# An Algorithm for Decoupling and Maximal Pole Assignment in Multivariable Systems by the Use of State Feedback

A. I. G. VARDULAKIS AND P. N. R. STOYLE

*Abstract*—**A simple algorithm to decouple a multivariable system and simultaneously assign the maximum possible number of the poles of the resulting closed-loop decoupled system by the use of state variable feedback is presented. Various questions regarding the minimality of the decoupled system are resolved. The whole procedure is illustrated by an example.**

## I. Introduction

We consider a linear time-invariant finite-dimensional multivariable system $\Sigma$ described by a set of state space equations

$$\dot{x} = Ax + Bu \tag{1a}$$

$$y = Cx \tag{1b}$$

where $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$, $C \in \mathbf{R}^{m\times n}$. We assume that $\Sigma = (A,B,C)$ is completely controllable and observable, that $B$ and $C$ are of full rank $m < n$ and that $\Sigma$ gives rise to a square $(m \times m)$ nonsingular (over $\mathbf{R}(s)$: the field of rational functions) transfer function matrix $T(s) = C(sI-A)^{-1}B$.

In this paper a simple algorithm is presented to compute a linear state variable feedback (LSVF) control law

$$u = F^*x + G^*v \tag{2}$$

$F^* \in \mathbf{R}^{m\times n}$, $G^* \in \mathbf{R}^{m\times m}$, and $\det G^* \neq 0$, such that the closed-loop system $\Sigma_{F^*,G^*} = (A + BF^*, BG^*, C)$ gives rise to a transfer function matrix $T_{F^*,G^*}(s) = C(sI - A - BF^*)^{-1}BG^*$ which is diagonal, nonsingular (over $\mathbf{R}(s)$) (i.e., the closed-loop system $\Sigma_{F^*,G^*}$ is "decoupled"), and the maximum possible number of the poles of $T_{F^*,G^*}(s)$ is arbitrarily assigned.

As it has been shown in [1], the existence of a "decoupling" LSVF control law $(F^*, G^*)$ entirely depends on: 1) the special form—$\hat{C}$ which the output matrix $C$ takes when $(A,B,C)$ is transformed to its Luenberger controllable canonical form [2]; and 2) the controllability indexes of the pair $(A,B)$. This fact is formally restated here in Theorem 1 which gives a new form of the classical necessary and sufficient condition [3] for the existence of a decoupling LSVF control law $(F^*, G^*)$. The algorithm which we will develop is based on the results reported in [1] and the approach adopted also gives information about the structure of the closed-loop system $\Sigma_{F^*,G^*}$. In particular, we examine under what conditions $\Sigma_{F^*,G^*}$ is minimal (i.e., the pair $A + BF^*, C$ is observable) and if it is not minimal, how unobservability of $\Sigma_{F^*,G^*}$ is related to the eigenvalues and eigenvectors of $A + BF^*$ and the "zeros" of $T(s)$ [4]. In the following, we let $\boldsymbol{m}$ denote the set of integers $\{1, 2, \cdots, m\}$.

## II. Decoupling and Pole Assignment

Under the controllability assumption of $(A,B)$ it is known [2], [5] that there always exists a nonsingular coordinate transformation $\hat{x} = Tx$ such that $TAT^{-1} = \hat{A}$, $TB = \hat{B}$, $CT^{-1} = \hat{C}$, and the pair $(\hat{A}, \hat{B})$ is in the Luenberger controllable canonical form. If $v_m \geq v_{m-1} \geq \cdots \geq v_1 \geq 1$ are the controllability indexes of $(\hat{A}, \hat{B})$ and $p_i = \Sigma_{j=1}^{i} v_{m+1-j}$, $i \in \boldsymbol{m}$ then let $\hat{B}_m$ be the $m \times m$ matrix consisting of the $p_i$ $i \in \boldsymbol{m}$ ordered rows of $\hat{B}$, then $\hat{B} = \tilde{B}\hat{B}_m$ [6], where $\tilde{B} = \text{block diag}[\tilde{b}_1, \tilde{b}_2, \cdots, \tilde{b}_m]$, $\tilde{b}_i = (0, 0, \cdots, 0, 1)^T \in \mathbf{R}^{v_{m+1-i} \times 1}$ $i \in \boldsymbol{m}$. Now if we take in (2)

Manuscript received September 22, 1978.

A. I. G. Vardulakis is with the Department of Computing and Control, Imperial College of Science and Technology, London, England, on leave at the Control and Management Systems Division, Department of Engineering, Cambridge University, Cambridge, England.

P. N. R. Stoyle is with the Department of Computing and Control, Imperial College of Science and Technology, London, England.

0018-9286/79/0400-0362$00.75 

$$G^* = \hat{B}_m^{-1}\tilde{G}^* \tag{4}$$

then it can be verified that

$$T_{F^*,G^*}(s) = C(sI - A - BF^*)^{-1}BG^* = \tilde{C}(sI - \tilde{A} - \tilde{B}\tilde{F}^*)^{-1}\tilde{B}\tilde{G}^* \tag{5}$$

where $\tilde{A} \equiv \hat{A}$, $\tilde{C} \equiv \hat{C}$, and now in order that $T_{F^*,G^*}(s)$ is diagonal nonsingular and the maximum possible number of its poles have desired values we must choose $\tilde{F}^*$ and $\tilde{G}^*$. In view of (5) and the LSVF form of the Wolovich and Falb [5], [6] structure theorem, $T_{F^*,G^*}(s)$ can be written as

$$T_{F^*,G^*}(s) = \tilde{C}(sI - \tilde{A} - \tilde{B}\tilde{F}^*)^{-1}\tilde{B}\tilde{G}^* = \tilde{C}S(s)\delta_{\tilde{F}^*}^{-1}(s)\tilde{G}^* \tag{6}$$

where $S(s) = \text{block diag}[\hat{s}_1, \hat{s}_2, \cdots, \hat{s}_m]$, $\hat{s}_i = (1, s, s^2, \cdots, s^{v_{m+1-i}-1})^T$, $i \in \mathbf{m}$

$$\delta_{\tilde{F}^*}(s) = [s^v] - (\tilde{A}_m + \tilde{F}^*)S(s) \tag{7}$$

$[s^v] = \text{diag}[s^{v_m}, s^{v_{m-1}}, \cdots, s^{v_1}]$ and $\tilde{A}_m$ the $m \times n$ matrix consisting of the $m$ ordered $p_i$, $i \in \mathbf{m}$ rows of $\tilde{A}$.

Let $\tilde{C}(s) = \tilde{C}S(s)$ be the "numerator matrix" in the matrix fraction description of (6) of $T_{F^*,G^*}(s)$, then we have the following.

*Theorem 1:* A necessary and sufficient condition for the existence of the LSVF control law $(\tilde{F}^*, \tilde{G}^*)$ such that $T_{F^*,G^*}(s)$ is diagonal and nonsingular is that the matrix

$$K = [\tilde{C}(s)[s^{v_m - v}]]_r^h \tag{8}$$

is nonsingular. Where $[s^{v_m - v}] = \text{diag}[1, s^{v_m - v_{m-1}}, \cdots, s^{v_m - v_1}]$ and $[\ \ ]_r^h$ denotes the matrix with elements in $\mathbf{R}$ consisting of the coefficients of the highest degree $s$ terms in each row of the expression inside the brackets.

The proof of the above theorem is given in [1] and will not be repeated here. We only mention that the relation of $K$ with the classical $B^*$ matrix [3], [5] is given by $K = B^*\hat{B}_m^{-1}$.

Assume now that the necessary and sufficient condition of Theorem 1 is satisfied. We will describe a way to compute $\tilde{F}^*$ and $\tilde{G}^*$. Let us assume that $\tilde{C}(s)$ is nonsingular (over $\mathbf{R}(s)$), i.e., that $\det \tilde{C}(s) \neq 0$ (which is equivalent to the assumption made in the introduction that $T(s)$ is nonsingular) and let $D_C(s)$ be the $m \times m$ nonsingular diagonal polynomial matrix

$$D_C(s) = \text{diag}[d_1(s), d_2(s), \cdots, d_m(s)] \tag{9}$$

with each $d_i(s)$, $i \in \mathbf{m}$ equal to the greatest common divisor (gcd) (monic) of the corresponding $i$th row $\tilde{c}_i^T(s)$ of $\tilde{C}(s)$, i.e., let

$$d_i(s) = \text{gcd}[\tilde{c}_{i1}(s), \tilde{c}_{i2}(s), \cdots, \tilde{c}_{im}], \qquad i \in \mathbf{m}$$

where $\tilde{c}_{ij}(s)$ $i \in \mathbf{m}$, $j \in \mathbf{m}$ denotes the $i,j$th polynomial element of $\tilde{C}(s)$. Also let $d_i = \deg\{d_i(s)\}$, $i \in \mathbf{m}$ and write

$$\tilde{C}(s) = D_C(s)\tilde{C}_R(s). \tag{10}$$

Also define [1]

$$f_i = \min_{j \in m}\{v_{m+1-j} - 1 - \deg \tilde{c}_{ij}(s)\}, \qquad i \in \mathbf{m} \tag{11}$$

and let

$$\Delta(s) = \text{diag}[\delta_1(s), \delta_2(s), \cdots, \delta_m(s)] \tag{12}$$

with $\delta_i(s)$, $i \in \mathbf{m}$ arbitrary monic polynomials of degree $\deg\{\delta_i(s)\} = d_i + f_i + 1 = \delta_i$, $i \in \mathbf{m}$. Now take

$$\tilde{G}^* = K^{-1} \tag{13}$$

and assume that there exists an $\tilde{F}^*$ such that it satisfies the equation

$$\delta_{\tilde{F}^*}(s) = K^{-1}\Delta(s)\tilde{C}_R(s) \tag{14}$$

or in view of (7) such that

$$[s^v] - (\tilde{A}_m + \tilde{F}^*)S(s) = K^{-1}\Delta(s)\tilde{C}_R(s). \tag{15}$$

Then

$$\begin{aligned} T_{F^*,G^*}(s) &= C(s)\delta_{\tilde{F}^*}^{-1}(s)\tilde{G}^* \\ &= D_C(s)\tilde{C}_R(s)[K^{-1}\Delta(s)\tilde{C}_R(s)]^{-1}K^{-1} \\ &= D_C(s)\Delta(s)^{-1}, \end{aligned} \tag{16}$$

i.e., $T_{F^*,G^*}(s)$ is diagonal nonsingular and it has zeros which are the zeros of $d_i(s)$, $i \in \mathbf{m}$, and poles which are the zeros of $\delta_i(s)$, $i \in \mathbf{m}$.

*Remark:* The assumption made above that there always exists an $\tilde{F}^*$ satisfying (14) [or (15)] will be fully justified below. In order to show that this assumption is always valid we need the following.

*Proposition 1:* The number $q$ of the zeros of the multivariable system $\Sigma = (A, B, C) \equiv (\tilde{A}, \tilde{B}, \tilde{C})$ or equally (since $\Sigma$ is by assumption completely controllable and observable) the number $q$ of the zeros of $T(s)$ [4], [7]–[9] is given by

$$q = n - m - f \tag{17}$$

where $f = \sum_{i=1}^m f_i$.

*Proof:* It is known that the zeros of $\Sigma$ or equally of $T(s)$ are equal to the zeros of the polynomials $\epsilon_i(s)$, $i \in \mathbf{m}$ of the Smith normal form $E(s) = \text{diag}[\epsilon_1(s), \epsilon_2(s), \cdots, \epsilon_m(s)]$ of $\tilde{C}(s)$, [8], [9], i.e., $q = \deg\{\prod_{i=1}^m \epsilon_i(s)\} = \deg\{\det \tilde{C}(s)\}$. Considering the transfer function matrix $T_{F^*,G^*}(s) = D_C(s)\Delta(s)^{-1}$ of the decoupled system we see that if no zero of $\delta_i(s)$ coincides with a zero of $d_i(s)$[1] then the order of $T_{F^*,G^*}(s)$ is

$$\sum_{i=1}^m \delta_i = \sum_{i=1}^m (f_i + d_i + 1) = f + d + m, \qquad \left(d = \sum_{i=1}^m d_i\right).$$

Hence the number of poles of $T(s)$ that have been canceled out with zeros by the action of the decoupling LSVF $(F^*, G^*)$ is

$$n - \sum_{i=1}^m \delta_i = n - (f + d + m).$$

Also from (16), $T_{F^*,G^*}(s)$ has $d$ zeros hence the number of the zeros that have been canceled out is

$$q - d = \lambda. \tag{18}$$

As the number of zeros and poles canceled out must be the same, we see that $\lambda = n - (f + d + m)$ or $q = n - m - f$. By taking determinants and then degrees in both sides of (10) we obtain

$$q = d + \deg\{\det \tilde{C}_R(s)\} \tag{19}$$

and comparing (18) and (19) we see that

$$\lambda = \deg\{\det \tilde{C}_R(s)\}. \tag{20}$$

Returning to Remark 1 we see that by taking determinants and then degrees in both sides of (15) we have the identity

$$\begin{aligned} n &= \deg\{\det \delta_{F^*}(s)\} = \deg\{\det[K^{-1}\Delta(s)\tilde{C}_R(s)]\} \\ &= \sum_{i=1}^m \delta_i + \lambda = f + d + m + q - d \\ &= f + m + n - m - f = n. \end{aligned}$$

This shows that an $\tilde{F}^*$ satisfying (15) always exists. From the analysis above we have the following.

*Proposition 2:* The triple $(A + BF^*, BG^*, C)$ is minimal if and only if $\lambda = 0$, i.e., if and only if $\tilde{C}_R(s)$ is a unimodular matrix.

[1] So that no further cancellation occurs in $d_i(s)/\delta_i(s)$, $i \in \mathbf{m}$.

## III. On the Structure of the Decoupled Closed-Loop System

From (6) we easily obtain the expression

$$(sI-\tilde{A}-\tilde{B}\tilde{F}^*)S(s)=\tilde{B}\delta_{F^*}(s)$$

which in view of (14) gives

$$(sI-\tilde{A}-\tilde{B}\tilde{F}^*)S(s)=\tilde{B}K^{-1}\Delta(s)\tilde{C}_R(s). \tag{21}$$

Let $\lambda>0$ and assume for simplicity that $\det\tilde{C}_R(s)$ has $\lambda$ distinct zeros[2] $s_i\in\mathbf{C}, i\in\boldsymbol{\lambda}$. If by $E_R(s)$ we denote the Smith normal form of $\tilde{C}_R(s)$, then $E_R(s)$ will have the form $E_R(s)=[1,1,\cdots,1,\epsilon(s)]$ where $\epsilon(s)=\det\tilde{C}_R(s)$ and rank $\tilde{C}_R(s_i)=$ rank $E_R(s_i)=m-1$. Hence $\dim\{\ker\tilde{C}_R(s_i)\}=1$ and there will be some vector $\beta_i\neq 0\in R^{m\times 1}$ such that

$$\tilde{C}_R(s_i)\beta_i=0. \tag{22}$$

From (21) and (22) we obtain

$$(s_iI-\tilde{A}-\tilde{B}\tilde{F}^*)S(s_i)\beta_i=\tilde{B}K^{-1}\Delta(s_i)\tilde{C}_R(s_i)\beta_i=0, \qquad i\in\boldsymbol{\lambda} \tag{23}$$

and if we put

$$\tilde{x}_i=S(s_i)\beta_i, \qquad i\in\boldsymbol{\lambda} \tag{24}$$

from (23) we have

$$(s_iI-\tilde{A}-\tilde{B}\tilde{F}^*)\tilde{x}_i=0, \qquad i\in\boldsymbol{\lambda}. \tag{25}$$

Equation (25) says that the $s_i$, $i\in\boldsymbol{\lambda}$ [i.e., the zeros of $\det\tilde{C}_R(s)$ which are a subset of the zeros of $T(s)$] are eigenvalues of $\tilde{A}+\tilde{B}\tilde{F}^*$ and the corresponding eigenvectors are $\tilde{x}_i$, $i\in\boldsymbol{\lambda}$. Obviously $\tilde{x}_i$ are independent as they correspond to distinct (by assumption) eigenvalues $s_i$, $i\in\boldsymbol{\lambda}$. If we consider the subspace

$$\mathcal{V}=\operatorname{span}\{\tilde{x}_i, i\in\boldsymbol{\lambda}\} \tag{27}$$

then obviously $\dim\mathcal{V}=\lambda$ and $\mathcal{V}$ is an $(\tilde{A}+\tilde{B}\tilde{F}^*)$-invariant subspace or simply an $(\tilde{A},\tilde{B})$-invariant subspace [10]. Now since

$$\tilde{C}\tilde{x}_i=\tilde{C}S(s_i)\beta_i=\tilde{C}(s_i)\beta_i=D_C(s_i)\tilde{C}_R(s_i)\beta_i=0, \qquad i\in\boldsymbol{\lambda} \tag{28}$$

$$\mathcal{V}\subseteq\ker\tilde{C} \tag{29}$$

and therefore $\mathcal{V}$ is the unobservable subspace of the state space $\mathbf{R}^n$. We give now another expression for $\mathcal{V}$. Consider the $(f+d+m)\times m$ polynomial matrix

$$Z(s)=\begin{bmatrix} 1 & 0 & \cdots & 0\\ s & 0 & \cdots & 0\\ \vdots & \vdots & & \vdots\\ s^{f_1+d_1} & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ 0 & s & \cdots & 0\\ \vdots & \vdots & & \vdots\\ 0 & s^{f_2+d_2} & \cdots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & 1\\ 0 & 0 & \cdots & s\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & s^{f_m+d_m} \end{bmatrix}.$$

We define the $(f+d+m)\times m$ polynomial matrix $\tilde{C}_E(s)$ via

$$\tilde{C}_E(s)=Z(s)\tilde{C}_R(s)$$

and let $\tilde{C}_E$ be the $(f+d+m)\times n$ matrix defined uniquely via the equation

$$\tilde{C}_E(s)=\tilde{C}_ES(s).$$

*Proposition 3:* If $\delta_i(s)$, $i\in\boldsymbol{m}$, are chosen so that no zero of $\delta_i(s)$ coincides with a zero of $d_i(s)$, $i\in\boldsymbol{m}$, then the unobservable subspace $\mathcal{V}\in\mathbf{R}^n$ is given by $\mathcal{V}=\ker\tilde{C}_E$.

*Proof:* $\tilde{C}_E\tilde{x}_i=\tilde{C}_ES(s_i)\beta_i=\tilde{C}_E(s_i)\beta_i=Z(s_i)\tilde{C}_R(s_i)\beta_i=0$, $i\in\boldsymbol{\lambda}$, i.e., $\mathcal{V}\subseteq\ker\tilde{C}_E$. From the structure of $\tilde{C}_E$ (see example below), rank $\tilde{C}_E=f+d+m$, therefore $\dim\{\ker\tilde{C}_E\}=n-(f+d+m)=\lambda$, i.e., $\mathcal{V}=\ker\tilde{C}_E$.

*Proposition 4:* In order to decouple a system $\Sigma=(A,B,C)$ that can be decoupled[3] at least $\lambda=\deg\{\det\tilde{C}_R(s)\}$ of its zeros have to be canceled out with coincident poles. (This is done automatically by the use of the decoupling state feedback $F^*$.) If the desired poles, i.e., the zeros of the $\delta_i(s)$ $i\in\boldsymbol{m}$, are chosen so that no further cancellation occurs between the $d_i(s)$ and $\delta_i(s)$ $i\in\boldsymbol{m}$ in one of the subsystems with transfer function $d_i(s)/\delta_i(s)$, and if $\lambda>0$, then the resulting closed-loop system $\Sigma_{F^*,G^*}=(A+BF^*,BG^*,C)$ is controllable[4] but unobservable. The observable subspace of the state space is spanned by the rows of $\tilde{C}_E$ and has dimension $d+f+m=n-\lambda$, and the unobservable subspace is $\mathcal{V}=\ker\tilde{C}_E$ and has dimension $\lambda$.

## IV. Algorithm for Decoupling and Pole Assignment

1) Enter $A,B,C$. If the system is controllable and observable, then go to step 2), otherwise exit.

2) Compute the Luenberger controllable canonical form $\hat{A},\hat{B},\hat{C}$. Store the transformation matrix $T$, the $m\times m$ matrix $\hat{B}_m$, and the controllability indexes $v_m\geq v_{m-1}\geq\cdots\geq v_1$.

3) Form $S(s)$, compute $\tilde{C}(s)$, $D_C(s)$, $\tilde{C}_R(s)$. Put $d_i=\deg\{d_i(s)\}$, $i\in\boldsymbol{m}$ and $d=\sum_{i=1}^m d_i$.

4) Compute $K$ and test if $\det K\neq 0$. If YES then GOTO (5) otherwise exit (system can not be decoupled by LSVF alone).

5) Compute $\tilde{G}^*=K^{-1}$ and $G^*=\hat{B}_m^{-1}\tilde{G}^*$.

6) Compute:

a) $f_i=\min_{j\in\boldsymbol{m}}\{v_{m+1-j}-1-\deg\tilde{c}_{ij}(s)\} \qquad i\in\boldsymbol{m}$

b) $\delta_i=d_i+f_i+1 \qquad i\in\boldsymbol{m}$

and choose $m$ monic polynomials $\delta_i(s)$ of degree $\delta_i$ such that they have desired roots. Form $\Delta(s)=\operatorname{diag}[\delta_1(s),\delta_2(s),\cdots,\delta_m(s)]$.

7) Using (15), compute $\tilde{F}^*$ by inspection and $F^*$ via $F^*=\hat{B}_m^{-1}\tilde{F}^*T$.

## IV. Example

Consider the system in its Luenberger controllable canonical form

$$\hat{A}=\tilde{A}=\left[\begin{array}{cccc|ccc} 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 1 & -1 & 0 & 1 & 2 & 1 & 2\\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1\\ 2 & 1 & -3 & 0 & 2 & -1 & 2 \end{array}\right], \quad \hat{B}=\left[\begin{array}{cc} 0 & 0\\ 0 & 0\\ 0 & 0\\ 1 & 2\\ \hline 0 & 0\\ 0 & 0\\ 0 & 1 \end{array}\right],$$

$$\hat{C}=\tilde{C}=\begin{bmatrix} -2 & -1 & 0 & 0 & -2 & 0 & 0\\ 2 & 3 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

so that

$$v_2=4, \quad v_1=3, \quad n=7, \quad m=2$$

$$\hat{B}_m=\begin{bmatrix} 1 & 2\\ 0 & 1 \end{bmatrix}\hat{B}_m^{-1}=\begin{bmatrix} 1 & -2\\ 0 & 1 \end{bmatrix}$$

$$S(s)=\begin{bmatrix} 1 & s & s^2 & s^3 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & s & s^2 \end{bmatrix}^T$$

$$\tilde{C}(s)=\tilde{C}S(s)=\begin{bmatrix} -(s+2) & -2\\ (s+1)(s+2) & s+1 \end{bmatrix}$$

[2] The more general case of a noncyclic $\tilde{C}_R(s)$ is dealt with in [11].

[3] It satisfies, i.e., the necessary and sufficient condition of Theorem 1.

[4] Controllability is invariant under state feedback.

$$=\begin{bmatrix}1 & 0\\ 0 & s+1\end{bmatrix}\begin{bmatrix}-(s+2) & -2\\ s+2 & 1\end{bmatrix}=D_C(s)\tilde{C}_R(s)$$

and

$$d_1(s)=1,\ d_1=0,\ d_2(s)=(s+1),\ d_2=1,\ d=1$$

$$[s^{a_m-v}]=\begin{bmatrix}1 & 0\\ 0 & s\end{bmatrix}$$

and so

$$K=\left[\begin{bmatrix}-(s+2) & -2\\ (s+1)(s+2) & (s+1)\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & s\end{bmatrix}\right]_r^h$$

$$=\begin{bmatrix}-1 & -2\\ 1 & 1\end{bmatrix},\qquad \det K=1.$$

Hence the system can be decoupled.

$$f_1=\min\{4-1-1,3-1-0\}=2,$$

$$f_2=\min\{4-1-2,3-1-1\}=1,$$

$$q=n-m-(f_1+f_2)=2$$

[number of zeros of $(\hat{A},\hat{B},\hat{C})$, $\delta_1=f_1+d_1+1=3$, $\delta_2=f_2+d_2+1=3$ so let us take $\delta_1(s)=(s+2)^3$, $\delta_2(s)=(s+2)^3$, $\Delta(s)=\operatorname{diag}[\delta_1(s),\delta_2(s)]$. From $\tilde{A}$ by inspection

$$\tilde{A}_m=\begin{bmatrix}1 & -1 & 0 & 1 & 2 & 1 & 2\\ 2 & 1 & -3 & 0 & 2 & -1 & 2\end{bmatrix}$$

$$[s^v]-\tilde{A}_mS(s)=\begin{bmatrix}s^4-s^3+s-1 & -2s^2-s-2\\ 3s^2-s-2 & s^3-2s^2+s-2\end{bmatrix}$$

and

$$K^{-1}\Delta(s)\tilde{C}_R(s)=\begin{bmatrix}(s+2)^4 & 0\\ 0 & (s+2)^3\end{bmatrix}.$$

From

$$[s^v]-\tilde{A}_mS(s)-K^{-1}\Delta(s)\tilde{C}_R(s)$$

$$=\begin{bmatrix}-17-31s-24s^2-9s^3 & -2-s-2s^2\\ -2-s+3s^2 & -10-11s-8s^2\end{bmatrix}=\tilde{F}^*S(s).$$

By inspection we have

$$\tilde{F}^*=\begin{bmatrix}-17 & -31 & -24 & -9 & -2 & -1 & -2\\ -2 & -1 & 3 & 0 & -10 & -11 & -8\end{bmatrix}.$$

Also

$$\tilde{G}^*=K^{-1}=\begin{bmatrix}1 & 2\\ -1 & -1\end{bmatrix}.$$

Furthermore

$$Z(s)=\begin{bmatrix}1 & 0\\ s & 0\\ s^2 & 0\\ 0 & 1\\ 0 & s\\ 0 & s^2\end{bmatrix}$$

and from $\tilde{C}_E(s)=Z(s)\tilde{C}_R(s)=\tilde{C}_ES(s)$ we obtain

$$\tilde{C}_E=\left[\begin{array}{ccccccc}-2 & -1 & 0 & 0 & -2 & 0 & 0\\ 0 & -2 & -1 & 0 & 0 & -2 & 0\\ 0 & 0 & -2 & -1 & 0 & 0 & -2\\ \hline 2 & 1 & 0 & 0 & 1 & 0 & 0\\ 0 & 2 & 1 & 0 & 0 & 1 & 0\\ 0 & 0 & 2 & 1 & 0 & 0 & 1\end{array}\right]$$

and

$$\mathcal{V}=\operatorname{span}\{(1\quad -2\quad 4\quad -8\quad 0\quad 0\quad 0)^T\}.$$

## V. Conclusions

An algorithm to compute an LSVF control law $(F^*,G^*)$ that will decouple an $m$-input, $m$-output multivariable system into $m$ independent subsystems and simultaneously assign the poles of the resulting $m$ subsystems has been presented. The questions concerning the minimality of the resulting decoupled system have been also examined.

## References

[1] P. N. R. Stoyle and A. I. G. Vardulakis, "The mechanism of decoupling," Imperial College, London, England, CCD Research Rep. 78/3; also *Int. J. Contr.*, to be published.
[2] D. G. Luenberger, "Canonical forms for linear multivariable systems," *IEEE Trans. Automat. Contr.*, vol. AC-12, p. 290, 1967.
[3] P. L. Falb and W. A. Wolovich, "Decoupling in the design and synthesis of multivariable control systems," *IEEE Trans. Automat. Contr.*, vol. AC-12, p. 651, 1967.
[4] A. I. G. Vardulakis, "The structure of multivariable systems under the action of constant output feedback control laws," *Int. J. Contr.*, vol. 27, no. 2, p. 261, 1978.
[5] W. A. Wolovich and P. L. Falb, "On the structure of multivariable systems," *SIAM J. Contr.*, no. 7, p. 437, 1969.
[6] W. A. Wolovich, *Linear Multivariable Systems.* Berlin: Springer, 1974.
[7] H. H. Rosenbrock, *State-Space and Multivariable Theory.* London: Nelson, 1970.
[8] ——, "The zeros of a system," *Int. J. Contr.*, vol. 18, no. 2, p. 297, 1973; also "Correction to 'The zeros of a system,'" *Int. J. Contr.*, vol. 20, no. 3, p. 525, 1973.
[9] W. A. Wolovich, "On the numerator and zeros of rational transfer matrices," *IEEE Trans. Automat. Contr.*, vol. AC-18, p. 544, 1973.
[10] W. M. Wonham, *Linear Multivariable Control, A Geometric Approach.* Berlin: Springer, 1974.
[11] P. N. R. Stoyle and A. I. G. Vardulakis, "Polynomial matrix algebra and $(A,B)$-invariant subspaces—towards a more integrated approach," Imperial College, London, England CCD, Research Rep. (under preparation).

0018-9286/79/0400-0365$00.75 