INT. J. CONTROL, 1991, VOL. 53, NO. 2, 431–443

Computation of the inverse of a polynomial matrix and evaluation of its Laurent expansion

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An algorithm is given which constitutes a generalization of the algorithm for the inversion of a pencil $sE-A$ due to Mertzios (1984) in the case of general systems described in differential operator form. Recursive formulae are obtained for the calculation of both the coefficient matrices of the adjoint of the polynomial matrix, as well as for the coefficients of the characteristic polynomial of the polynomial matrix. A simple method is also presented that allows the evaluation of the Laurent expansion for the inverse of a polynomial matrix of any order. Specifically, an ARMA model is given for the computation of the coefficient matrices of the Laurent expansion in terms of the coefficient matrices of the polynomial matrix, without inverting the latter. The Laurent expansion of a polynomial matrix is used for the analysis and synthesis of the polynomial matrix descriptions that constitute a further generalization of the singular (or generalized) systems.

1. Introduction

This paper gives an algorithm that computes the inverse of a polynomial matrix $A(s)$ in terms of the system's matrices and the determinant $|A(s)|$. This algorithm is a generalization of the Leverrier type algorithm presented by Mertzios (1984) for generalized or singular systems of the form:

$$\left.\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}\right\} \tag{1.1}$$

where $E \in \mathbb{R}^{r\times r}$ is a singular matrix. The importance of Mertzios (1984) is that he introduced an auxilliary matrix sequence that allows a recursive solution of the algorithm reducing simultaneously the computational cost. Also an algorithm for the calculation of a generalized system is presented by Lewis (1986).

The paper also refers to the problem of the evaluation of the Laurent expansion around infinity of the inverse of a polynomial matrix. The main motivation behind this effort is the development of an algorithm for the calculation of the fundamental matrix sequence of polynomial matrix descriptions (PMDs). PMDs are described by the model due to Callier and Desoer (1982) and Kailath (1980):

$$\left.\begin{aligned} A(\rho)x(t) &= B(\rho)u(t) \\ y(t) &= C(\rho)x(t) \end{aligned}\right\} \tag{1.2}$$

Received 10 October 1989. Revised 7 June 1990.

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where $\rho = d/dt$ is the differential operator, $A(\rho)$ is an $r \times r$ non-singular (over the field $\mathbb{R}(\rho)$ of real rational functions in ρ) polynomial matrix having the form:

$$A(\rho) = A_{q_1}\rho^{q_1} + A_{q_1-1}\rho^{q_1-1} + \dots + A_1\rho + A_0 \tag{1.3}$$

where $A_i \in \mathbb{R}^{r \times r}$, $i = 0, 1, 2, \dots, q_1 > 0$ and A_{q_1} possibly singular, $x(t) \in \mathbb{R}^r$ is the pseudo state, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^l$ are vector valued functions and $B(\rho) \in \mathbb{R}^{r \times m}[\rho]$, $C(\rho) \in \mathbb{R}^{l \times r}[\rho]$ are polynomial matrices.

PMDs actually constitute a further generalization of generalized or singular state-space systems. Therefore PMDs may be considered as the more general model for linear systems. The class of state-space systems constitute a special case of singular systems and therefore of PMDs.

The importance of the fundamental matrix sequence for the analysis and synthesis of state-space systems is well known. Moreover, the definition of the fundamental matrix sequence for singular systems has been given by Mertzios and Lewis (1989) and an efficient algorithm for its computation has been proposed. In this paper, the fundamental matrix sequence for PMDs is defined and a recursive algorithm is developed for its computation in terms of the system's matrices.

In § 2 we give some background results regarding the relations between the Smith–McMillan form at $s = \infty$ and the Laurent expansion of a polynomial matrix around infinity. Note that the notion of the Smith–McMillan form of a rational matrix at $s = \infty$, as well as an improved calculation procedure for finding this form, was introduced by Vardulakis *et al.* (1982). In § 3 the algorithm for the computation of the inverse of a polynomial matrix is given. In § 4 the proposed recursive algorithm for the computation of the fundamental matrix sequence of the PMD systems is presented.

2. Background material

The next proposition and its corollaries generalize to the matrix case some well-known properties of scalar rational functions that will be used in what follows.

Proposition 1 (Vardulakis and Fragulis 1990)

Let $A(s) \in \mathbb{R}(s)^{r \times m}$, $\text{rank}_{\mathbb{R}} A(s) = r$ and let $S^{\infty}_{A(s)}(s)$ be the Smith–McMillan form of $A(s)$ at $s = \infty$, i.e.

$$S^{\infty}_{A(s)}(s) = \text{block diag}\left[s^{q_1}, s^{q_2}, \dots, s^{q_k}, \frac{1}{s^{\hat{q}_{k+1}}}, \dots, \frac{1}{s^{\hat{q}_r}}, 0_{p-r, m-r}\right] \tag{2.1}$$

where $1 \leqslant k \leqslant r$ and $\hat{q}_i = -q_i$, $i = k+1, \dots, r$, so that $q_1 \geqslant q_2 \geqslant \dots \geqslant q_k \geqslant 0$ are the orders of the poles of $A(s)$ at $s = \infty$ and $\hat{q}_r \geqslant \hat{q}_{r-1} \geqslant \dots \geqslant \hat{q}_{k+1} \geqslant 0$ are the orders of the zeros of $A(s)$ at $s = \infty$. Let $q_1 > 0$, i.e. assume that $A(s)$ is non-proper, and let

$$A(s) = A_k s^k + A_{k-1}s^{k-1} + \dots + A_1 s + A_0 + A_{-1}s^{-1} + A_{-2}s^{-2} + \dots \tag{2.2}$$

be the Laurent expansion at $s = \infty$ of $A(s)$ where $A_i \in \mathbb{R}^{r \times m}$, $i = k - j$,

$j = 0, 1, 2, \ldots, A_k \neq 0, k > 0$ and $A_{k+l} = 0, l = 1, 2, \ldots$ Then it may be proved that

$$k = q_1 \tag{2.3}$$

Corollary 1 (Vardulakis and Fragulis 1990)

Let $A(s) \in \mathbb{R}^{r \times m}[s]$ and write it as a matrix polynomial, i.e. write

$$A(s) = A_k s^k + A_{k-1} s^{k-1} + \ldots + A_1 s + A_0 \tag{2.4}$$

$A_i \in \mathbb{R}, i = 0, 1, 2, \ldots, k > 0$. Let $S^{\infty}_{A(s)}(s)$ be as in (2.1). Then

$$k = q_1 \tag{2.5}$$

i.e.

$$A(s) = A_{q_1} s^{q_1} + A_{q_1 - 1} s^{q_1 - 1} + \ldots + A_1 s + A_0 \tag{2.6}$$

Corollary 2 (Vardulakis and Fragulis 1990)

Let $A(s) \in \mathbb{R}^{r \times r}[s]$ and write it as a matrix polynomial as in (2.4), let $\text{rank}_{\mathbb{R}} A(s) = r$ and let $S^{\infty}_{A(s)}$ be given by (2.1), $r = m$. Let $\hat{q}_r > 0$, i.e. assume that $A(s)$ has at least one zero at $s = \infty$ of order $|\hat{q}_r|$. Let

$$A^{-1}(s) = H_v s^v + H_{v-1} s^{v-1} + \ldots + H_1 s + H_0 + H_{-1} s^{-1} + H_{-2} s^{-2} + \ldots$$
$$= \sum_{i=-v}^{\infty} H_{-i} s^{-i} \tag{2.7}$$

be the Laurent expansion of $A^{-1}(s)$ at $s = \infty$, where $H_i \in \mathbb{R}^{r \times r}$, $i = v - j$, $j = 0, 1, 2, \ldots, H_v \neq 0$ and $H_{v+j} = 0$ for all $j = 1, 2, \ldots, v > 0$. Then

$$v = \hat{q}_r \tag{2.8}$$

i.e.

$$A^{-1}(s) = H_{\hat{q}_r} s^{\hat{q}_r} + H_{\hat{q}_r - 1} s^{\hat{q}_r - 1} + \ldots + H_1 s + H_0 + H_{-1} s^{-1} + H_{-2} s^{-2} + \ldots \tag{2.9}$$

Definition 1

The coefficient matrices H_j, $j \leq v$ in (2.9) constitute the *fundamental matrix sequence* of the polynomial matrix $A(s)$.

3. Calculation of the inverse of the polynomial matrix $A(s)$

Now a recently developed algorithm is given that calculates the inverse of a polynomial matrix $A(s)$ in terms of the coefficients of the adjoint matrix and of the determinant of $A(s)$.

Consider the linear time-invariant multivariable system in (1.2). If we take the Laplace transform with zero initial conditions, i.e. $\beta^{(i)}(0^-) = 0, i = 0, 1, \ldots, q_1 - 1$, of (1.1) we obtain the following:

$$\left.\begin{aligned} A(s)X(s) &= B(s)U(s) \\ Y(s) &= C(s)X(s) \end{aligned}\right\} \tag{3.1}$$

which gives rise to the transfer function matrix

$$H(s) = C(s)A^{-1}(s)B(s) \tag{3.2}$$

If we take the Laplace transform of (1.2) we obtain

$$A(s) = A_{q_1}s^{q_1} + \dots + A_1 s + A_0 \tag{3.3}$$

Therefore, in actuality the problem of computing $H(s)$ is reduced to the calculation of $A^{-1}(s)$ in terms of the A_i, $i = 0, 1, \dots, q_1$.

The transfer function (3.2) can be written as

$$\begin{aligned} H(s) &= C(s)[Iz - [Iz - A(s)]]^{-1}B(s) \\ &= C(s)[Iz - \bar{A}]^{-1}B(s) \end{aligned} \tag{3.4}$$

where

$$\bar{A} = Iz - A(s) = Iz - (A_{q_1}s^{q_1} + \dots + A_1 s + A_0) \tag{3.5}$$

and z is a new *pseudo-variable*, which does not affect $H(s)$ since it is always eliminated. Now the Leverrier's algorithm (see Zadeh and Desoer 1963, p. 303) can be applied to compute the inverse of the matrix $Iz - \bar{A}$. Then the transfer function $H(s)$ can be expanded as

$$H(s) = p^{-1}(s)C(s)[z^{r-1}R_0(s) + z^{r-2}R_1(s) + \dots + zR_{r-2}(s) + R_{r-1}(s)]B(s) \tag{3.6}$$

where

$$p(s) = z^r + p_1(s)z^{r-1} + p_2(s)z^{r-2} + \dots + p_r(s) = \det[Iz - \bar{A}] = \det[A(s)] \tag{3.7}$$

and

$$\left.\begin{aligned} &R_0(s) = I_r, && p_1(s) = -\operatorname{tr}[\bar{A}] \\ &R_1(s) = \bar{A}R_0(s) + p_1 I_r, && p_2(s) = -\frac{1}{2}\operatorname{tr}[\bar{A}R_1(s)] \\ &R_2(s) = \bar{A}R_1(s) + p_2 I_r, && p_3(s) = -\frac{1}{3}\operatorname{tr}[\bar{A}R_2(s)] \\ &\quad\vdots && \quad\vdots \\ &R_{r-1}(s) = \bar{A}R_{r-2}(s) + p_{r-1}I_r, && p_r(s) = -\frac{1}{r}\operatorname{tr}[\bar{A}R_{r-1}(s)] \end{aligned}\right\} \tag{3.8}$$

The matrices $R_i(s) \in \mathbb{R}^{r\times r}[\rho]$, $i = 0, 1, \dots, r-1$ can also be computed by the following expression:

$$R_i(s) = \bar{A}^i + p_1(s)\bar{A}^{i-1} + p_2(s)\bar{A}^{i-2} + \dots + p_i(s)I_r \tag{3.9}$$

The matrices $R_i(s)$ are no longer the coefficient matrices of the powers of s, but depend on the variable s itself. This can be seen from (3.8), since the matrix $\bar{A}$ depends on s.

Since $H(s)$ in (3.6) is independent of z, for the sake of simplicity we take $z = 0$. Therefore the relations (3.5)–(3.7) can be written:

$$\bar{A} = -A(s) = -(A_{q_1}s^{q_1} + \dots + A_1 s + A_0) \tag{3.10}$$

$$H(s) = -C(s)\bar{A}^{-1}(s)B(s) = p^{-1}(s)C(s)R_{r-1}(s)B(s) \tag{3.11}$$

$$p(s) = p_r(s) \tag{3.12}$$

It is seen from (3.3) and (3.8) that $R_i(s)$ and $p_i(s)$ can be written as:

$$R_i(s) = \sum_{v=0}^{iq_1} R_{iv} s^v \quad i = 0, 1, \ldots, r-1 \tag{3.13}$$

$$p_i(s) = \sum_{k=0}^{iq_1} p_{ik} s^k \quad i = 1, 2, \ldots, r \tag{3.14}$$

where R_{ik}, p_{ik} are constant coefficient matrices and scalars of the powers of s^k, respectively. It is seen from (3.11) that for the computation of $H(s)$ we need only the quantities $R_{r-1}(s)$, $p_r(s)$, i.e. the coefficient matrices $R_{r-1,k}$ and the coefficients p_{rk} defined by

$$R(s) = R_{r-1}(s) = \text{adj}\, \bar{A} = \sum_{k=0}^{(r-1)q_1} R_{r-1,k} s^k \tag{3.15}$$

and

$$p(s) = p_r(s) = \det[\bar{A}(s)] = \sum_{k=0}^{rq_1} p_{rk} s^k \tag{3.16}$$

Substituting (3.13) and (3.14) in the recursive relations (3.8), taking into account (3.10) and equating the coefficients of the powers of s in the two sides of each equation, we obtain the following recursive equations that determine $p_{i+1,k}$, $R_{i,k}$, $i = 0, 1, \ldots, r-1$, $k = 1, 2, \ldots, iq_1$.

Initialize:

$$R_{00} = I_r \tag{3.17}$$

Boundary conditions:

$$R_{ik} = 0, \quad \text{for } k > iq_1 \quad \text{and} \quad i = 0, 1, \ldots, r-1 \tag{3.18}$$

(*a*) *Recursive relation for* $p_i(s)$:

$$p_{i+1,k} = \frac{1}{(i+1)} \operatorname{tr}\left[\sum_{j=0}^{k} A_j R_{i,k-j}\right] \quad \text{if } k = 0, 1, \ldots, (i+1)q_1 \tag{3.19}$$

for $i = 0, 1, \ldots, r-1$.

(*b*) *Recursive relation for* $R_i(s)$:

$$R_{i+1,k} = -\left[\sum_{j=0}^{k} A_j R_{i,k-j}\right] + p_{i+1,k} I_r, \quad \text{if } k = 0, 1, \ldots, (i+1)q_1 \tag{3.20}$$

for $i = 0, 1, \ldots, r-2$.

Terminate:

$$R_k = R_{r-1,k} \quad k = 0, 1, \ldots, (r-1)q_1 \tag{3.21}$$

$$p_k = p_{r,k} \qquad k = 0, 1, \ldots, rq_1 \tag{3.22}$$

It is readily seen that the inversion algorithm is a two-dimensional algorithm, since it depends on the two independent variables i, k.

The formulae (3.19) and (3.20) are readily reduced to the Leverrier-type algorithm for singular systems (equations (14), (15) of Mertzios 1984) if we assume that $q_1 = 1$, i.e. when the polynomial matrix is a singular pencil $A(s) = A_1 s + A_0$.

Example 1

Let a multivariable system have the form (1.2), or in the s-domain the form (3.1), where

$$A(s) = \begin{bmatrix} 1 & s^3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s \end{bmatrix}$$

For this system we have $q_1 = 3$, $r = 3$ and therefore we are seeking $R_2(s)$ and $p_3(s)$ in order to evaluate $A^{-1}(s)$. Applying the presented algorithm, we obtain

$$R_{00} = I_3 \quad R_{10} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad R_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_{12} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad R_{13} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_{20} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \quad R_{22} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad R_{24} = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad R_{25} = R_{26} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$p_{10} = 1 \quad p_{21} = 2 \quad \text{and} \quad p_{22} = p_{23} = p_{24} = p_{25} = p_{26} = 0$$

$$p_{30} = 0 \quad p_{31} = 1 \quad \text{and} \quad p_{32} = p_{33} = p_{34} = p_{35} = p_{36} = p_{37} = p_{38} = p_{39} = 0$$

Hence

$$R_2(s) = R_{20} + R_{21}s + R_{22}s^2 + R_{23}s^3 + \dots + R_{26}s^6$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} s + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} s^2 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} s^3$$

$$+ \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} s^4 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} s^5 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} s^6$$

$$= \begin{bmatrix} s & -s^4 & s^4 \\ 0 & s & -s \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$p_3(s) = p_{30} + p_{31}s + \dots + p_{39}s^9 = s$$

Finally it is found that

$$A^{-1}(s) = [Iz - \bar{A}] = -\bar{A}^{-1} = \frac{1}{s}\begin{bmatrix} s & -s^4 & s^4 \\ 0 & s & -s \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -s^3 & s^3 \\ 0 & 1 & -1 \\ 0 & 0 & \frac{1}{s} \end{bmatrix}$$

4. Evaluation of the Laurent expansion

In this section we derive an explicit formula for the evaluation of the Laurent expansion of $A^{-1}(s)$ of the form in terms of the matrices A_i, $i = 0, 1, \dots, q_1$.

By definition, it holds that

$$A^{-1}(s) = \det(A(s))^{-1} \cdot \operatorname{adj}(A(s)) = p_r^{-1}(s)R_{r-1}(s) \tag{4.1}$$

or equivalently

$$p_r(s)A^{-1}(s) = R_{r-1}(s) \tag{4.2}$$

Substituting $R_{r-1}(s)$, $p_r(s)$ and $A^{-1}(s)$ by (3.14), (3.15) and (2.7), respectively, in (4.2) and equating the coefficient matrices of each power of s, we obtain the following relations:

$$\left.\begin{aligned} p_{r,rq_1}H_\nu &= 0 \\ p_{r,rq_1}H_{\nu-1} + p_{r,rq_1-1}H_\nu &= 0 \\ &\vdots \\ p_{r,rq_1}H_{-q_1+1} + p_{r,rq_1-1}H_{-q_1} + \dots + p_{r,(r-1)q_1-(\nu-1)}H_\nu &= 0 \end{aligned}\right\} \tag{4.3}$$

$$\left.\begin{aligned} p_{r,rq_1}H_{-q_1} + p_{r,rq_1-1}H_{-q_1+1} + \dots + p_{r,(r-1)q_1-\nu}H_\nu &= R_{r-1,(r-1)q_1} \\ p_{r,rq_1}H_{-q_1-1} + p_{r,rq_1-1}H_{-q_1} + \dots + p_{r,(r-1)q_1-(\nu+1)}H_\nu &= R_{r-1,(r-1)q_1-1} \\ &\vdots \\ p_{r,rq_1}H_{-rq_1} + p_{r,rq_1-1}H_{-rq_1+1} + \dots + p_{r,0}H_0 &= R_{r-1,0} \end{aligned}\right\} \tag{4.4}$$

$$\left.\begin{aligned} p_{r,rq_1}H_{-rq_1-1} + p_{r,rq_1-1}H_{-rq_1} + \dots + p_{r,0}H_{-1} &= 0 \\ p_{r,rq_1}H_{-rq_1-2} + p_{r,rq_1-1}H_{-rq_1-1} + \dots + p_{r,0}H_{-2} &= 0 \\ &\vdots \end{aligned}\right\} \tag{4.5}$$

Equations (4.3) can be written in matrix form as

$$\begin{bmatrix} p_{r,rq_1} & 0 & \cdots & 0 \\ p_{r,rq_1-1} & p_{r,rq_1} & \cdots & 0 \\ \vdots & & & \vdots \\ p_{r,(r-1)q_1-(\nu-1)} & \cdots & & p_{r,rq_1} \end{bmatrix} \begin{bmatrix} H_\nu \\ H_{\nu-1} \\ \vdots \\ H_{-q_1+1} \end{bmatrix} = 0 \tag{4.6}$$

The first relation gives $p_{r,rq_1}H_\nu = 0$. Since we have already considered that $H_\nu \neq 0$, this results in

$$p_{r,rq_1} = 0 \tag{4.7}$$

Similarly, if we take the second equation of (4.5), taking into account (4.7) and that $H_v \neq 0$, we have the result

$$p_{r,rq_1-1}=0 \tag{4.8}$$

Taking similar observations for the rest of the equations of (4.6), we finally arrive at the following results:

$$p_{r,i}=0, \quad i=rq_1, rq_1-1, \ldots, (r-1)q_1-(v-1) \tag{4.9}$$

In the light of (4.9), (2.18) can be rewritten

$$p_r(s)=\sum_{k=0}^{(r-1)q_1-v} p_{r,k}s^k \tag{4.10}$$

which is the characteristic polynomial of $A(s)$. Hence we obtain the following corollary.

Corollary 3

Let $A(s)$ be a polynomial matrix as in (2.6). Then the degree μ of the characteristic polynomial of its inverse $A(s)$ is

$$\mu=(r-1)q_1-v \tag{4.11}$$

Using (4.9), the equations (4.4) can be written in matrix form as:

$$P \times H = R \tag{4.12}$$

where

$$\left.\begin{aligned}
P&=\begin{bmatrix}
p_{r,(r-1)q_1-v} & 0 & & & 0\\
p_{r,(r-1)q_1-(v+1)} & p_{r,(r-1)q_1-v} & 0 & \cdots & 0\\
\vdots & \vdots & & & \vdots\\
p_{r,0} & p_{r,1} & p_{r,(r-1)q_1-v} & & 0\\
0 & p_{r,0} & \ddots & & \\
\vdots & \ddots & & \ddots & \\
0 & \cdots & 0 & p_{r,0} & p_{r,(r-1)q_1-v}
\end{bmatrix}\\
H&=\begin{bmatrix}
H_v\\ H_{v-1}\\ \vdots\\ H_{v-[(r-1)q_1-v]}\\ \vdots\\ H_{-[(r-1)q_1-v]}
\end{bmatrix} \quad \text{and} \quad R=\begin{bmatrix}
R_{r-1,(r-1)q_1}\\ R_{r-1,(r-1)q_1-1}\\ \vdots\\ R_{r-1,v}\\ \vdots\\ R_{r-1,0}
\end{bmatrix}
\end{aligned}\right\} \tag{4.13}$$

Since $R_{r-1,(r-1)q_1}\neq 0$ (see (3.21)) and $H_v \neq 0$, from the first of the above equations we have that

$$p_{r,(r-1)q_1-v}\neq 0 \tag{4.14}$$

Thus the square Toeplitz matrix P in (4.12) is always non-singular. Hence we can find a unique solution $H=P^{-1}R$ that determines the first $(r-1)q_1+1$ matrices H_i, $i=v, v-1, \ldots, -[(r-1)q_1-v]$. Owing to the Toeplitz form of P, the inverse P^{-1} may be written in the form (Mertzios and Lewis 1989):

$$D = P^{-1} = \begin{bmatrix} d_0 I_r & 0 & \dots & 0 \\ d_1 I_r & d_0 I_r & \dots & 0 \\ \vdots & \vdots & & \\ d_{(r-1)q_1} I_r & d_{(r-1)q_1-1} I_r & \dots & d_0 I_r \end{bmatrix} \tag{4.15}$$

where

$$d_0 = \frac{1}{p_{r,(r-1)q_1-v}} \tag{4.16}$$

and

$$\left.\begin{aligned} d_j = (-1)^j \frac{1}{p_{r,(r-1)q_1-v}} \\ \times \begin{bmatrix} p_{r,(r-1)q_1-(v+1)} & p_{r,(r-1)q_1-(v+2)} & \dots & p_{r,(r-1)q_1-(v+j)} \\ p_{r,(r-1)q_1-v} & p_{r,(r-1)q_1-(v+1)} & \dots & p_{r,(r-1)q_1-(v+j-1)} \\ 0 & p_{r,(r-1)q_1-v} & \dots & p_{r,(r-1)q_1-(v+j-2)} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdot\; p_{r,(r-1)q_1-v} & p_{r,(r-1)q_1-(v+1)} \end{bmatrix} \\ \text{for } j = 1, 2, \dots, (r-1)q_1 \end{aligned}\right\} \tag{4.17}$$

Using (4.15), (4.12) gives the following equation:

$$H = P^{-1} R \tag{4.18}$$

which gives the first $(r-1)q_1 + 1$ matrices H_i, $i = v, v-1, \dots, -[(r-1)q_1 + 1]$. The matrices H_j, $j = -[(r-1)q_1 - v + 1], -[(r-1)q_1 - v + 2], \dots, -\infty$, can be found recursively from the equations (4.5), which, in the light of (4.9), can be written in matrix form:

$$[p_{r,0} \;\; p_{r,1} \;\; \cdots \;\; p_{r,(r-1)q_1-v}] \begin{bmatrix} H_{-1} & H_{-2} & \dots \\ H_{-2} & H_{-3} & \dots \\ \vdots & \vdots & \\ H_{-[(r-1)q_1-v+1]} & H_{-[(r-1)q_1-v+2]} & \dots \end{bmatrix} = 0 \tag{4.19}$$

The first equation of (4.19) gives

$$p_{r,0} H_{-1} + p_{r,1} H_{-2} + \dots + p_{r,(r-1)q_1-v} H_{-[(r-1)q_1-v+1]} = 0 \tag{4.20}$$

In (4.20) we know already the matrices H_k, $k = -1, -2, \dots, -[(r-1)q_1 - v]$. Also, from (4.14) it is known that $p_{r,(r-1)q_1-v} \neq 0$. Therefore (4.20) may be solved for $H_{-[(r-1)q_1-v+1]}$ as follows:

$$\begin{aligned} H_{-[(r-1)q_1-v+1]} = & -\frac{p_{r,0}}{p_{r,(r-1)q_1-v}} H_{-1} - \frac{p_{r,1}}{p_{r,(r-1)q_1-v}} H_{-2} - \dots \\ & -\frac{p_{r,(r-1)q_1-(v-1)}}{p_{r,(r-1)q_1-v}} H_{-[(r-1)q_1-v]} \end{aligned} \tag{4.21}$$

If we take the second equation of (4.19) and make similar considerations as above we obtain

$$H_{-[(r-1)q_1-v+2]} = -\frac{p_{r,0}}{p_{r,(r-1)q_1-v}} H_{-2} - \frac{p_{r,1}}{p_{r,(r-1)q_1-v}} H_{-3} - \cdots$$
$$- \frac{p_{r,(r-1)q_1-(v+1)}}{p_{r,(r-1)q_1-v}} H_{-[(r-1)q_1-v+1]} \tag{4.22}$$

The general recursive relation for the evaluation of the matrices H_k, $k = -[(r-1)q_1 - v + 1], -[(r-1)q_1 - v + 2], \ldots, -\infty$ is given by the following ARMA model:

$$H_{-[\mu+k]} = \sum_{i=0}^{\mu-1} \frac{p_{r,i}}{p_{r,m}} H_{-[i+k]} \tag{4.23}$$

for $k = 1, 2, \ldots, +\infty$ where $\mu = (r-1)q_1 - v$.

Example 2

Let

$$A(s) = \begin{bmatrix} s+1 & s^3 \\ 0 & s+1 \end{bmatrix}$$

The Smith–McMillan form (Vardulakis *et al.* 1982) is

$$S^{\infty}_{A(s)} = \begin{bmatrix} s^3 & 0 \\ 0 & 1/s \end{bmatrix}$$

Hence $q_1 = 3$, $r = 2$, $\hat{q}_2 = v = 1$.

We shall determine the matrices H_i, $i = 1, 0, -1, -2, \ldots, -\infty$, using the proposed technique. It is found using the algorithm (3.17)–(3.22) that

$$R_1(s) = R_{10} + R_{11}s + R_{12}s^2 + R_{13}s^3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} s + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} s^2 + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} s^3$$
$$= \begin{bmatrix} s+1 & -s^3 \\ 0 & s+1 \end{bmatrix}$$

and

$$p_2(s) = p_{20} + p_{21}s + p_{22}s^2 = 1 + 2s + s^2 = (s+1)^2$$

From the analysis of the method it is confirmed that

$$p_{2i} = 0$$

for $i = rq_1, \ldots, (r-1)q_1 - (v-1)$ or $i = 6, 5, 4, 3$. Hence $p_{23} = p_{24} = p_{25} = p_{26} = 0$. From (4.12) we obtain

$$\begin{bmatrix} p_{22} & 0 & 0 & 0 \\ p_{21} & p_{22} & 0 & 0 \\ p_{20} & p_{21} & p_{22} & 0 \\ 0 & p_{20} & p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} H_1 \\ H_0 \\ H_{-1} \\ H_{-2} \end{bmatrix} = \begin{bmatrix} R_{1,3} \\ R_{1,2} \\ R_{1,1} \\ R_{1,0} \end{bmatrix}$$

Solving for H where $H = P^{-1}R$ we obtain

$$P^{-1} = D = \begin{bmatrix} d_0 I_2 & 0 & 0 & 0 \\ d_1 I_2 & d_0 I_2 & 0 & 0 \\ d_2 I_2 & d_1 I_2 & d_0 I_2 & 0 \\ d_3 I_2 & d_2 I_2 & d_1 I_2 & d_0 I_2 \end{bmatrix}$$

where

$$d_0 = \frac{1}{p_{22}} = 1$$

$$d_1 = (-1)^1 \left(\frac{1}{p_{22}}\right)^2 |p_{21}| = (-1)1\,|2| = -2$$

$$d_2 = (-1)^2 \left(\frac{1}{p_{22}}\right)^3 \begin{vmatrix} p_{21} & p_{20} \\ p_{22} & p_{21} \end{vmatrix} = 3$$

$$d_3 = (-1)^3 \left(\frac{1}{p_{22}}\right)^4 \begin{vmatrix} p_{21} & p_{20} & p_{2,-1} \\ p_{22} & p_{21} & p_{20} \\ 0 & p_{22} & p_{21} \end{vmatrix} = (-1) \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = -4$$

Therefore

$$P^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 3 & 0 & -2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & -2 & 0 & 1 & 0 & 0 \\ -4 & 0 & 3 & 0 & -2 & 0 & 1 & 0 \\ 0 & -4 & 0 & 3 & 0 & -2 & 0 & 1 \end{bmatrix}$$

and from (4.18) we obtain

$$\begin{bmatrix} H_1 \\ \\ H_0 \\ \\ H_{-1} \\ \\ H_{-2} \\ \\ \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \\ 0 & 2 \\ 0 & 0 \\ 1 & -3 \\ 0 & 1 \\ -1 & 4 \\ 0 & -1 \end{bmatrix}$$

i.e.

$$H_1 = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \quad H_0 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \quad H_{-1} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}, \quad H_{-2} = \begin{bmatrix} -1 & 4 \\ 0 & -1 \end{bmatrix}$$

Finally, from (4.23) for $\mu = 2$ we have

$$H_{-[2+k]} = \sum_{i=0}^{1} \frac{p_{2i}}{p_{22}} H_{-[i+k]}$$

for $k = 1, 2, \ldots, +\infty$. Now for $k = 1$ we have:

$$H_{-3} = \sum_{i=0}^{1} \frac{p_{2,i}}{p_{2,2}} H_{-(i+1)} = -\frac{p_{2,0}}{p_{2,2}} H_{-1} - \frac{p_{2,1}}{p_{2,2}} H_{-2}$$

i.e.

$$H_{-3} = \begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix}$$

For $k = 2$

$$H_{-4} = \sum_{i=0}^{1} \frac{p_{2,i}}{p_{2,2}} H_{-(i+1)} = -\frac{p_{2,0}}{p_{2,2}} H_{-2} - \frac{p_{2,1}}{p_{2,2}} H_{-3}$$

i.e.

$$H_{-4} = \begin{bmatrix} -1 & 6 \\ 0 & -1 \end{bmatrix}$$

For $k = 3$

$$H_{-5} = \sum_{i=0}^{1} \frac{p_{2,i}}{p_{2,2}} H_{-(i+1)} = -\frac{p_{2,0}}{p_{2,2}} H_{-3} - \frac{p_{2,1}}{p_{2,2}} H_{-4}$$

i.e.

$$H_{-5} = \begin{bmatrix} 1 & -7 \\ 0 & 1 \end{bmatrix}$$

5. Conclusions

An algorithm is given for the inversion of a polynomial matrix $A(s)$ in terms of the coefficient matrices of the adjoint matrix of $A(s)$ and of the determinant $|A(s)|$. This algorithm constitutes an extension of a Leverrier type algorithm that has been developed for singular systems. Specifically, recursive formulae are obtained for the calculation of both the coefficient matrices of the adjoint as well as for the coefficients of the characteristic polynomial of the given polynomial matrix. Also, a method has been presented that allows the evaluation of the Laurent expansion for the inverse of a polynomial matrix. The coefficient matrices of the Laurent expansion constitute the fundamental matrix sequence of the considered PMD systems, which are very important for various analysis and synthesis problems.

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