Proper and Stable, Minimal MacMillan Degree Bases of Rational Vector Spaces

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Abstract—**The structure of proper and stable bases of rational vector spaces is investigated. We prove that if $\mathfrak{I}(s)$ is a rational vector space, then among the proper bases of $\mathfrak{I}(s)$ there is a subfamily of proper bases which are 1) stable, 2) have no zeros in $\mathbb{C}\cup\{\infty\}$ and therefore are column (row) reduced at infinity, and 3) their MacMillan degree is minimum among the MacMillan degrees of all other proper bases of $\mathfrak{I}(s)$ and it is given by the sum of the MacMillan degrees of their columns taken separately. It is shown that this notion is the counterpart of Forney's concept of a minimal polynomial basis for the family of proper and stable bases of $\mathfrak{I}(s)$.**

I. INTRODUCTION

In this paper, the structure of proper and stable bases of rational vector spaces is investigated. The algebraic structure of the set of all proper rational vectors, which have no poles inside a "forbidden" region Ω of the finite complex plane, and which are also contained in a given rational vector space $\mathfrak{I}(s)$, is known to be that of a Noetherian $\mathbb{R}_{\mathcal{P}}(s)$-module M^* [1]-[3] [$\mathbb{R}_{\mathcal{P}}(s)$: the ring of rational function with no poles in $\mathcal{P} := \Omega \cup \{\infty\}$]. The structure of the various bases of M^* is examined via the notion of a "*simple*" basis T_1 of M^* [3]. A simple basis of M^* has the property that its MacMillan degree is given by the sum of the MacMillan degrees of its columns taken separately. Based on this concept, the existence and construction of "simple, proper, and Ω-stable" bases of $\mathfrak{I}(s)$ having minimal MacMillan degree among all other proper bases of $\mathfrak{I}(s)$ is established. These proper bases we call "simple, minimal, MacMillan degree, proper, and Ω-stable" bases of $\mathfrak{I}(s)$, and it is shown that this notion is the counterpart to Forney's [4] concept of a minimal polynomial basis for $\mathfrak{I}(s)$ for the case of the $\mathbb{R}_{\mathcal{P}}(s)$-module M^*.

II. NOTATION AND PRELIMINARIES

Let $\mathbb{R}$ be the field of reals, $\mathbb{R}[s]$ the ring of polynomials, $\mathbb{R}(s)$ the field of rational functions and $\mathbb{R}_{pr}(s)$ the ring of proper rational functions. Let Ω be a region of the finite complex plane $\mathbb{C}$, symmetrically located with respect to the real axis and which excludes at least one point $\alpha \in \mathbb{R}$, and let Ω^c be the complement of Ω with respect to $\mathbb{C}$, i.e., $\mathbb{C} = \Omega\cup\Omega^c$. Let $t \in \mathbb{R}(s)$, and factorize it as: $t = (n_\Omega \cdot \hat{n})/(d_\Omega \cdot \hat{d})$ where $n_\Omega, d_\Omega \in \mathbb{R}[s]$ coprime with all their zeros not outside Ω and $\hat{n}, \hat{d} \in \mathbb{R}[s]$ coprime with all their zeros outside Ω. Let $\mathcal{P} := \Omega\cup\{\infty\}$, and denote by $\mathbb{R}_{\mathcal{P}}(s)$ the subring of $\mathbb{R}(s)$ consisting of all $t \in \mathbb{R}(s)$ with no poles in $\mathcal{P}$, i.e., of "proper and Ω-stable" rational functions. It is known [5], [6] that with "*degree*" function $\delta_{\mathcal{P}}(\cdot)$: $\mathbb{R}_{\mathcal{P}}(s) \to \mathbb{Z}\cup\{+\infty\}$ defined by $\delta_{\mathcal{P}}(t) := \deg \cdot \hat{d} - \deg \cdot \hat{n} \geq 0$, $\delta_{\mathcal{P}}(0) := +\infty$, $\mathbb{R}_{\mathcal{P}}(s)$ is a Euclidean domain. Two matrices $(T_1, T_2) \in \mathbb{R}^{p\times m}(s) \times \mathbb{R}^{p\times m}(s)$ are called "*equivalent in* $\mathcal{P}$" if there exist $\mathbb{R}_{\mathcal{P}}(s)$-unimodular matrices T_L, T_R such that $T_L T_1 T_R = T_2$. Consequently, every $T \in \mathbb{R}^{p\times m}(s)$ is equivalent in $\mathcal{P}$ to its "Smith–MacMillan form in $\mathcal{P}$" [8]: $S_T^{\mathcal{P}}$ whose invariant factors ϵ_i/ψ_i, $\epsilon_i \in \mathbb{R}_{\mathcal{P}}(s)$, $\psi_i \in \mathbb{R}_{\mathcal{P}}(s)$ give the pole-zero structure of T in $\mathcal{P}$ (see [7], [8], [1] for details). The following is a direct consequence of the above.

Proposition 1 [8]: Let $A \in \mathbb{R}_{\mathcal{P}}^{l\times m}(s)$, $B \in \mathbb{R}_{\mathcal{P}}^{t\times m}(s)$ with $p := l + t \geq m$. Then the following statements are equivalent: 1) A and B are *right coprime in* $\mathcal{P}$; 2) $T := \begin{bmatrix} A \\ B \end{bmatrix} \in \mathbb{R}_{\mathcal{P}}^{p\times m}(s)$ *has no zeros in* $\mathcal{P}$, 3) there exists an $\mathbb{R}_{\mathcal{P}}(s)$-unimodular matrix $T_L \in \mathbb{R}_{\mathcal{P}}^{p\times p}(s)$ such that $T_L T = \begin{bmatrix} I_m \\ O \end{bmatrix} = S_T^{\mathcal{P}}$; 4) there exist $X \in \mathbb{R}_{\mathcal{P}}^{m\times l}(s)$, $Y \in \mathbb{R}_{\mathcal{P}}^{m\times t}(s)$ such that $XA + YB = I_m$; and 5) $\operatorname{rank}_{\mathbb{C}} \begin{bmatrix} A(s_0) \\ B(s_0) \end{bmatrix} = m \ \forall s_0 \in \Omega$ *and* $\lim_{s\to\infty} \begin{bmatrix} A(s) \\ B(s) \end{bmatrix} = E \in \mathbb{R}^{p\times m}$ with $\operatorname{rank}_{\mathbb{R}} E = m$.

A matrix $T \in \mathbb{R}_{\mathcal{P}}^{p\times m}(s)$ satisfying the equivalent conditions of Proposition 1 is called $\mathbb{R}_{\mathcal{P}}(s)$-*left unimodular*. Notice that a $\mathbb{R}_{\mathcal{P}}(s)$-left unimodular matrix might have only finite poles and zeros in Ω^c. If $T \in \mathbb{R}_{\mathcal{P}}^{p\times m}(s)$, $T_1 \in \mathbb{R}_{\mathcal{P}}^{p\times m}(s)$, $T_R \in \mathbb{R}_{\mathcal{P}}^{m\times m}(s)$ are related via

$$T = T_1 T_R \tag{1}$$

then T_R is called a *right divisor in* $\mathcal{P}$ of (the rows of) T. If $p \geq m$ and $\operatorname{rank}_{\mathbb{R}(s)} T = m$, then any $T_R \in \mathbb{R}_{\mathcal{P}}^{m\times m}(s)$ that satisfies (1) for some $\mathbb{R}_{\mathcal{P}}(s)$-left unimodular matrix $T_1 \in \mathbb{R}_{\mathcal{P}}^{p\times m}(s)$ is called a *greatest (common) right divisor in* $\mathcal{P}$ *of (the rows of)* T. Notice, that in such a case, T_R "contains" all the zeros of T in $\mathcal{P}$ [8]. Finally, for a $T \in \mathbb{R}_{\mathcal{P}}^{p\times m}(s)$ with $\operatorname{rank}_{\mathbb{R}(s)} T = r$ we define: $\delta_{\mathcal{P}}(T) = \min\{\delta_{\mathcal{P}}(\cdot)$, among the $\delta_{\mathcal{P}}(\cdot)$'s of all rth-order minors of $T\}$ if $r > 0$ and $\delta_{\mathcal{P}}(T) := +\infty$ if $r = 0$. It can be easily verified that if $p = m$ then $\delta_{\mathcal{P}}(T)$ represents the total number of zeros of T in $\mathcal{P}$ (see [8] for details).

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III. Proper and Ω-Stable, Minimal MacMillan Degree Bases of Rational Vector Spaces

We examine now the algebraic structure of the set of all $t \in \mathbb{R}_{\mathcal{P}}^{p\times 1}(s)$ which are contained in the rational vector space $\mathfrak{J}(s)$ spanned by the columns $t_j \in \mathbb{R}^{p\times 1}(s)$, $j \in \mathbf{m}$ of a general rational matrix $T \in \mathbb{R}^{p\times m}(s)$. First, the existence of proper and Ω-stable bases for $\mathfrak{J}(s)$ follows from the results in [9], [10], i.e., if $T \in \mathbb{R}^{p\times m}(s)$ is a basis for $\mathfrak{J}(s)$ and $T = BA^{-1}$ is a "fractional representation" of T where $B \in \mathbb{R}_{\mathcal{P}}^{p\times m}(s)$, $A \in \mathbb{R}_{\mathcal{P}}^{m\times m}(s)$ then clearly B is a proper and Ω-stable basis for $\mathfrak{J}(s)$. Thus, let $T_1 \in \mathbb{R}_{\mathcal{P}}^{p\times m}(s)$ be a basis for $\mathfrak{J}(s)$ and consider the $\mathbb{R}_{\mathcal{P}}(s)$-module M_1, generated by (the columns of) T_1. If T_1 is not $\mathbb{R}_{\mathcal{P}}(s)$-left unimodular, then by extracting from it non-$\mathbb{R}_{\mathcal{P}}(s)$-unimodular right divisors in $\mathcal{P}$: T_{1R}, T_{2R}, $\cdots T_{iR} \cdots \in \mathbb{R}_{\mathcal{P}}^{m\times m}(s)$ such that $0 < \delta_{\mathcal{P}}(T_{1R}) < \delta_{\mathcal{P}}(T_{2R}) < \cdots < \delta_{\mathcal{P}}(T_{iR})$ and $T_1 = T_{i+1}T_{iR}$ for some (not necessarily $\mathbb{R}_{\mathcal{P}}(s)$-left unimodular) $T_{i+1} \in \mathbb{R}_{\mathcal{P}}^{p\times m}(s)$, $i = 1, 2, \cdots$, then the $\mathbb{R}_{\mathcal{P}}(s)$-modules M_{i+1}, $i = 1, 2, \cdots$ generated by (the columns of) T_{i+1}, $i = 1, 2, \cdots$ form an ascending sequence of submodules [1]–[3]: $M_1 \subset M_2 \subset \cdots \subset M_{i+1}$. If now for some $i = 1, 2, \cdots$ $T_{iR} =: T_R$ is a $g(c\cdot)R\cdot D\cdot$ in $\mathcal{P}$ of T_1, so that $T_1 = \hat{T}T_R$ for some $\mathbb{R}_{\mathcal{P}}(s)$-left unimodular $\hat{T} \in \mathbb{R}_{\mathcal{P}}^{p\times m}(s)$, then the $\mathbb{R}_{\mathcal{P}}(s)$-module generated by $\hat{T}$, and which we denote by M^*, satisfies an ascending chain condition on submodules [11], i.e., $M_1 \subset M_2 \subset \cdots \subset M_{i+1} = M^*$ for some $i = 1, 2, \cdots$ and coincides with the set of all proper and Ω-stable rational vectors $t \in \mathbb{R}_{\mathcal{P}}^{p\times 1}(s)$ which are contained in $\mathfrak{J}(s)$ [1], [2].

In the sequel, we examine the structure of the various ($\mathbb{R}_{\mathcal{P}}(s)$-left unimodular) bases of M^*. As we show, these bases can be further classified according to properties of their MacMillan degree $\delta_M(\cdot)$. In order to proceed, we need a few more known results. We start with the following.

Proposition 2 [3]: Let $T \in \mathbb{R}_{pr}^{p\times m}(s)$, $p \geq m$, $\operatorname{rank}_{\mathbb{R}(s)} T = m$ *be column reduced at* $s = \infty$ [12], [13]. Let $d_j \in \mathbb{R}[s]$ be the monic least common multiple of the p denominators which appear in the jth column t_j of T and write: $t_j = n_j/d_j$, $n_j \in \mathbb{R}^{p\times 1}[s]$, $j \in \mathbf{m}$. Let $N := [n_1, \cdots, n_m] \in \mathbb{R}^{p\times m}[s]$ and $D := \operatorname{diag}(d_1, \cdots, d_m) \in \mathbb{R}^{m\times m}[s]$. Then

$$\sum_{j=1}^{m} \delta_M(t_j) \geq \delta_M(T) \tag{2}$$

with equality holding iff N, D are right coprime.

Proof: See [3, Proposition 15].

Definition 1 [3]: A column reduced at $s = \infty$ matrix $T = [t_1, \cdots, t_m]$ $\mathbb{R}_{pr}^{p\times m}(s)$ with $p \geq m$ and $\operatorname{rank}_{\mathbb{R}(s)} T = m$ which satisfies (2) with equality is called a "*simple*" basis of the $\mathbb{R}_{pr}(s)$-module M generated by t_j.

Remark 1: It can be easily seen[1] that given a column reduced at $s = \infty$ basis $T \in \mathbb{R}_{pr}^{p\times m}(s)$ of M, then we can always determine an $\mathbb{R}_{pr}(s)$-unimodular (biproper) $U_R \in \mathbb{R}_{pr}^{m\times m}(s)$ such that $\hat{T} = TU_R$ is column reduced at $s = \infty$ and *simple* basis of M with $\delta_M(\hat{T}) = \delta_M(T)$.

Lemma 1: Let 1) $T \in \mathbb{R}_{pr}^{p\times m}(s)$, $p \geq m$, $\operatorname{rank}_{\mathbb{R}(s)} T = m$ and 2) $\lim_{s\to\infty} T = E \in \mathbb{R}^{p\times m}$ with $\operatorname{rank}_{\mathbb{R}} E = m$. Let $v_j \geq 0$, $j \in \mathbf{m}$ the (Forney) invariant dynamical indexes of the rational vector space $\mathfrak{J}(s)$ spanned by the columns of T, $\operatorname{ord}_F(\mathfrak{J}(s)) := \sum_{j=1}^{m} v_j$ the (Forney) invariant dynamical order of T and $Z_f(T)$ the number of finite zeros of T. Then

$$\delta_M(T) = \operatorname{ord}_F(\mathfrak{J}(s)) + Z_f(T). \tag{3}$$

Proof: 1) and 2) imply, respectively, that T has no poles and no zeros at $s = \infty$ [3]. Hence, $\delta_M(T)$ is the number of finite poles of T. The lemma then follows as a particular case of a more general result (see [12], and [14, Theorem 2.1], or [15, Theorem 3] or [3, Corollary 8].

Finally, we will need the following.

Lemma 2: Let $T_i \in \mathbb{R}_{pr}^{p\times m}(s)$, with $\lim_{s\to\infty} T_i =: E_i \in \mathbb{R}^{p\times m}$ and $\operatorname{rank}_{\mathbb{R}} E_i = m$, $i = 1, 2$. If there exist a $Q \in \mathbb{R}^{m\times m}(s)$ such that $T_1 = T_2Q$, then $Q \in \mathbb{R}_{pr}^{m\times m}(s)$ and $\lim_{s\to\infty} Q = Q_0 \in \mathbb{R}^{m\times m}$ with $\operatorname{rank}_{\mathbb{R}} Q_0 = m$.

Having introduced the above concepts and results, we return now to the examination of the various $\mathbb{R}_{\mathcal{P}}(s)$-left unimodular bases of the maximal $\mathbb{R}_{\mathcal{P}}(s)$-module M^*. First, and unlike the proper submodules M_i of M^*, all bases of M^* are column reduced at $s = \infty$, since (by the definition of M^*) they are all $\mathbb{R}_{\mathcal{P}}(s)$-left unimodular (see [3, Proposition 9]). Second, and due to the above fact, it also follows from Remark 1 that if $T \in \mathbb{R}_{\mathcal{P}}^{p\times m}(s)$ is a basis of M^*, then we can always determine an $\mathbb{R}_{\mathcal{P}}(s)$-unimodular matrix $U_R \in \mathbb{R}_{\mathcal{P}}^{m\times m}(s)$ such that $\hat{T} := TU_R \in \mathbb{R}_{\mathcal{P}}^{p\times m}(s)$ is a *simple* basis of M^* which satisfies: $\delta_M(\hat{T}) = \delta_M(T)$. The following theorem, which is our main result, proves the fact that given a basis T of M^* we can always determine an $\mathbb{R}_{\mathcal{P}}(s)$-unimodular matrix $U_R \in \mathbb{R}_{\mathcal{P}}^{m\times m}(s)$ such that $\hat{T} := TU_R^{-1}$ is a *simple* basis of M^* which has *desired* poles (in Ω^c) and whose MacMillan degree $\delta_M(\hat{T})$ is *minimum* among the MacMillan degrees of all other proper or proper and Ω-stable bases of $\mathfrak{J}(s)$, i.e., that $\delta_M(\hat{T}) \leq \delta_M(T)$ $\forall$ $T \in \mathbb{R}_{pr}^{p\times m}(s)$ basis of $\mathfrak{J}(s)$. We thus prove that if $\mathfrak{J}(s)$ is a rational vector space, then among the proper and Ω-stable bases of $\mathfrak{J}(s)$ which have no zeros in $\mathcal{P}$ (i.e., they are $\mathbb{R}_{\mathcal{P}}(s)$-left unimodular) there is a subfamily of *simple* $\mathbb{R}_{\mathcal{P}}(s)$-left unimodular bases which have any desired set of poles and whose MacMillan degrees are minimum among the MacMillan degrees of all other proper bases of $\mathfrak{J}(s)$. This result is formally stated in the following.

Theorem 1: Let $T \in \mathbb{R}_{\mathcal{P}}^{p\times m}(s)$, $p \geq m$, be $\mathbb{R}_{\mathcal{P}}(s)$-left unimodular and let M^* be the $\mathbb{R}_{\mathcal{P}}(s)$-module generated by its columns $t_j(s)$. Then $T(s)$ can always be factorized (in a nonunique way) as

$$T = \hat{T}U_R$$

where $\hat{T} = [\hat{t}_1, \cdots, \hat{t}_m] \in \mathbb{R}_{\mathcal{P}}^{p\times m}(s)$ is $\mathbb{R}_{\mathcal{P}}(s)$-left unimodular and *simple* basis of M^* which has no *finite zeros* and $U_R(s) \in \mathbb{R}_{\mathcal{P}}^{m\times m}(s)$ is $\mathbb{R}_{\mathcal{P}}(s)$-unimodular and the set of its finite zeros[2] contains as a subset the set of finite zeros (which, if any, are in Ω^c) of T. Furthermore, if $v_j \geq 0$, $j \in \mathbf{m}$, are the (Forney) *invariant dynamical indexes* of the rational vector space $\mathfrak{J}(s)$ spanned by the columns of T, and also of $\hat{T}$, $\operatorname{ord}_F(\mathfrak{J}(s)) := \sum_{j=1}^{m} v_j$ is the (Forney *invariant dynamical order* of $\mathfrak{J}(s)$ then 1) $\delta_M(\hat{t}_j) = v_j$, $j \in \mathbf{m}$, and 2) $\delta_M(\hat{T}) = \sum_{j=1}^{m} \delta_M(\hat{T}_j) = \sum_{j=1}^{m} v_j = \operatorname{ord}_F(\mathfrak{J}(s))$ and $\delta_M(\hat{T})$ is minimum among the MacMillan degrees of all other proper bases of $\mathfrak{J}(s)$.

Proof: Let $T = ND^{-1}$, $N \in \mathbb{R}^{p\times m}[s]$, $D \in \mathbb{R}^{m\times m}[s]$ be a right coprime *MFD* of T, where, due to the assumption that T is $\mathbb{R}_{\mathcal{P}}(s)$-left unimodular, the finite zeros of N and D are confined in Ω^c, and factorize N as $N = N_LN_R$ where $N_R \in \mathbb{R}^{m\times m}[s]$ is a $g\cdot c\cdot r\cdot d\cdot$ of (the rows of) N and N_L is a minimal polynomial basis of the rational vector space $\mathfrak{J}(s)$ [4] (i.e., N_L is 1) column proper and 2) has relatively right prime rows [4], [16]). If $n_j(s) = [n_{1j}, \cdots, n_{pj}]^T \in \mathbb{R}^{p\times 1}[s]$, $j \in \mathbf{m}$, are the columns of N_L then, by definition [4], $\deg\cdot n_j := \max_{i\in\mathbf{p}} \{\deg\cdot n_{ij}\} = v_j$, $j \in \mathbf{m}$.

Let now $\hat{D} := \operatorname{diag}(\hat{d}_1, \cdots, \hat{d}_m)$ with $\deg\cdot\hat{d}_j = v_j$ and $\hat{d}_j$ arbitrary monic polynomials with zeros in Ω^c. Then, T can be written as

$$T = [N_L\hat{D}^{-1}][\hat{D}N_RD^{-1}] := \hat{T}U_R. \tag{4}$$

Now, since $p \geq m$ and $N_L(s)$ is a minimal basis: $\operatorname{rank}_{\mathbb{R}} [N_L]_c^h = m$[3] and $\operatorname{rank}_{\mathbb{C}} [N_L^T, \hat{D}^T]^T = \operatorname{rank}_{\mathbb{C}} N_L = m$ for every $s \in \mathbb{C}$, i.e., N_L, $\hat{D}$ are right coprime. By construction, $\hat{T} := N_L\hat{D}^{-1}$ is proper and *simple* and $\lim_{s\to\infty} \hat{T} = [N_L]_c^h$. Therefore, $\hat{T}$ is proper and has no zeros in $\mathbb{C} \cup \{\infty\}$. Also, by construction, $\hat{T}$ has no poles in Ω, therefore, $\hat{T} \in \mathbb{R}_{\mathcal{P}}^{p\times m}(s)$ is $\mathbb{R}_{\mathcal{P}}(s)$-left unimodular with no finite zeros at all. From Lemma 2, it follows that $U_R := \hat{D}N_RD^{-1}$ has $\lim_{s\to\infty} U_R =: Q_0 \in \mathbb{R}^{m\times m}$ with $\operatorname{rank}_{\mathbb{R}} Q_0 = m$. However, U_R has all its finite zeros and poles in Ω^c and so $U_R \in \mathbb{R}_{\mathcal{P}}^{m\times m}(s)$ is $\mathbb{R}_{\mathcal{P}}(s)$-unimodular and some of its zeros, i.e., the zeros of N_R are zeros of T. Now, $\delta_M(\hat{t}_j) := \deg\cdot\hat{d}_j = v_j$ and $\sum_{j=1}^{m} \delta_M(\hat{t}_j) = \sum_{j=1}^{m} v_j =: \operatorname{ord}_F(\mathfrak{J}(s)) = \deg\cdot\det\hat{D} =: \delta_M(\hat{T})$. Finally, to see that $\delta_M(\hat{T})$ is a minimum, notice that from Lemma 1 we have that for every proper basis T of $\mathfrak{J}(s)$ with $\lim_{s\to\infty} T(s) = E$ and $\operatorname{rank}_{\mathbb{R}} E = m$, $\delta_M(T) = \operatorname{ord}_F(\mathfrak{J}(s)) + Z_f(T)$ where now $Z_f(T) \geq 0$ denotes the number of finite zeros (if any) of $T(s)$ in Ω^c. Hence, from 2) it follows that $\delta_M(\hat{T}) \leq \delta_M(T)$ for every proper basis $T(s)$ of $\mathfrak{J}(s)$.

Definition 2: An $\mathbb{R}_{\mathcal{P}}(s)$*-left unimodular and simple* basis $\hat{T} \in \mathbb{R}_{\mathcal{P}}^{p\times m}(s)$ of M^* which has no finite zeros (and thus satisfies 1) and 2) of Theorem 1) is defined as a *simple, minimal MacMillan degree, proper* and Ω*-stable basis* of the rational vector space $\mathfrak{J}(s)$ spanned by its columns [SMMD-$\mathbb{R}_{\mathcal{P}}(s)$ basis].

[1] See proof and notation of Theorem 1 below and write: $T = ND^{-1} = N\hat{D}^{-1}(D\hat{D}^{-1})^{-1} = \hat{T}U_R^{-1}$.

[2] Notice that an $\mathbb{R}_{\mathcal{P}}(s)$-unimodular matrix has (possibly) only finite poles and zeros not outside Ω^c.

[3] By $[N]_c^h$ we denote the "highest column degree coefficient matrix" of $N(s) \in \in \mathbb{R}^{p\times m}[s]$ [16].

IV. Conclusions

In this paper, we have investigated the algebraic structure of the set of all rational vectors which are contained in a given rational vector space $\mathfrak{I}(s)$ and which also have no poles in a set $\mathcal{P} := \Omega \cup \{\infty\}$, i.e., they are "proper and Ω-stable." Relying on the fact that the above set is an $R_{\mathcal{P}}(s)$-module M^*, and by investigating the structure of the various bases of M^*, we have established the notion of "simple, proper, Ω-stable and minimal MacMillan degree" bases of $\mathfrak{I}(s)$ as the counterpart to Forney's concept of minimal polynomial bases of $\mathfrak{I}(s)$, for the case of the $R_{\mathcal{P}}(s)$-module M^*.

These concepts and results can be used for the resolution of algebraic control problems which involve questions of properness, stability, and/or minimality of solutions of rational matrix equations [17]. For example, the "stable exact model matching problem" (SEMMP) can be tackled and the difficulties involved in the construction of solutions to the stable minimal design problem (SMDP) can be elucidated [18].

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