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A. I. G. VARDULAKIS [a]
[a] Department of Computing and Control, Imperial College of Science and Technology, London, SW7 2BZ
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On the structure of the bases of all possible controllability subspaces $\mathscr{R}$ of a controllable pair $[A, B]$ in canonical form

A. I. G. VARDULAKIS†

The problem of determining the structure of the basis matrices of all possible controllability subspaces $\mathscr{R}$ of a controllable pair $[\tilde{A}, \tilde{B}]$ in the Brunovski (1966) and Luenberger (1967) controllable canonical form is considered. Departing from a characterization of the c.s.'s of $[\tilde{A}, \tilde{B}]$ given by Warren and Eckberg (1975) it is shown that to every pair A, B in the Brunovski (1966) and Luenberger (1967) controllable canonical form, there corresponds a unique polynomial matrix $X(s)$ which has a canonical structure. Using the results on rational vector spaces obtained by Forney (1975) it is seen that this polynomial matrix qualifies as a minimal basis which uniquely identifies a rational vector space $\mathscr{X}(s)$. A correspondence between the polynomial n-tuples $\mathbf{x}(s) \in \mathscr{X}(s)$ and the c.s.'s $\mathscr{R}$ of $[\tilde{A}, \tilde{B}]$ leads to simple expressions that describe the structure of the bases of all c.s. $\mathscr{R}$ of $[\tilde{A}, \tilde{B}]$ of all possible dimensions.

1. Introduction and notation

A very important concept in the geometric state-space theory developed by Wonham and Morse (1970, 1971) and Wonham (1974) is that of a controllability subspace (c.s.) of a pair $[A, B]$.

This concept was introduced by Wonham and Morse (1970, 1971) who intended to describe input–output properties of a linear multivariable system without using transfer-function matrices, so that a more general state-space theory of the decoupling problem (Falb and Wollovich (1967)) could be obtained. Through their geometric approach, Wonham and Morse (1970, 1971) showed that solvability of the decoupling problem becomes equivalent to finding suitable sets of c.s.'s. The results of this abstract theory are however not always given in simple form, which directly can be used for computations. For example, computing c.s.'s satisfying certain restrictive requirements is done by an algorithmic procedure which involves the calculation of sums, intersections, unions, etc. of subspaces.

In this paper the problem of determining the structure of the basis matrices of all possible c.s.'s $\mathscr{R}$ of a controllable pair $[\tilde{A}, \tilde{B}]$ in the Brunovski (1966) and Luenberger (1967) controllable canonical form is investigated. By combining the results of Warren and Eckberg (1975) on a characterization of c.s.'s via polynomial matrices and controllability indices (Kronecker invariants) with those on minimal bases of rational vector spaces obtained by Forney (1975) one arrives at a simple algorithm through which one may construct matrices that describe the structure of the bases of all c.s.'s of all possible dimensions. The questions regarding the possible dimensions and the number of the c.s.'s of a certain dimension, originally examined by Warren and Eckberg (1975), are

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† Department of Computing and Control, Imperial College of Science and Technology, London SW7 2BZ.

also examined using some observations regarding the structure of the bases that originate from the algorithm.

Let $\Re$ denote the field of real numbers, $\Re(s)$ the field of rational functions in s with coefficients in $\Re$ and $\Re[s]$ the ring of polynomials in s with coefficients in $\Re$. Let script letters $\mathscr{X}$, $\mathscr{R}$, $\mathscr{L}$, ... denote vector spaces or subspaces over $\Re$, and italic letters A, B, C, ... denote linear transformations (maps) between vector spaces or their associated matrices. The set of all $p\times q$ matrices with elements in $\Re$ will be denoted by $\Re^{p\times q}$. Also by $\mathbf{a}$, $\mathbf{b}$, $\mathbf{x}$ ... let us denote elements of vector spaces and the image of a map (e.g. B) by the corresponding script capital (e.g. $\mathscr{B}$). Let $A:\mathscr{X}\to\mathscr{X}$ $B:\mathscr{U}\to\mathscr{X}$ be the maps associated with the linear constant system described by the state-space equation $\dot{\mathbf{x}}=A\mathbf{x}+B\mathbf{u}$ where $\mathscr{U}$, $\mathscr{X}$, denote the input and state spaces respectively. Writing dim $(\cdot)$ for the dimension we shall fix dim $\mathscr{X}=n$, dim $\mathscr{U}=m$. If k is a positive integer, then $\mathbf{k}$ denotes the set of integers 1, 2, ..., k. $\langle A\,|\,\mathscr{B}\rangle$ will denote (Wonham and Morse 1970) the controllable subspace of $\mathscr{X}$ generated by a pair $[A, B]$ and is defined as $\langle A\,|\,\mathscr{B}\rangle=\mathscr{B}+A\mathscr{B}+\ldots+A^{n-1}\mathscr{B}$. So that (A, B) is a controllable pair if and only if $\langle A\,|\,\mathscr{B}\rangle=\mathscr{X}$. A subspace $\mathscr{V}\subset\mathscr{X}$ is (A, B)-invariant if there exists a map $F:\mathscr{X}\to\mathscr{U}$ such that $(A+BF)\mathscr{V}\subset\mathscr{V}$. A subspace $\mathscr{R}\subset\mathscr{X}$ is a controllability subspace (c.s.) of (A, B) if $\mathscr{R}$ is (A, B)-invariant and $\mathscr{R}=\langle A+BF\,|\,\mathscr{B}\cap\mathscr{R}\rangle$ for some $F:\mathscr{X}\to\mathscr{U}$ (Wonham and Morse 1970).

The following results and definitions are taken from Forney (1975). Consider the set of all n-tuples $\mathbf{x}(s)$ of rational functions, i.e. all $\mathbf{x}^{\mathrm{T}}(s)=[x_1(s), \ldots, x_n(s)]$ with $x_i(s)\in\Re(s)$ $i\in\mathbf{n}$. This set represents a vector space over the field $\Re(s)$ which we denote by $\mathscr{W}(s)$, and which is called a rational vector space (Forney 1975). Let $X(s)=[\mathbf{x}_1(s), \ldots, \mathbf{x}_m(s)]$ be an $n\times m$ $(n\geqslant m)$ matrix with elements in $\Re(s)$. Its column space, i.e. the set of all linear combinations (over $\Re(s)$) of its columns $\mathbf{x}_i(s)$ $i\in\mathbf{m}$, defines a vector space $\mathscr{X}(s)$ over the field $\Re(s)$ (which is a subspace of $\mathscr{W}(s)$) and the dim $\mathscr{X}(s)$ is equal to the rank of $X(s)$. Conversely if $\mathscr{X}(s)$ is a subspace of $\mathscr{W}(s)$ of dimension m, it has a basis of m linearly independent (over $\Re(s)$) n-tuples of rational functions $\mathbf{x}_i(s)$ $i\in\mathbf{m}$ which can be arranged to form an $n\times m$ rational matrix $X(s)$ of rank m. We refer to such a matrix as a rational basis for $\mathscr{X}(s)$. Any rational basis of $\mathscr{X}(s)$ gives rise to a polynomial basis (we multiply each element of a rational basis vector by the least common denominator of all its elements).

Definition 1

The degree deg $\mathbf{x}(s)$ of an n-tuple $\mathbf{x}(s)$, $\mathbf{x}^{\mathrm{T}}(s)=[x_1(s), \ldots, x_n(s)]$, with $x_i(s)\in\Re[s]$ $i\in\mathbf{n}$, is the greatest degree of its components $x_i(s)$ $i\in\mathbf{n}$.

Definition 2

If $X(s)$ is an $n\times m$ polynomial matrix with columns $\mathbf{x}_i(s)$ $i\in\mathbf{m}$, the ith index v_i of $X(s)$ is defined as $v_i=\deg\mathbf{x}_i(s)$ $i\in\mathbf{m}$, and the order v of $X(s)$ is defined as $v=\sum_{i=1}^{m} v_i$.

Definition 3

If $\mathscr{X}(s)$ is an m-dimensional vector space of n-tuples over $\Re(s)$ (i.e. a subspace of $\mathscr{W}(s)$), a minimal basis of $\mathscr{X}(s)$ is an $n\times m$ polynomial matrix $X(s)$ such that

$X(s)$ is a basis for $\mathscr{X}(s)$ and $X(s)$ has least order among all polynomial bases for $\mathscr{X}(s)$.

Let the degree d' of an $n \times m$ polynomial matrix $X(s)$ be the highest degree occurring among all $m \times m$ minors of $X(s)$ (Rosenbrock and Hayton 1974).

Forney (1975) (main theorem) shows that $X(s)$ is a minimal basis of the vector space $\mathscr{X}(s)$ spanned by its columns if and only if the greatest common divisor of all $m \times m$ minors of $X(s)$ is 1 and $v = d'$.

2. Analysis

Warren and Eckberg (1975) (Lemma 2) gave the following characterization of controllability subspaces. Let $A \in \Re^{n \times n}$, $B \in \Re^{n \times m}$, rank $B = m$ be a pair describing a linear system in state space form, and let

$$\mathbf{z}(s) = \begin{bmatrix} \mathbf{x}(s) \\ \hdashline -\mathbf{u}(s) \end{bmatrix} \in \ker\, [sI - A \,|\, B]$$

where

$$\mathbf{x}(s) = \mathbf{x}_0 + \mathbf{x}_1 s + \ldots + \mathbf{x}_{d-1} s^{d-1}, \qquad \mathbf{x}_k \in \mathscr{X},\ k = 0, 1, \ldots, d-1$$

$$\mathbf{u}(s) = \mathbf{u}_0 + \mathbf{u}_1 s + \ldots + \mathbf{u}_d s^d, \qquad \mathbf{u}_{k'} \in \mathscr{U},\ k' = 0, 1, \ldots, d$$

(d : integer), then $\mathscr{R} \triangleq \text{span}\, \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{d-1}\}$ is a c.s. of A, B.

We consider now the problem of determining the structure of the bases of all c.s. $\mathscr{R}$ of A, B of all possible dimensions d. Before we proceed any further we state here, for easy reference, what is meant by a ' fundamental series ' of solutions to equation

$$[sI - A \,|\, B] \begin{bmatrix} \mathbf{x}(s) \\ \hdashline -\mathbf{u}(s) \end{bmatrix} = 0 \tag{1}$$

Equation (1) admits polynomial solution vectors (Rosenbrock 1970). A fundamental series of solutions

$$\mathbf{z}_1(s) = \begin{bmatrix} \mathbf{x}_1(s) \\ \hdashline -\mathbf{u}_1(s) \end{bmatrix}, \ldots, \mathbf{z}_m(s) = \begin{bmatrix} \mathbf{x}_m(s) \\ \hdashline -\mathbf{u}_m(s) \end{bmatrix}$$

to eqn. (1) is defined as follows (Gantmacher 1959, Rosenbrock 1970). Among all solutions of eqn. (1) we choose a non-zero solution $\mathbf{z}_1^{\mathrm{T}}(s) = [\mathbf{x}_1^{\mathrm{T}}(s), -\mathbf{u}_1^{\mathrm{T}}(s)]$ of least possible degree k_1. Among all solutions of the same equation that are linearly independent (over $\Re[s]$, see also Rosenbrock 1970, p. 20) of $\mathbf{z}_1(s)$ we take a solution $\mathbf{z}_2^{\mathrm{T}}(s) = [\mathbf{x}_2^{\mathrm{T}}(s), -\mathbf{u}_2^{\mathrm{T}}(s)]$ of least degree k_2. (Obviously $k_2 \geqslant k_1$, because otherwise we should have that $k_2 < k_1$ contrary to our assumption, that $\mathbf{z}_1(s)$ is of least possible degree.) We continue the process of choosing among the solutions that are linearly independent of $\mathbf{z}_1(s)$ and $\mathbf{z}_2(s)$ a solution $\mathbf{z}_3(s)$ of least degree k_3, etc. Since the number of linearly independent solutions of (1) is m, the process will come to an end. We thus obtain a ' fundamental series ' of solutions of eqn. (1) : $\mathbf{z}_1(s), \mathbf{z}_2(s), \ldots, \mathbf{z}_m(s)$ having degrees $k_1 \leqslant k_2 \leqslant \ldots \leqslant k_m$.

We return now to the problem set originally of determining the structure of the bases of all c.s.'s of A, B of all possible dimensions. To this end we first change the basis in the state space. Let this change of basis in $\mathscr{X}$ be defined by $\tilde{\mathbf{x}} = Q\mathbf{x}$, and let us take Q so that our system is now described by the pair $(\tilde{A}, \tilde{B})$ which is in the Luenberger (1967) and Brunovski (1966) controllable canonical form and $\tilde{A} = QAQ^{-1}$, $\tilde{B} = QB$. After such a change of basis in $\mathscr{X}$, eqn. (1) takes the form

$$[sI - \tilde{A} \mid \tilde{B}] \begin{bmatrix} \tilde{\mathbf{x}}(s) \\ \hline -\mathbf{u}(s) \end{bmatrix} = 0 \tag{2}$$

Assume now that a 'fundamental series' of solutions to eqn. (2) has been determined by some (yet unknown) algorithm and is given by

$$\tilde{\mathbf{z}}_1(s) = \begin{bmatrix} \tilde{\mathbf{x}}_1(s) \\ \hline -\mathbf{u}_1(s) \end{bmatrix}, \ldots, \mathbf{z}_m(s) = \begin{bmatrix} \tilde{\mathbf{x}}_m(s) \\ \hline -\mathbf{u}_m(s) \end{bmatrix} \tag{3}$$

with corresponding degrees $k_1 \leqslant k_2 \leqslant \ldots \leqslant k_m$. Form a polynomial matrix $\tilde{Z}(s)$ by arranging the fundamental series (3) in the reverse order, i.e. let

$$\tilde{Z}(s) = \left[\begin{array}{c|c|c|c} \tilde{\mathbf{x}}_m(s) & \tilde{\mathbf{x}}_{m-1}(s) & \cdots & \tilde{\mathbf{x}}_1(s) \\ \hline -\mathbf{u}_m(s) & -\mathbf{u}_{m-1}(s) & \cdots & -\mathbf{u}_1(s) \end{array}\right] = \begin{bmatrix} \tilde{X}(s) \\ \hline -U(s) \end{bmatrix} \tag{4}$$

Then eqn. (2) gives

$$[sI - \tilde{A} \mid \tilde{B}] \begin{bmatrix} \tilde{X}(s) \\ \hline -U(s) \end{bmatrix} \tag{5}$$

or

$$\tilde{X}(s) U^{-1}(s) = (sI - \tilde{A})^{-1} \tilde{B} \tag{6}$$

and from eqn. (6) and the structure theorem by Wolovich and Falb (1969) and Wolovich (1974) we obtain that

$$\tilde{X}(s) = [\tilde{\mathbf{x}}_m(s), \ldots, \tilde{\mathbf{x}}_1(s)] = \begin{bmatrix} 1 & 0 \\ s & 0 \\ \vdots & \vdots \\ s^{k_m - 1} \ldots & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 1 \\ 0 & s \\ \vdots & \vdots \\ \vdots & \ldots s^{k_1 - 1} \end{bmatrix} \tag{7}$$

and

$$U(s) = [\mathbf{u}_m(s), \ldots, \mathbf{u}_1(s)] = \tilde{B}_m^{-1} \, \delta_0(s) \tag{8}$$

where

$$\delta_0(s) = \text{diag}\,[s^{k_m}, \ldots, s^{k_1}] - \tilde{A}_m \tilde{X}(s) \tag{9}$$

$\tilde{A}_m$ is an $m \times n$ matrix consisting of the m ordered $p_i = \sum_{j=1}^{i} k_{m+1-j}$ $i \in \mathbf{m}$ rows of $\tilde{A}$, and $\tilde{B}_m$ an $m \times m$ matrix consisting of the m ordered p_i rows of $\tilde{B}$, and $k_1 \leqslant k_2, \ldots \leqslant k_m$ are the controllability indices of the pair $[\tilde{A}, \tilde{B}]$.

From this analysis we may conclude that having a pair $[\tilde{A}, \tilde{B}]$ in the Brunovski (1966) and Luenberger (1967) controllable canonical form, a fundamental series of solutions to eqn. (2) can be obtained by inspection from the columns $\tilde{\mathbf{z}}_m(s), \ldots, \tilde{\mathbf{z}}_1(s)$ of the polynomial matrix $\tilde{Z}(s)$.

We also note that the only data needed in order to form the matrix $\tilde{X}(s)$ are the values of the controllability indices $k_1 \leqslant k_2 \leqslant \ldots \leqslant k_m$ of the pair $[\tilde{A}, \tilde{B}]$. From eqn. (7) we have that the ith index v_i of $\tilde{X}(s)$ is $v_i = k_i - 1$ $i \in \mathbf{m}$, and the order v of $\tilde{X}(s)$ is $v = \sum_{i=1}^{m} v_i = \sum_{i=1}^{m} k_i - 1 = n - m$. Considering the results by Forney (1975) that we mentioned earlier we see that $\tilde{X}(s)$ as given by eqn. (7) qualifies as a minimal basis and therefore uniquely identifies the rational vector space $\mathscr{X}(s)$ spanned by its columns $\tilde{\mathbf{x}}_i(s)$ $i \in \mathbf{m}$, as it satisfies all the equivalent conditions that a minimal basis matrix of an m-dimensional rational vector space $\mathscr{X}(s)$ must satisfy and which are given in Forney's (1975) main theorem.

Following again Forney (1975) we denote by $M_{\mathscr{X}(s)}$ the set of all polynomial n-tuples $\tilde{\mathbf{x}}(s) \in \mathscr{X}(s)$. $M_{\mathscr{X}(s)}$ is a free $\Re[s]$ module and (see remark 2 by Forney (1975)) $\tilde{X}(s)$ is a basis for the free $\Re[s]$ module $M_{\mathscr{X}(s)}$, i.e. all $\tilde{\mathbf{x}}(s) \in M_{\mathscr{X}(s)}$ can be expressed as

$$\tilde{\mathbf{x}}(s) = \sum_{i\,:\,v_i \leqslant \deg \tilde{\mathbf{x}}(s)} \tilde{\mathbf{x}}_i(s) a_i(s) = \tilde{X}(s)\mathbf{a}(s) \tag{10}$$

where $\mathbf{a}^{\mathrm{T}}(s) = [a_m(s), \ldots, a_1(s)]$ is some polynomial m-tuple with appropriate $a_i(s) = a_{i0} + a_{i1}s + \ldots + a_{ir_i}s^{r_i}$, such that

$$\deg a_i(s) \leqslant \deg \tilde{\mathbf{x}}(s) - v_i \tag{11}$$

We also denote by $\mathscr{X}_d$ the set of all $\tilde{\mathbf{x}}(s) \in M_{\mathscr{X}(s)}$ with degree r less than d, for all integers $d \geqslant 0$, i.e.

$$\mathscr{X}_d = \{\tilde{\mathbf{x}}(s) \mid \tilde{\mathbf{x}}(s) \in M_{\mathscr{X}(s)} \quad \text{and} \quad \deg \tilde{\mathbf{x}}(s) = r < d\} \tag{12}$$

$\mathscr{X}_d$ is a vector space over $\Re$ (but not a vector space over $\Re(s)$) (Forney 1975), and if we denote the dimension of $\mathscr{X}_d$ by $\dim_{\Re} \mathscr{X}_d$, then Forney (1975) showed that

$$\dim_{\Re} \mathscr{X}_d = \sum_{i\,:\,v_i \leqslant d} (d - v_i) \tag{13}$$

Proposition 1

Let $\tilde{\mathbf{x}}(s) \in M_{\mathscr{X}(s)}$, and write $\tilde{\mathbf{x}}(s)$ as $\tilde{\mathbf{x}}(s) = \tilde{\mathbf{x}}_0 + \tilde{\mathbf{x}}_1 s + \ldots + \tilde{\mathbf{x}}_{d-1}s^{d-1}$. Then if $\{\tilde{\mathbf{x}}_0, \tilde{\mathbf{x}}_1, \ldots, \tilde{\mathbf{x}}_{d-1}\} \in \mathscr{X}$ are independent over $\Re$, $\mathscr{R} = \text{span}\,\{\tilde{\mathbf{x}}_0, \tilde{\mathbf{x}}_1, \ldots, \tilde{\mathbf{x}}_{d-1}\}$ is a c.s. of $\tilde{A}$, $\tilde{B}$ and $\dim \mathscr{R} = d$.

Proof

$\tilde{\mathbf{x}}(s) \in M_{\mathscr{X}(s)}$ implies there exists a polynomial m-tuple $\mathbf{a}^{\mathrm{T}}=[a_m(s), \ldots, a_1(s)]$ such that $\tilde{\mathbf{x}}(s)$ can be expressed as in (10). Define a polynomial m-tuple $\mathbf{u}(s)$ by $\mathbf{u}(s) \triangleq U(s)\mathbf{a}(s)$, then Proposition 1 follows from

$$[sI-\tilde{A} \mid \tilde{B}]\left[\begin{array}{c}\tilde{\mathbf{x}}(s) \\ \hdashline -\mathbf{u}(s)\end{array}\right]=[sI-\tilde{A} \mid \tilde{B}]\left[\begin{array}{c}\tilde{X}(s) \\ \hdashline -U(s)\end{array}\right] \mathbf{a}(s)=0 \tag{14}$$

and Lemma 2 by Warren and Eckberg (1975).

Proposition 2

Let $\mathscr{R}=\operatorname{span}\{\tilde{\mathbf{x}}_0, \tilde{\mathbf{x}}_1, \ldots, \tilde{\mathbf{x}}_{d-1}\}$ be a c.s. of $\tilde{A}$, $\tilde{B}$. If $\tilde{\mathbf{x}}(s)$ is defined by $\tilde{\mathbf{x}}(s) \triangleq \tilde{\mathbf{x}}_0+\tilde{\mathbf{x}}_1 s+\ldots+\tilde{\mathbf{x}}_{d-1} s^{d-1}$, then $\tilde{\mathbf{x}}(s) \in M_{\mathscr{X}(s)}$.

Proof

Using again Lemma 2 by Warren and Eckberg we have that there exists a polynomial m-tuple $u(s)$, such that

$$\tilde{\mathbf{z}}(s) \triangleq\left[\begin{array}{c}\tilde{\mathbf{x}}(s) \\ \hdashline -\mathbf{u}(s)\end{array}\right] \in \ker\,[sI-\tilde{A} \mid \tilde{B}]$$

Furthermore, any element $\mathbf{z}(s) \in \ker\,[sI-\tilde{A} \mid \tilde{B}]$ can be expressed as

$$\tilde{\mathbf{z}}(s)=\left[\begin{array}{c}\tilde{\mathbf{x}}(s) \\ \hdashline -\mathbf{u}(s)\end{array}\right]=\sum_{i\,:\,k_i \leqslant \deg \tilde{z}(s)} \tilde{\mathbf{z}}_i(s) a_i(s)=\left[\begin{array}{c}\tilde{X}(s) \\ \hdashline U(s)\end{array}\right]\left[\begin{array}{c}a_m(s) \\ \vdots \\ a_1(s)\end{array}\right]=\tilde{Z}(s)\mathbf{a}(s)$$

for appropriate $a_i(s) \in \Re[s]$ and such that

$$\deg a_i(s) \leqslant \deg \tilde{\mathbf{z}}(s)-k_i$$

i.e. $\tilde{x}(s)=\tilde{X}(s)\mathbf{a}(s)$, i.e. $\tilde{x}(s) \in M_{\mathscr{X}(s)}$.

From Propositions 1 and 2 we conclude that there is a correspondence between the elements $\tilde{\mathbf{x}}(s)$ of $M_{\mathscr{X}(s)}$ and the controllability subspaces $\mathscr{R}$ of $\tilde{A}$, $\tilde{B}$. Summarizing the above analysis we can state the following.

Proposition 3

There exists a c.s. $\mathscr{R}$ of $\tilde{A}$, $\tilde{B}$ of $\dim \mathscr{R}=d$ $(1 \leqslant d \leqslant n)$ if there exists a $\tilde{\mathbf{x}}(s) \in \mathscr{X}_d$ with degree $r=\deg \tilde{\mathbf{x}}(s)=d-1$, such that it can be written as

$$\tilde{\mathbf{x}}(s)=\sum_{i\,:\,v_i \leqslant \deg \tilde{\mathbf{x}}(s)} \tilde{\mathbf{x}}_i(s) a_i(s)=\tilde{\mathbf{x}}_0+\tilde{\mathbf{x}}_1 s+\ldots+\tilde{\mathbf{x}}_{d-1} s^{d-1}$$

with $\{\tilde{\mathbf{x}}_0, \tilde{\mathbf{x}}_1, \ldots, \tilde{\mathbf{x}}_{d-1}\} \in \mathscr{X}$ and independent over $\Re$. This c.s. $\mathscr{R}$ of $\tilde{A}$, $\tilde{B}$ is then given by :

$$\mathscr{R}=\operatorname{span}\{\tilde{\mathbf{x}}_0, \tilde{\mathbf{x}}_1, \ldots, \tilde{\mathbf{x}}_{d-1}\}$$

Example

Let $\tilde{A}$, $\tilde{B}$ be such, so that $k_1 = 1$, $k_2 = 3$. Then $v_1 = 0$, $v_2 = 2$ and

$$\tilde{X}(s) = [\tilde{\mathbf{x}}_2(s), \tilde{\mathbf{x}}_1(s)] = \begin{bmatrix} 1 & 0 \\ s & 0 \\ s^2 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and } \mathscr{X}(s) = \text{span}_{\Re(s)}[\tilde{\mathbf{x}}_2(s), \tilde{\mathbf{x}}_1(s)]$$

Let $d = 1$ and consider $\mathscr{X}_1 = \{\tilde{\mathbf{x}}(s) \mid \tilde{\mathbf{x}}(s) \in M_{\mathscr{X}(s)} \text{ and } \deg \tilde{\mathbf{x}}(s) = r = 0\}$. According to eqn. (10) and the fact that only for $i = 1$ we have $v_i \leqslant \deg \tilde{\mathbf{x}}(s) = 0$, an arbitrary $\tilde{\mathbf{x}}(s) \in \mathscr{X}_1$ can be written as

$$\tilde{\mathbf{x}}(s) = \sum_{i\,:\, v_i \leqslant \deg \mathbf{x}(s)} \tilde{\mathbf{x}}_i(s) a_i(s) = \tilde{\mathbf{x}}_1(s) a_1(s) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} a_1(s)$$

From eqn. (11) and $i = 1$ we have that $\deg a_1(s) \leqslant 0$, i.e. $a_1(s) \equiv a_{10} \in \Re$, so that

$$\tilde{\mathbf{x}}(s) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} a_{10} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_{10} \end{bmatrix} = \tilde{\mathbf{x}}_0 \tag{15}$$

and so there is a unique c.s. $\mathscr{R}$ of $\tilde{A}$, $\tilde{B}$ of dim $\mathscr{R} = 1$, given by

$$\mathscr{R} = \text{span} \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_{10} \end{bmatrix} \right\}, \quad (a_{10} \neq 0)$$

From eqn. (15)

$$\mathscr{X}_1 = \text{span}_{\Re} \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \tag{16}$$

Let $d=2$ and consider an arbitrary $\tilde{\mathbf{x}}(s)\in\mathscr{X}_2$ with $\deg\tilde{\mathbf{x}}(s)=1$. Again only for $i=1$ we have $v_i\leqslant\deg\tilde{\mathbf{x}}(s)=1$ and $\tilde{\mathbf{x}}(s)\in\mathscr{X}_2$ can be written according to eqn. (10) as

$$\tilde{\mathbf{x}}(s)=\sum_{i:v_i\leqslant\deg\tilde{\mathbf{x}}(s)}\tilde{\mathbf{x}}_i(s)a_i(s)=\tilde{\mathbf{x}}_1(s)a_1(s)=\begin{bmatrix}0\\0\\0\\1\end{bmatrix}a_1(s)$$

From eqn. (11): $\deg a_1(s)\leqslant 1-0=1$ we conclude that the degree of $a_1(s)$ must be equal to 1, i.e. $a_1(s)=a_{10}+a_{11}s$, so that

$$\tilde{\mathbf{x}}(s)=\begin{bmatrix}0\\0\\0\\1\end{bmatrix}(a_{10}+a_{11}s)=\begin{bmatrix}0\\0\\0\\a_{10}\end{bmatrix}+\begin{bmatrix}0\\0\\0\\a_{11}\end{bmatrix}s=\tilde{\mathbf{x}}_0+\tilde{\mathbf{x}}_1s \tag{17}$$

As there exist no values of $a_{10}, a_{11}\in\Re$, such that $\tilde{\mathbf{x}}_0$, $\tilde{\mathbf{x}}_1$ are independent (over $\Re$), we conclude that there exists no c.s. of $\tilde{A}$, $\tilde{B}$ of dimension $d=2$.

Writing $\tilde{\mathbf{x}}(s)$ in eqn. (17) as

$$\tilde{\mathbf{x}}(s)=\begin{bmatrix}0\\0\\0\\1\end{bmatrix}a_{10}+\begin{bmatrix}0\\0\\0\\s\end{bmatrix}a_{11} \tag{18}$$

we have that

$$\mathscr{X}_2=\operatorname{span}_{\Re}\left\{\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\begin{bmatrix}0\\0\\0\\s\end{bmatrix}\right\} \tag{19}$$

Let $d=3$ and consider an arbitrary $\tilde{\mathbf{x}}(s)\in\mathscr{X}_3$ with $\deg\tilde{\mathbf{x}}(s)=2$. Now for both $i=1$ and $i=2$, $v_i\leqslant\deg\tilde{\mathbf{x}}(s)=2$ and $\tilde{\mathbf{x}}(s)\in\mathscr{X}_3$ can be written according to eqn. (10) as

$$\tilde{\mathbf{x}}(s)=\sum_{i:v_i\leqslant\deg\tilde{\mathbf{x}}(s)}\tilde{\mathbf{x}}_i(s)a_i(s)=\begin{bmatrix}0\\0\\0\\1\end{bmatrix}a_1(s)+\begin{bmatrix}1\\s\\s^2\\0\end{bmatrix}a_2(s) \tag{20}$$

From eqn. (11): $\deg a_1(s) \leqslant 2 - v_i = 2 - 0 = 2$ and $\deg a_2(s) \leqslant 2 - v_2 = 2 - 2 = 0$ hence $a_1(s) = a_{10} + a_{11}s + a_{12}s^2$, $a_2(s) = a_{20}$, so that

$$\tilde{\mathbf{x}}(s) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} (a_{10} + a_{11}s + a_{12}s^2) + \begin{bmatrix} 1 \\ s \\ s^2 \\ 0 \end{bmatrix} a_{20} = \begin{bmatrix} a_{20} \\ 0 \\ 0 \\ a_{10} \end{bmatrix} + \begin{bmatrix} 0 \\ a_{20} \\ 0 \\ a_{10} \end{bmatrix} s + \begin{bmatrix} 0 \\ 0 \\ a_{20} \\ a_{12} \end{bmatrix} s^2$$

$$= \tilde{\mathbf{x}}_0 + \tilde{\mathbf{x}}_1 s + \tilde{\mathbf{x}}_2 s^2$$

and the general form of all c.s.'s $\mathscr{R}$ of $\tilde{A}$, $\tilde{B}$ of dim $\mathscr{R} = 3$ is given by

$$\mathscr{R} = \text{span} \left\{ \begin{bmatrix} a_{20} \\ 0 \\ 0 \\ a_{10} \end{bmatrix}, \begin{bmatrix} 0 \\ a_{20} \\ 0 \\ a_{11} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ a_{20} \\ a_{12} \end{bmatrix} \right\} \tag{21}$$

By giving values to $a_{ij} \in \mathfrak{R}$ so that the vectors obtained are independent (over $\mathfrak{R}$) we obtain an infinity of c.s. $\mathscr{R}$ of $\tilde{A}$, $\tilde{B}$ of dimension 3. Writing $\tilde{\mathbf{x}}(s)$ as

$$\tilde{\mathbf{x}}(s) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} a_{10} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ s \end{bmatrix} a_{11} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ s^2 \end{bmatrix} a_{12} + \begin{bmatrix} 1 \\ s \\ s^2 \\ 0 \end{bmatrix} a_{20}$$

we have

$$\mathscr{X}_3 = \text{span}_{\mathfrak{R}} \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ s \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ s^2 \end{bmatrix}, \begin{bmatrix} 1 \\ s \\ s^2 \\ 0 \end{bmatrix} \right\} \tag{22}$$

Finally for $d = 4$ and by analogous arguments as above an arbitrary $\tilde{\mathbf{x}}(s) \in \mathscr{X}_4$ with $\deg \tilde{\mathbf{x}}(s) = 3$ can be written as

$$\tilde{\mathbf{x}}(s) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} (a_{10} + a_{11}s + a_{12}s^2 + a_{13}s^3) + \begin{bmatrix} 1 \\ s \\ s^2 \\ 0 \end{bmatrix} (a_{20} + a_{21}s)$$

$$= \begin{bmatrix} a_{20} \\ 0 \\ 0 \\ a_{10} \end{bmatrix} + \begin{bmatrix} a_{21} \\ a_{20} \\ 0 \\ a_{11} \end{bmatrix} s + \begin{bmatrix} 0 \\ a_{21} \\ a_{20} \\ a_{12} \end{bmatrix} s^2 + \begin{bmatrix} 0 \\ 0 \\ a_{21} \\ a_{13} \end{bmatrix} s^3 = \tilde{\mathbf{x}}_0 + \tilde{\mathbf{x}}_1 s + \tilde{\mathbf{x}}_2 s + \tilde{\mathbf{x}}_3 s^3$$

so that the (unique) c.s. $\mathscr{R}$ of $\tilde{A}$, $\tilde{B}$ of dim $\mathscr{R}=4$ (i.e. the state space $\mathscr{X}$) is given by

$$\mathscr{R}=\operatorname{span}\left\{\begin{bmatrix}a_{20}\\0\\0\\a_{10}\end{bmatrix},\begin{bmatrix}a_{21}\\a_{20}\\0\\a_{11}\end{bmatrix},\begin{bmatrix}0\\a_{21}\\a_{20}\\a_{12}\end{bmatrix},\begin{bmatrix}0\\0\\a_{21}\\a_{13}\end{bmatrix}\right\} \tag{23}$$

Writing $\mathbf{x}(s)$ as

$$\mathbf{x}(s)=\begin{bmatrix}0\\0\\0\\1\end{bmatrix}a_{10}+\begin{bmatrix}0\\0\\0\\s\end{bmatrix}a_{11}+\begin{bmatrix}0\\0\\0\\s^2\end{bmatrix}a_{12}+\begin{bmatrix}0\\0\\0\\s^3\end{bmatrix}a_{13}+\begin{bmatrix}1\\s\\s^2\\0\end{bmatrix}a_{20}+\begin{bmatrix}s\\s^2\\s^3\\0\end{bmatrix}a_{21}$$

we see that

$$\mathscr{X}_4=\operatorname{span}_{\Re}\left\{\begin{bmatrix}0\\0\\0\\1\end{bmatrix},\begin{bmatrix}0\\0\\0\\s\end{bmatrix},\begin{bmatrix}0\\0\\0\\s^2\end{bmatrix},\begin{bmatrix}0\\0\\0\\s^3\end{bmatrix},\begin{bmatrix}1\\s\\s^2\\0\end{bmatrix},\begin{bmatrix}s\\s^2\\s^3\\0\end{bmatrix}\right\} \tag{24}$$

3. Structure of the basis matrices of all c.s.'s $\mathscr{R}$ of A, B of all possible dimensions

The controllability indices $k_1 \leqslant k_2 \leqslant \ldots \leqslant k_m$ that correspond to a controllable pair $[\tilde{A}, \tilde{B}]$, in the Brunovski (1966) and Luenberger (1967) controllable canonical form, define $\mathscr{X}(s)$ and $M_{\mathscr{X}(s)}$ uniquely. Let some $\tilde{\mathbf{x}}(s)\in M_{\mathscr{X}(s)}$ with $\deg \tilde{\mathbf{x}}(s)=d-1$ $(n \geqslant d \geqslant 1)$. Using eqn. (10) $\tilde{\mathbf{x}}(s)$ can be written in its most general form as

$$\tilde{\mathbf{x}}(s)=\sum_{i\,:\,v_i \leqslant d-1}\tilde{\mathbf{x}}_i(s)a_i(s)=\tilde{\mathbf{x}}_0+\tilde{x}_1 s+\ldots+\tilde{\mathbf{x}}_{d-1}s^{d-1} \tag{25}$$

for $a_i(s)=a_{i0}+a_{i1}s+\ldots+a_{ir_i}s^{r_i}\in\Re[s]$, for all i, such that $v_i \leqslant d-1$ and $r_i=\deg a_i(s)=\deg \mathbf{x}(s)-v_i=d-1-v_i$.

Let $\mathbf{a}$ denote the ordered set of coefficients a_{ij} of all $a_i(s)\in\Re[s]$ appearing in eqn. (25), i.e. let

$$\mathbf{a}=\{a_{ij}\,|\,i: v_i \leqslant d-1 \quad \text{and} \quad j=0,1,\ldots,d-1-v_i\} \tag{26}$$

and consider the $n \times d$ matrix

$$[\tilde{\mathbf{x}}_0, \tilde{\mathbf{x}}_1, \ldots, \tilde{\mathbf{x}}_{d-1}] \tag{27}$$

From the structure of the vectors $\tilde{\mathbf{x}}_i(s)$ $i\in\mathbf{m}$ spanning $\mathscr{X}(s)$ and eqn. (25) (see also Example 1) we see that the components of the d column vectors $\tilde{\mathbf{x}}_k\in\mathscr{X}$ $k=0, 1, \ldots, d-1$ are either zeros or possibly non-zero entries a_{ij} corresponding to the coefficients of the polynomials $a_i(s)$, $i: v_i \leqslant d-1$.

Writing out in full the $n \times d$ matrix $[\mathbf{x}_0, \ldots, \mathbf{x}_{d-1}]$ using either zeros or symbols a_{ij} for the possibly non-zero entries we obtain a matrix which we call a basis structure matrix (BSM) and denote by $R_d(\mathbf{a})$.

Definition 4

If a set $\mathbf{a}$ of $a_{ij} \in \Re$ can be found, such that $R_d(\mathbf{a})$ has full rank (i.e. d), then $R_d(\mathbf{a})$ is called a c.s. basis structure matrix (CSBSM).

A c.s. basis structure matrix describes the structure of all basis matrices of all c.s.'s $\mathscr{R}$ of $\tilde{A}$, $\tilde{B}$ of dim $\mathscr{R} = d$. If no set $\mathbf{a}$ of $a_{ij} \in \Re$ exists, such that the basis structure matrix $R_d(\mathbf{a})$ has full rank, then there exists no c.s. $\mathscr{R}$ of $\tilde{A}$, $\tilde{B}$ of dim $\mathscr{R} = d$.

A BSM $R_d(\mathbf{a})$ will involve as many possibly non-zero and different unspecified entries a_{ij} as the total number of the coefficients of the $a_i(s) \in \Re[s]$ appearing in eqn. (25). Let q_i be the number of coefficients a_{ij} in $a_i(s)$, then

$$q_i = \deg a_i(s) + 1 = d - 1 - v_i + 1 = d - v_i$$

For $q_i \geq 0$ we have that i is such, so that $d - v_i \geq 0$. If l_d denotes the total number of coefficients a_{ij} of all $a_i(s) \in \Re[s]$ appearing in a basis structure matrix $R_d(\mathbf{a})$, then

$$l_d = \sum_{i\,:\,v_i \leq d} q_i = \sum_{i\,:\,v_i \leq d} (d - v_i) = \dim_{\Re} \mathscr{X}_d \tag{28}$$

So we have

Proposition 4

The number l_d of the different possibly non-zero unspecified entries a_{ij} appearing in a basis structure matrix $R_d(\mathbf{a})$ is equal to $\dim_{\Re} \mathscr{X}_d$.

We give now an algorithm to construct a basis structure matrix from the data $k_1 \leq \ldots \leq k_m$ and an integer $1 \leq d \leq n$.

Algorithm to form a basis structure matrix

(1) Form an $n \times n$ crate CR.
(2) $i = 0$, $j = 0$.
(3) $i = i + 1$.
(4) If $i \leq m$ CONTINUE, OTHERWISE GO TO 12.
(5) $f_i = n - v_{m+1-i}$.
(6) $k = 0$.
(7) $k = k + 1$.
(8) If $k \leq k_{m+1-i}$ CONTINUE, OTHERWISE GO TO 3.
(9) $j = j + 1$.
(10) Put the f_i symbols : $a_{m+1-i,0}, a_{m+1-i,1}, \ldots, a_{m+1-i,f_i-1}$ in the jth row of the crate starting in the kth columns of it.
(11) GO TO 7.
(12) Fill the remaining compartments of the crate with zeros.
(13) Keep the first d columns of the crate and put equal to zeros all symbols $a_{m+1-i,j}$ in the columns $d+1, d+2, \ldots, n$ of the crate.
(14) Make the necessary adjustments in the elements of the first d columns of the crate by putting equal to zero all the symbols $a_{m+1-i,j}$ of columns $1, 2, \ldots, d$ that also appeared in the columns $d+1, d+2, \ldots, n$ of the crate and call the resulting sub-crate $R_d(\mathbf{a})$ a basis structure matrix.

The resulting form of the crate that we obtain after step (12) of the above algorithm has been completed is the following :

$$
\begin{array}{c|llllllllll}
 & a_{m0} & a_{m1} & \cdots\cdots & a_{mf_1-1} & 0 & 0 & & & & 0 \\
k_m & 0 & a_{m0} & a_{m1} & \cdots\cdots & a_{mf_1-1} & 0 & & & & \vdots \\
 & \vdots & & & & & & & & & \\
 & 0 & 0 & 0 & a_{m0} & a_{m1} & \cdots\cdots & a_{mf_1-1} & 0 & 0 & 0 \\
\hline
 & a_{m-1,0} & a_{m-1,1} & & & a_{m-1,f_2-1} & 0 & & & & \\
k_{m-1} & 0 & a_{m-1,0} & a_{m-1,1} & & & a_{m-1,f_2-1} & 0 & 0 & \cdots\cdots & 0 \\
 & \vdots & & & & & & & & & \vdots \\
 & 0 & 0 & 0 & a_{m-1,0} & a_{m-1,1} & & & a_{m-1,f_2-1} & 0 \cdots\cdots & 0 \\
\hline
 & \vdots & & & \vdots & & & & \vdots & \vdots & \vdots \\
\hline
 & a_{10} & a_{11} & \cdots\cdots & & & & & a_{1f_m-1} & 0 \cdots\cdots & 0 \\
k_1 & 0 & a_{10} & a_{11} & \cdots\cdots & & & & & a_{1f_m-1} & \vdots \\
 & \vdots & & & & & & & & & \vdots \\
 & 0 \cdots\cdots & 0 & a_{10} & a_{11} & \cdots\cdots & & & & & a_{1f_m-1}
\end{array}
\qquad (29)
$$

so that the crate has the form

$$\mathrm{CR}=\begin{bmatrix} A_m \\ \hdashline A_{m-1} \\ \hdashline \vdots \\ \hdashline A_1 \end{bmatrix} \tag{30}$$

where A_{m+1-i} $i\in\mathbf{m}$ are $k_{m+1-i}\times n$ $i\in\mathbf{m}$ blocks each containing $f_i=n-v_{m+1-i}$ possibly non-zero entries a_{m+1-ij} $i\in\mathbf{m}$; $j=0, 1, \ldots, f_i-1$ arranged as shown in (29). If MPR (A_{m+1-i}) denotes the maximum possible rank of the matrix A_{m+1-i} obtained by giving appropriate numerical values to the entries a_{m+1-ij} of the blocks A_{m+1-i}, then from the structure of these blocks we have : MPR $(A_{m+1-i})=$ number of rows of $A_{m+1-i}=k_{m+1-i}$, $i\in\mathbf{m}$.

The resulting form of a BSM $R_d(\mathbf{a})$ obtained from the algorithm is

$$R_d(\mathbf{a})=\begin{bmatrix} D_m{}^d \\ \hdashline D_{m-1}{}^d \\ \hdashline \vdots \\ \hdashline D_1{}^d \end{bmatrix} \tag{31}$$

where $D_{m+1-i}{}^d$ is a $k_{m+1-i}\times d$, $i\in\mathbf{m}$, block which : (i) contains only zeros if $1\leqslant d<k_{m+1-i}$ $i\in\mathbf{m}$ and (ii) if $k_{m+1-i}\leqslant d\leqslant n$ contains $d-v_{m+1-i}$ possibly non-zero entries a_{m+1-ij} $i\in\mathbf{m}$, $j=0, 1, \ldots, d-v_{m+1-i}-1$ arranged so that if a possibly non-zero entry a_{m+1-ij} appears in a certain row and column of a block $D_{m+1-i}{}^d$ $i\in\mathbf{m}$, then it reappears in the next row and column (if there is any) of the same block $D_{m+1-i}{}^d$.

If $F_{m+1-i}=\mathrm{MPR}\ (D_{m+1-i}{}^d)$ $i\in\mathbf{m}$, then from the particular structure of the blocks $D_{m+1-i}{}^d$ we have

Remark 1

$$F_{m+1-i}{}^d=\begin{cases} =0 & \text{if} \quad 1\leqslant d<k_{m+1-i} \\ =k_{m+1-i} & \text{if} \quad k_{m+1-i}\leqslant d\leqslant n \end{cases} \tag{32}$$

Remark 2

For every integer d : $1\leqslant d\leqslant n$ the BSM $R_d(\mathbf{a})$ has the general form

$$R_d(\mathbf{a})=\begin{bmatrix} \mathbf{0} \\ \hline T_d(\mathbf{a}) \end{bmatrix} \tag{33}$$

where $T_d(\mathbf{a})$ is a $(k_1+k_2+\ldots+k_e)\times d$ matrix consisting of possibly non-zero entries a_{m+1-ij}, $\mathbf{0}$ a $[n-(k_1+k_2+\ldots+k_e)]\times d$ zero matrix and $1\leqslant e\leqslant m$.

From these remarks

$$\text{[the number of possibly non-zero rows of] } R_d(\mathbf{a}) \triangleq e^d = \sum_{i=1}^{m} F_{m+1-i}{}^d \tag{34}$$

so that we can state the following.

Proposition 5

The number of the possible c.s.'s $\mathscr{R}$ of $\tilde{A}$, $\tilde{B}$ of dim $R=d$ is determined by the number e^d and if

(i) $0 \leqslant e^d < d$: There is no c.s. of $\tilde{A}$, $\tilde{B}$ of dimension d.

(ii) $e^d = d$: There is a unique c.s. of $\tilde{A}$, $\tilde{B}$ of dimension d given by

$$\mathscr{R} = \text{span}\begin{bmatrix} \mathbf{0} \\ \\ I_d \end{bmatrix}.$$

(iii) $e^d > d$: There is an infinity of c.s. of $\tilde{A}$, $\tilde{B}$ of dimension d.

Proof

(i) From (34) $R_d(\mathbf{a})$ has not full rank hence is not a CSBSM.

(ii) We will prove that under the assumptions made, for any numerical choice of the l_d-vector $\mathbf{a}$ that makes rank $(R_d(\mathbf{a}))=d$, $R_d(\mathbf{a})$ is a basis matrix of a unique d-dimensional c.s. $\mathscr{R}$ if $\tilde{A}$, $\tilde{B}$ given by $\mathscr{R} = \text{span } R_d(\mathbf{a})$. Let $\mathbf{a}_1 \neq \mathbf{a}_2$ arbitrary but such that rank $R_d(\mathbf{a}_i)=d$ $i=1, 2$ and consider the matrix equation :

$$R_d(a_1) = R_d(a_2)M \tag{35}$$

where $M \in \Re^{d \times d}$. If eqn. (35) has a solution with respect to M, such that $\det M \neq 0$, then (ii) follows. From Remark 2 and the fact that $k_1 + k_2 + \ldots + k_e = e^d = d$, eqn. (34) can be written as

$$\begin{bmatrix} \mathbf{0} \\ \hline I_d \end{bmatrix} T_d(\mathbf{a}_1) = \begin{bmatrix} \mathbf{0} \\ \hline I_d \end{bmatrix} T_d(\mathbf{a}_2)M \tag{36}$$

or equivalently $T_d(\mathbf{a}_1) = T_d(\mathbf{a}_2)M$, hence $M = T_d(\mathbf{a}_2)^{-1}T_d(\mathbf{a}_1)$, since $T_d(\mathbf{a}_i)$ $i=1, 2$ is non-singular by assumption, i.e. M is non-singular.

Q.E.D.

(iii) Let $\mathbf{a}_1 \neq \mathbf{a}_2$ arbitrary but such that rank $R_d(\mathbf{a}_i)=d$ $i=1, 2$, then span $R_d(\mathbf{a}_1) = \text{span } R_d(\mathbf{a}_2)$ iff all non-zero $d \times d$ minors of $R_d(\mathbf{a}_1)$ are equal to those of $R_d(\mathbf{a}_2)$ up to a multiplicative constant.

Consider the list $m_1, m_2, \ldots$ of distinct non-zero $d \times d$ minors of $R_d(\mathbf{a})$; these minors consist of polynomials in the components of $\mathbf{a}$, at least two of which will be distinct if $e_d > d$. Hence, if we normalize the list $m_1, m_2, \ldots$ by dividing through by (say) m_1, we obtain a list of rational functions, at least one of which will be non-trivial and so will traverse an infinity of values as $\mathbf{a}$ varies continuously. Such a list is in 1–1 correspondence with all d-dimensional subspaces of $R_d(\mathbf{a})$ and this shows that there is indeed an infinity of distinct c.s. of $\tilde{A}$, $\tilde{B}$ of dimension d if $e_d > d$.

Finally let that for some $1 \leqslant d \leqslant n$, $R_d(\mathbf{a})$ be a BSM and let that $l_d = 1$, i.e. $\mathbf{a} \equiv a_{10}$. Then for every non-zero value of $a_{10} \in \Re$ rank $R_d(a_{10}) = d$, i.e. $R_d(a_{10})$ is a CSBSM of the same d-dimensional c.s. $\mathscr{R}$ of A, B.

From this result and Proposition 4 we have

Proposition 6

A sufficient condition for the existence of a unique c.s. $\mathscr{R}$ of $\tilde{A}$, $\tilde{B}$ of dim $\mathscr{R} = d$ is that $\dim_{\Re} \mathscr{X}_d = 1$.

Example

Let that for some pair $[\tilde{A}, \tilde{B}]$, $n = 7$, $m = 2$, $k_1 = 3$, $k_2 = 4$. Then $v_1 = 2$, $v_2 = 3$ and the algorithm gives the crate

$$\begin{bmatrix} A_2 \\ \cdots \\ A_1 \end{bmatrix} = \begin{bmatrix} a_{20} & a_{21} & a_{22} & a_{23} & 0 & 0 & 0 \\ 0 & a_{20} & a_{21} & a_{22} & a_{23} & 0 & 0 \\ 0 & 0 & a_{20} & a_{21} & a_{22} & a_{23} & 0 \\ 0 & 0 & 0 & a_{20} & a_{21} & a_{22} & a_{23} \\ \hdashline a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & 0 & 0 \\ 0 & a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & 0 \\ 0 & 0 & a_{10} & a_{11} & a_{12} & a_{13} & a_{14} \end{bmatrix}$$

For $d = 1$, 2 the resulting BSM's are zero matrices, so that no c.s.'s of dimensions 1 and 2 exist. For $d = 3$, $\dim_{\Re} \mathscr{X}_3 = \sum_{i : v_i \leqslant d} (d - v_i) = (3-3) + (3-2) = 1$ (see Proposition 6). So that there exists a unique c.s. $\mathscr{R}$ of $\tilde{A}$, $\tilde{B}$ of dim $\mathscr{R} = 3$. From the algorithm we obtain that

$$R_3(\mathbf{a}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hdashline a_{10} & 0 & 0 \\ 0 & a_{10} & 0 \\ 0 & 0 & a_{10} \end{bmatrix} = \begin{bmatrix} 0 \\ \cdots \\ T_3(\mathbf{a}) \end{bmatrix}$$

where $\mathbf{a} \equiv a_{10}$ and hence this unique three-dimensional c.s. $\mathscr{R}$ of $\tilde{A}$, $\tilde{B}$ is given by $\mathscr{R} = \text{span}\,(\mathbf{e}_5, \mathbf{e}_6, \mathbf{e}_7)$ (where $\mathbf{e}_i$ is the ith standard unit vector in $\mathscr{X}$).

For $d=4$ the resulting CSBSM has the form

$$R_4(\mathbf{a})=\begin{bmatrix} a_{20} & 0 & 0 & 0 \\ 0 & a_{20} & 0 & 0 \\ 0 & 0 & a_{20} & 0 \\ 0 & 0 & 0 & a_{20} \\ a_{10} & a_{11} & 0 & 0 \\ 0 & a_{10} & a_{11} & 0 \\ 0 & 0 & a_{10} & a_{11} \end{bmatrix}$$

and so there is an infinity of c.s. $\mathscr{R}$ of $\tilde{A}$, $\tilde{B}$ of dim $R=4$, e.g. the sub-spaces $\mathscr{R}=\text{span}\,(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$, $\mathscr{R}=(\mathbf{e}_1+\mathbf{e}_5, \mathbf{e}_2+\mathbf{e}_6, \mathbf{e}_3+\mathbf{e}_7, \mathbf{e}_4)$, etc., are examples of four-dimensional c.s.'s of $\tilde{A}$, $\tilde{B}$. Similarly for $d=5$, 6 there is infinity of c.s.'s of dimensions 5, 6 and for $d=7$ there is a unique c.s. of dimension 7 which is the (controllable) state space $\mathscr{X}$.

4. Conclusions

A parametrization of the basis matrices of all possible controllability subspaces $\mathscr{R}$ of a pair $\tilde{A}$, $\tilde{B}$ in the Brunovski (1966) and Luenberger (1967) controllable canonical form was presented. It was shown that the structure of these matrices depends entirely on the controllability indices $k_1 \leqslant k_2 \leqslant \ldots \leqslant k_m$ of the pair $\tilde{A}$, $\tilde{B}$ and is independent of the particular numerical values of the elements of the matrices $\tilde{A}$ and $\tilde{B}$.

An efficient algorithm to construct the parametric form of the bases of all possible c.s.'s of $\tilde{A}$, $\tilde{B}$ from the data $k_1 \leqslant k_2 \leqslant \ldots \leqslant k_m$ was given and the particular parametric structure of these bases was used in order to investigate the problem regarding the number and the existence of c.s.'s of $\tilde{A}$, $\tilde{B}$ of a certain dimension.

These results could be exploited in order to further investigate the existence of a solution to the restricted decoupling problem (RDP) (Wonham 1974). For example, the problem of finding the maximal c.s. $\mathscr{R}^*$ of $\tilde{A}$, $\tilde{B}$ contained in a subspace $\mathscr{K}_i$ of $\mathscr{X}$ could be solved directly by constructing all parametric, forms of the bases of c.s.'s of dimensions equal or less than dim $\mathscr{K}_i$ and then examining whether values of the parameters a_{ij} exist, so that an inclusion is satisfied. In the case of the RDP when, although a family of maximal c.s.'s $\mathscr{R}_i^*$ $i \in \mathbf{m}$ ($m \geqslant 2$) exists, such that $\mathscr{R}_i^* + \mathscr{K}_i = \mathscr{X}$ $i \in \mathbf{m}$ (where $\mathscr{K}_i = \ker \tilde{D}_i$ ($i \in \mathbf{m}$) and $\tilde{D}_i : \mathscr{X} \to \mathscr{Y}_i$, see Wonham (1974), chapter 9) but the $\mathscr{R}_i^*$ are not compatible, the algorithm could be used in order to scan over all families of compatible $\mathscr{R}_i$, such that $\mathscr{R}_i + \mathscr{K}_i = \mathscr{X}$ $i \in \mathbf{m}$ and $R_i \subset R_i^*$, with strict inclusion for some i. Another problem that probably could be solved using a similar approach is that of finding minimal dimension c.s. covering arbitrary subspaces $\mathscr{L}$ of $\mathscr{X}$.

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