{bmatrix}}_{B_2(s)} (z-1)(z-2)+$$

$$+\underbrace{\begin{bmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 5&3&15&9&6&3 \\ 0&0&1&0&1&1 \end{bmatrix}}_{B_1(s)} (z-1) + \underbrace{\begin{bmatrix} 1&0&2-s&0&0&0 \\ 0&1&0&2-s&0&0 \\ 0&0&1&0&1-s&0 \\ 0&0&0&1&0&1-s \\ 3&2&11&6&6&7 \\ 0&0&1&1&2&3 \end{bmatrix}}_{B_0(s)}$$

$$= B_2(s)(z-1)(z-2) + B_1(s)(z-1) + B_0(s).$$

So, we take the matrix pencil

$$T_{sz}(s,z) = \begin{bmatrix} I_6 & (2-z)I_6 & 0 \\ 0 & I_6 & (1-z)I_6 \\ B_2(s) & B_1(s) & B_0(s) \end{bmatrix}$$

According to Theorem 1, $T(s,z)$ and $T_{sz}(s,z)$ are zce and thus from Lemmata 1, 2 share the same determinantal and invariant zeros.

4 On the reduction of a bivariate polynomial matrix expressed in Lagrange basis to a bivariate matrix pencil

In this section, following similar lines with the previous section, we reduce a bivariate polynomial matrix expressed in Lagrange basis to a bivariate matrix pencil.

Consider a polynomial matrix $T(s,z) \in \mathbb{R}[s,z]^{m\times n}$ expressed in the Langrange base i.e.

$$T(s,z) = \sum_{i=0}^{p}\sum_{j=0}^{q} T_{ij}L_{ij}(s,z).$$

where

$$L_{ij}(s,z) = L_i(s)\cdot L_j(s) \;\; 0 \le i \le p\,, \;\; 0 \le j \le q$$

with

$$L_i(s) = \prod_{\substack{\lambda=0 \\ \lambda\neq i}}^{p} \frac{(s-s_\lambda)}{s_i - s_\lambda}, \quad L_j(s) = \prod_{\substack{\nu=0 \\ \nu\neq j}}^{q} \frac{(z-z_\nu)}{z_j - z_\nu}$$

and (s_λ, z_ν) are distinct interpolation points that do not intersect with the invariant zeros of $T(s,z)$ i.e. $T(s_\lambda, z_\nu)$ do not loose rank.

◦ **First Step**

Write $T(s,z)$ as a polynomial in the form

$$\begin{aligned}
T(s,z) &= \underbrace{\left[T_{00}(z-z_1)\cdots(z-z_q)w_0^{'}+\cdots+T_{0q}(z-z_1)\cdots(z-z_{q-1})w_q^{'}\right]}_{A_0(z)}\times \\
&\quad \times(s-s_1)\cdots(s-s_p)w_0+\cdots+ \\
&\quad +\underbrace{\left[T_{p0}(z-z_1)\cdots(z-z_q)w_0^{'}+\cdots+T_{pq}(z-z_1)\cdots(z-z_{q-1})w_q^{'}\right]}_{A_p(z)}\times \\
&\quad \times(s-s_0)\cdots(s-s_{p-1})w_p \\
&= A_0(z)(s-s_1)\cdots(s-s_p)w_0++\cdots+(s-s_0)\cdots(s-s_{p-1})w_p
\end{aligned}$$

where

$$w_i=\prod_{\substack{\lambda=0\\ \lambda\neq i}}^{p}\frac{1}{(s_i-s_\lambda)},w_j'=\prod_{\substack{\nu=0\\ \nu\neq i}}^{q}\frac{1}{(z_j-z_\nu)}$$

Consider the matrix $T_s(s,z)\in\mathbb{R}[s,z]^{[n(p+1)+m]\times n(p+2)}$

$$T_s(s,z)=\begin{bmatrix}(s-s_0)I_n & 0 & \cdots & 0 & w_0I_n\\ 0 & (s-s_1)I_n & \cdots & 0 & w_1I_n\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & (s-s_p)I_n & w_pI_n\\ -A_0(z) & -A_1(z) & \cdots & -A_p(z) & 0\end{bmatrix}$$

Lemma 5 *$T(s,z)$ is zero coprime equivalent to $T_s(s,z)$.*

Proof There exists matrices $M^{(1)}(s,z)$ and $N^{(1)}(s,z)$ of appropriate dimensions such that

$$\underbrace{\begin{bmatrix}0_{n(p+1)\times m}\\ I_m\end{bmatrix}}_{M^{(1)}(s,z)}T(s,z)=T_s(s,z)\underbrace{\begin{bmatrix}-w_0(s-s_1)\ldots(s-s_p)I_n\\ \vdots\\ -w_p(s-s_0)\ldots(s-s_{p-1})I_n\\ (s-s_0)\ldots(s-s_p)I_n\end{bmatrix}}_{N^{(1)}(s,z)}$$

Moreover the block matrix

$$\underset{[n(p+1)+m]\times[n(p+2)+m]}{\left[M^{(1)}(s,z)\ T_s(s,z)\right]}=\begin{bmatrix}0 & (s-s_0)I_n & 0 & \cdots & 0 & w_0I_n\\ 0 & 0 & (s-s_1)I_n & \cdots & 0 & w_1I_n\\ & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & (s-s_p)I_n & w_pI_n\\ I_m & -A_0(z) & -A_1(z) & \cdots & -A_p(z) & 0\end{bmatrix}$$

has full row rank since in case where : a) $s\neq s_i$ the first $p+1$ block column matrices gives rise to an invertible matrix, b) $s=s_i$ all the block column matrices except the ones that corresponds to $(s-s_i)I_n$ gives rise to an invertible matrix. Similarly, the block matrix

$$\underbrace{\begin{bmatrix} T(s,z) \\ N^{(1)}(s,z) \end{bmatrix}}_{[m+(p+2)n]\times n} = \begin{bmatrix} T(s,z) \\ \hline -w_0(s-s_1)\dots(s-s_p)I_n \\ \vdots \\ (s-s_0)\dots(s-s_p)I_n \end{bmatrix}$$

has full column rank for all $(s,z)\in\mathbb{C}^2$ since in case where a) $s\neq s_i$ the last $p+1$ block column matrix has full column rank, b) $s=s_i$ one of the $i-th$ (with $i=2,3,...,p+1$) row block matrices has full column rank. Hence, the matrices $T_s(s,z)$ and $M^{(1)}(s,z)$ (resp. $T(s,z)$ and $N^{(1)}(s,z)$) are zero left (resp. right) coprime. So, $T(s,z)$ and $T_s(s,z)$ are zero coprime equivalent which verifies the proof.

◦ **Second Step**

$T_s(s,z)$ is written as a polynomial in z such that

$$T_s(s,z) = \underbrace{\begin{bmatrix} (s-s_0)I_n & 0 & \cdots & 0 & w_0 I_n \\ 0 & (s-s_1)I_n & \cdots & 0 & w_1 I_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & (s-s_p)I_n & w_p I_n \\ -T_{00} & -T_{11} & \cdots & -T_{p0} & 0 \end{bmatrix}}_{B_0(s)} (z-z_1)\cdots(z-z_q)\, w_0^{'} + \cdots +$$

$$+ \underbrace{\begin{bmatrix} (s-s_0)I_n & 0 & \cdots & 0 & w_0 I_n \\ 0 & (s-s_1)I_n & \cdots & 0 & w_1 I_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & (s-s_p)I_n & w_p I_n \\ -T_{0q} & -T_{1q} & \cdots & -T_{pq} & 0 \end{bmatrix}}_{B_q(s)} (z-z_0)\cdots(z-z_{q-1})\, w_q^{'} =$$

$$= B_0(s)\,(z-z_1)\cdots(z-z_q)\, w_0^{'} + \cdots + B_q(s)\,(z-z_0)\cdots(z-z_{q-1})\, w_q^{'}$$

Consider the $[n(p+2)(q+1)+n(p+1)+m]\times[n(p+2)(q+2)]$ matrix

$$T_{sz}(s,z) = \begin{bmatrix} (z-z_0)I_{n(p+2)} & 0 & \cdots & 0 & w_0' I_{n(p+2)} \\ 0 & (z-z_1)I_{n(p+2)} & \cdots & 0 & w_1' I_{n(p+2)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & (z-z_q)I_{n(p+2)} & w_q' I_{n(p+2)} \\ -B_0(s) & -B_1(s) & \cdots & -B_q(s) & 0 \end{bmatrix}$$

Lemma 6 *$T_s(s,z)$ is zero coprime equivalent to $T_{sz}(s,z)$.*

Proof There exists $M^{(2)}(s,z)$ and $N^{(2)}(s,z)$ of appropriate dimensions such that

$$\underbrace{\begin{bmatrix} 0_{[n(p+2)(q+1)]\times[n(p+1)+m]} \\ I_{[n(p+1)+m]} \end{bmatrix}}_{M^{(2)}(s,z)} T_s(s,z) = T_{sz}(s,z) \underbrace{\begin{bmatrix} -w_0'(z-z_1)\dots(z-z_q)I_{n(p+2)} \\ \vdots \\ -w_q'(z-z_0)\dots(z-z_{q-1})I_{n(p+2)} \\ (z-z_0)\dots(z-z_q)I_{n(p+2)} \end{bmatrix}}_{N^{(2)}(s,z)}$$

Following similar lines with the proof of Lemma 5 we can show that the block matrices have full rank for all $(s,z) \in \mathbb{C}^2$. Thus $T_s(s,z)$ and $T_{sz}(s,z)$ are zero coprime equivalent.

From Lemma 5, Lemma 6 and the transitivity property of zero coprime equivalence we can conclude that $T_s(s,z)$ and $T_{sz}(s,z)$ are zero coprime equivalent. However, a direct connection between $T(s,z)$ an $T_{sz}(s,z)$ is given by the following Theorem.

Theorem 2 *$T(s,z)$ is zero coprime equivalent to the matrix $T_{sz}(s,z)$.*

Proof There exists two matrices $M^{(3)}(s,z)$ and $N^{(3)}(s,z)$ such that

$$\underbrace{\begin{bmatrix} 0_{[(pq+2p+2q+3)n]\times m} \\ I_m \end{bmatrix}}_{M^{(3)}(s,z)} T(s,z) = T_{sz}(s,z) \underbrace{\begin{bmatrix} -w_0'(z-z_1)\dots(z-z_q)I_{n(p+2)}X(s) \\ \vdots \\ -w_q'(z-z_0)\dots(z-z_{q-1})I_{n(p+2)}X(s) \\ (z-z_0)\dots(z-z_q)I_{n(p+2)}X(s) \end{bmatrix}}_{N^{(3)}(s,z)}$$

where

$$X(s) := \begin{bmatrix} -w_0(s-s_1)\dots(s-s_p)I_n \\ \vdots \\ -w_p(s-s_0)\dots(s-s_{p-1})I_n \\ (s-s_0)\dots(s-s_p)I_n \end{bmatrix}.$$

Moreover, the block matrices $\begin{bmatrix} M^{(3)}(s,z) \; T_{sz}(s,z) \end{bmatrix}$ and $\begin{bmatrix} T(s,z) \\ N^{(3)}(s,z) \end{bmatrix}$ have full rank for all $(s,z) \in \mathbb{C}^2$, for similar reasons that have been explained in Lemma 5. Thus, $T(s,z)$ and $T_{sz}(s,z)$ are zero coprime equivalent.

Example 2 Consider the bivariate polynomial matrix in Lagrange base

$$T(s,z) = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} \frac{(s-2)(s-3)}{(1-2)(1-3)} \frac{(z-2)(z-3)}{(1-2)(1-3)} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{(s-1)(s-3)}{(2-1)(2-3)} \frac{(z-1)(z-3)}{(2-1)(2-3)} + \\ + \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix} \frac{(s-1)(s-2)}{(3-1)(3-2)} \frac{(z-1)(z-2)}{(3-1)(3-2)}$$

with distinct interpolation points $s_0 = 1$, $s_1 = 2$, $s_2 = 3$ and $z_0 = 1$, $z_1 = 2$ $z_2 = 3$.

◦ **First Step**

We write $T(s, z)$ as a polynomial in s

$$T(s,z) = \underbrace{\begin{bmatrix} 2(z-2)(z-3) & 0 \\ 0 & 0 \end{bmatrix}}_{A_0(z)} \frac{(s-2)(s-3)}{2} +$$

$$+ \underbrace{\begin{bmatrix} -(z-1)(z-3) & 0 \\ 0 & -(z-1)(z-3) \end{bmatrix}}_{A_1(z)} \frac{(s-1)(s-3)}{-1} +$$

$$+ \underbrace{\begin{bmatrix} 0 & 2(z-1)(z-2) \\ 2(z-1)(z-2) & 0 \end{bmatrix}}_{A_2(z)} \frac{(s-1)(s-2)}{2}$$

where $w_0 = \frac{1}{2}$, $w_1 = -1$ and $w_2 = \frac{1}{2}$. Thus, we take the matrix pencil

$$T_s(s,z) = \begin{bmatrix} (s-s_0)I_2 & 0 & 0 & w_0 I_2 \\ 0 & (s-s_1)I_2 & 0 & w_1 I_2 \\ 0 & 0 & (s-s_2)I_2 & w_2 I_2 \\ -A_0(z) & -A_1(z) & -A_2(z) & 0 \end{bmatrix}$$

◦ **Second Step**

$T_s(s, z)$ is written as a polynomial in z.

$$T_s(s,z) = \underbrace{\begin{bmatrix} s-1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & s-1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & s-2 & 0 & 0 & 0 & -1 & \frac{1}{2} \\ 0 & 0 & 0 & s-2 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & s-3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s-3 & 0 & \frac{1}{2} \\ -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{B_0(s)} \frac{(z-2)(z-3)}{2} +$$

$$+ \underbrace{\begin{bmatrix} s-1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & s-1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & s-2 & 0 & 0 & 0 & -1 & \frac{1}{2} \\ 0 & 0 & 0 & s-2 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & s-3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s-3 & 0 & \frac{1}{2} \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}}_{B_1(s)} \frac{(z-1)(z-3)}{-1}$$

$$+\underbrace{\begin{bmatrix} s-1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & s-1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & s-2 & 0 & 0 & 0 & -1 & \frac{1}{2} \\ 0 & 0 & 0 & s-2 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & s-3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s-3 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 & 0 & 0 \end{bmatrix}}_{B_2(s)} \frac{(z-1)(z-2)}{2}$$

where $w'_0 = \frac{1}{2}$, $w'_1 = -1$ and $w'_2 = \frac{1}{2}$. So, we take the matrix pencil

$$T_{sz}(s,z) = \begin{bmatrix} (z-z_0)I_8 & 0 & 0 & w'_0 I_8 \\ 0 & (z-z_1)I_8 & 0 & w'_1 I_8 \\ 0 & 0 & (z-z_2)I_8 & w'_2 I_8 \\ -B_0(s) & -B_1(s) & -B_2(s) & 0 \end{bmatrix}$$

According to Theorem 2, $T(s,z)$ and $T_{sz}(s,z)$ possess the same determinantal and invariant zeros.

5 Conclusions

A two step reduction procedure has been described that transforms a bivariate $m \times n$ polynomial matrix expressed in Newton (Lagrange) base to a bivariate $[pn+m+qn(p+1)] \times [(p+1)(q+1)n]$ (resp. $[n(p+2)(q+1)+n(p+1)+m] \times [n(p+2)(q+2)]$) matrix pencil with the same zero structure by extending the results presented in (Amiraslani et al, 2009), (Corless and Litt, 2001) for the univariate case. By following the same methodology, we can work in other bases (Amiraslani et al, 2009).

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