it is well-known that this procedure does not usually provide us with the "global minimum." Further research remains to be done to solve this problem.

Furthermore, the computer application (and, in a way, solution) of the theory developed in this work and in [2] is to be done.

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A Sufficient Condition for n Specified Eigenvalues to be Assigned under Constant Output Feedback

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***Abstract*—A sufficient condition for n (n-dimension of the state vector) specified eigenvalues to be assigned under the use of constant output feedback is derived. The form of the output feedback constant gain matrix which results in the above eigenvalue assignment is also established.**

I. Introduction

Davison [1] and Davison and Chattergee [2] have shown that given a completely controllable and completely observable multivariable system described by a triple $[A,B,C]$ (where $A\in R^{n\times n}, B\in R^{n\times m}, C\in R^{r\times n}$) then a constant output feedback gain matrix $K_y\in R^{m\times r}$ can always be found so that $\max(m,r)$ eigenvalues of the closed-loop system, $[A-BK_yC,B,C]$, can be assigned desired values while the remaining [i.e., n-$\max(m,r)$] will assume arbitrary values. It was Davison [1, conclusions] who originally pointed out the need of establishing conditions under which it would be possible to guarantee that the remaining n-$\max(m,r)$ closed-loop system eigenvalues are confined to some known (stable) region in the s-plane. Fallside and Seraji [3] have established necessary and sufficient conditions for complete eigenvalue allocation by output feedback. Given a completely controllable and observable multivariable system in state-space form described by a triple $[A,B,C]$, the problem of complete eigenvalue assignment using constant output feedback is that of determining a constant matrix $K_y\in R^{m\times r}$ such that the resulting closed-loop system plant matrix: $A-BK_yC$ has desired eigenvalues. In this correspondence a sufficient condition for n specified eigenvalues to be assigned under constant output feedback is derived and the form of the output feedback gain matrix which results in the above eigenvalue assignment is established.

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II. Sufficient Condition

Consider a linear, time-invariant, completely controllable and observable multivariable system described by a set of first-order differential equations, and an output expression

$$\dot{x}=Ax+Bu \tag{1}$$

$$y=Cx \tag{2}$$

where $A\in R^{n\times n}$, $B\in R^{n\times m}$, $C\in R^{r\times n}$, and rank $B=m$.

Using a Luenberger state transformation defined by the equation

$$z=Qx \tag{3}$$

the above system can be brought to the Luenberger controllable canonical form

$$\dot{z}=\tilde{A}z+\tilde{B}u \tag{4}$$

$$y=\tilde{C}z \tag{5}$$

where

$$\tilde{A}=QAQ^{-1},\quad \tilde{B}=QB,\quad \tilde{C}=CQ^{-1} \tag{6}$$

and the matrices $[\tilde{A},\tilde{B}]$ have the special canonical form described in [4], [5], while the $\tilde{C}$ matrix has no special form. Let $\sigma_1\geqslant\sigma_2\geqslant\cdots\geqslant\sigma_m$ be the minimal indices (controllability indices) of the singular pencil of matrices $[sI-A,B]$ [6]. Define now m symmetric (with respect to the real axis) but otherwise arbitrary sets of points Λ_{σ_i} in the s-plane, each set containing σ_i points and let Λ be the union of these sets. Let also a matrix $\bar{A}\in R^{n\times n}$ have the following (block-lower triangular) form:

$$\bar{A}=\left[\begin{array}{c|c|c|c}
\begin{matrix}0&1&0&\cdots&0\\0&0&1&\cdots&0\\\vdots&&&&\vdots\\x&x&x&\cdots&x\end{matrix} & \bigcirc & & \bigcirc \\ \hline
\begin{matrix}0&0&&\cdots&0\\0&0&&\cdots&0\\\vdots&&&&\vdots\\+&+&&\cdots&+\end{matrix} & \begin{matrix}0&1&0&\cdots&0\\0&0&1&\cdots&0\\\vdots&\vdots&&&\vdots\\x&x&x&\cdots&x\end{matrix} & & \bigcirc \\ \hline
&&& \\ \hline
\begin{matrix}0&0&&\cdots&0\\0&0&&\cdots&0\\\vdots&&&&\vdots\\+&+&&\cdots&+\end{matrix} & \begin{matrix}0&0&0&\cdots&0\\0&0&0&\cdots&0\\\vdots&&&&\vdots\\+&+&+&\cdots&+\end{matrix} & & \begin{matrix}0&1&0&\cdots&0\\0&0&1&\cdots&0\\\vdots&&&&\vdots\\x&x&x&\cdots&x\end{matrix}
\end{array}\right] \tag{7}$$

with block sizes the same as the corresponding ones in the matrix $\tilde{A}$, the x's having such values so that $\bar{A}$ has as eigenvalues the set Λ, and the $+$'s having any arbitrary values.

Theorem 1:

1) Given a completely controllable and observable system described by a triple $[A,B,C]$, then a sufficient condition for the existence of an output feedback law defined by the equation $u=v-K_yy$ with $K_y\in R^{m\times r}$ and such that

$$spr[A-BK_yC]=\Lambda \tag{8}$$

is

$$(\tilde{A}-\bar{A})\tilde{C}^{g1}\tilde{C}=\tilde{A}-\bar{A} \tag{9}$$

where $\tilde{A}$ is the Luenberger controllable canonical form of the open-loop plant matrix A, $\bar{A}$ is the matrix having the form given by (7), $\tilde{C}$ is the corresponding (to the Luenberger controllable canonical form for the pair $[A,B]$) form of the output matrix C, and $\tilde{C}^{g1}$ is any $g1$-inverse of $\tilde{C}$ [7], [8].[1]

2) If (9) is satisfied, then the constant output feedback matrix K_y is given by

$$K_y=(\tilde{B}^T\tilde{B})^{-1}\tilde{B}^T(\tilde{A}-\bar{A})\tilde{C}^{g1}. \tag{10}$$

Proof: Let

$$(\tilde{A}-\bar{A})\tilde{C}^{g_1}\tilde{C}=(\tilde{A}-\bar{A}). \tag{9}$$

Then the matrix equation

$$\tilde{K}_z\tilde{C}=(\tilde{A}-\bar{A}) \tag{11}$$

has a solution with respect to $\tilde{K}_z\in R^{n\times r}$. (This is because (9) constitutes the necessary and sufficient condition that has to be satisfied so that (11) has a solution with respect to $\tilde{K}_z$ [7], [8].) When (9) is satisfied, the general solution of (11) is

$$\tilde{K}_z=(\tilde{A}-\bar{A})\tilde{C}^{g_1}+Z_1(I_r-\tilde{C}\tilde{C}^{g_1}) \tag{12}$$

where $Z_1\in R^{n\times r}$ and otherwise arbitrary. Taking Z_1 equal to the null matrix, (12) gives the particular solution

$$\tilde{K}_z=(\tilde{A}-\bar{A})\tilde{C}^{g_1}. \tag{13}$$

Consider now the matrix equation

$$\tilde{B}\tilde{K}_x=(\tilde{A}-\bar{A}) \tag{14}$$

with respect to $\tilde{K}_x\in R^{m\times n}$. The necessary and sufficient condition that has to be satisfied so that (14) has a solution with respect to $\tilde{K}_x$, is that for a g_1-inverse $\tilde{B}^{g_1}$ of $\tilde{B}$ we have

$$\tilde{B}\tilde{B}^{g_1}(\tilde{A}-\bar{A})=(\tilde{A}-\bar{A}) \tag{15}$$

the condition of consistency. Furthermore, if condition (15) is satisfied then the general solution of (14) is [7], [8], [9],

$$\tilde{K}_x=\tilde{B}^{g_1}(\tilde{A}-\bar{A})+(I_m-\tilde{B}^{g_1}\tilde{B})Z_2 \tag{16}$$

where $Z_2\in R^{m\times n}$ and otherwise arbitrary. Condition (15) is always satisfied if as $\tilde{B}^{g1}$ the left inverse of $\tilde{B}$ is used, i.e., with $\tilde{B}^{g1}=(\tilde{B}^T\tilde{B})^{-1}\tilde{B}^T$ then

$$\tilde{B}\tilde{B}^{g1}(\tilde{A}-\bar{A})=\tilde{B}(\tilde{B}^T\tilde{B})^{-1}\tilde{B}^T(\tilde{A}-\bar{A})$$
$$=\hat{B}P(P^T\hat{B}^T\hat{B}P)^{-1}P^T\hat{B}^T(\tilde{A}-\bar{A})=\tilde{A}-\bar{A}$$

where above we made use of the following facts: 1) $\tilde{B}=\hat{B}P$, $\hat{B}=$block diag$[\hat{b}_1,\hat{b}_2,\cdots,\hat{b}_m]$, $\hat{b}_i^T=[00\cdots01]$, $i=1,2,\cdots,m$, $P\in R^{m\times m}$ is upper-triangular matrix with 1,s on the main diagonal formed from the m (possibly nonzero) rows of $\tilde{B}$ [4], 2) $\hat{B}^T\hat{B}=I_m$, 3) $\hat{B}\hat{B}^T(\tilde{A}-\bar{A})=\tilde{A}-\bar{A}$. So (14) always has a solution with respect to K_x. In such a case, from (16), and setting the arbitrary matrix Z_2 equal to the null matrix, we have:

$$\tilde{K}_x=(\tilde{B}^T\tilde{B})^{-1}\tilde{B}^T(\tilde{A}-\bar{A}). \tag{17}$$

Combining (13) and (14) we have:

$$\tilde{K}_z=\tilde{B}\tilde{K}_x\tilde{C}^{g1} \tag{18}$$

and from (11) and (18), $\tilde{A}-\bar{A}=\tilde{K}_z\tilde{C}=\tilde{B}\tilde{K}_x\tilde{C}^{g1}\tilde{C}$ or $\tilde{A}-\tilde{B}\tilde{K}_x\tilde{C}^{g1}\tilde{C}=\bar{A}$, so that

$$K_y=\tilde{K}_x\tilde{C}^{g1}. \tag{19}$$

[1] A $q\times p$ matrix C^{g1} is said to be a g_1-inverse of the $p\times q$ matrix C if $CC^{g1}C=C$.

3) From (19) and (17)

$$K_y=(\tilde{B}^T\tilde{B})^{-1}\tilde{B}^T(\tilde{A}-\bar{A})\tilde{C}^{g1}. \tag{20}$$

III. Conclusions

As it can be seen through the proof of Theorem 1, (17), being a solution of (14), gives a rather general solution to the eigenvalue assignment problem using state feedback. An interesting question seems to be the relationship between controllability of $[A,B]$ and the existence of a g_1-inverse of $\tilde{B}$ such that $\tilde{B}\tilde{B}^{g1}(\tilde{A}-\bar{A})=(\tilde{A}-\bar{A})$. In this correspondence, except for the note that complete controllability is necessary to put A and B in the canonical form which simplifies the arithmetic, this notion is not explored. For an $[A,B]$ controllable pair, a solution to (14) always exists, but the reason must be deeper than just a convenient basis change. Further we note that if some or all of the minimal indices (controllability indices) σ_i $(i=1,2,\cdots,m)$ of the singular pencil of matrices $[sI-A,B]$ are odd numbers, then by forming the desired closed-loop plant matrix $\bar{A}$, as described by (7), it is not possible to choose the elements denoted by x in (7) to be real numbers and such so that the eigenvalues of $\bar{A}$ are all arbitrary complex conjugate pairs. For example if $n=4$, $m=2$, $\sigma_1=3$, $\sigma_2=1$ it does not permit the assignment of four complex eigenvalues; i.e., two of the eigenvalues must be real. This difficulty can be circumvented by simply choosing $\bar{A}$ to be an $n\times n$ companion matrix. With $\bar{A}$ in this form we still note that $\hat{B}\hat{B}^T(\tilde{A}-\bar{A})=\tilde{A}-\bar{A}$.

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A Mixed Method for Multivariable System Reduction

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Abstract—**A mixed method which combines dominant-eigenvalue concept and matrix-continued-fraction approach is proposed to obtain a stable reduced model from a stable high degree multivariable system.**

I. Introduction

When the analysis and synthesis of a high degree multivariable system are required, a low degree reduced model is often searched for so that an analog or digital simulation of the system is possible. Model-reduction techniques have been proposed by numerous authors [1]–[7]. For example, in the time domain, the dominant-pole approach proposed by Davison [1] and Chidambara [3] is well known. For a very high order dynamic system, the order of a reduced model can be determined by the dominant poles of the original system. The reduced model is always stable and dominant performance of the original system may be maintained. However, the processes of reduction technique involve complicated linear transformation, matrix diagonalization, and steady-state value matching, etc. In other words, if the order of a multivariable

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