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On certain connections between the geometric and the polynomial matrix approaches to linear system theory

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On certain connections between the geometric and the polynomial matrix approaches to linear system theory

P. N. R. STOYLE† and A. I. G. VARDULAKIS†

Certain connections between the geometric theory of Wonham and Morse and the Rosenbrock–Wolovich–Forney polynomial matrix approaches to linear system theory are indicated. The results of Warren and Eckberg (1975), worked by Vardulakis (1977) into an algorithm which constructively exhibits all controllability subspaces of a given controllable pair (A, B) in canonical form, are extended and turned into a theoretical tool of considerable power. A new theory is developed which translates certain statements of the geometric theory into a succinct algebraic form.

1. Introduction

There are many aspects of the Wonham and Morse geometric theory (Wonham and Morse 1970, Morse and Wonham 1971) which are involved in the actual process of solution of a given problem rather than its mere formulation by geometric means, and which in the literature are treated in a way not especially attractive from a computational point of view. Such are the checking of compatibility for a set of controllability subspaces, or the determination of certain elements in the family of all controllability subspaces contained in some fixed subspace of the state space.

This paper introduces a new and incisive tool in control theory especially well suited for analysing problems such as the ones mentioned above. The new approach put forward here is in effect a synthesis of the 'geometric approach' due to Wonham and Morse (Wonham and Morse 1970, Morse and Wonham 1971, Wonham 1974) and the polynomial matrix methods of Rosenbrock (1970) and Wolovich (1974) in the field of non-interacting control.

Moreover, the method of analysis employed is of independent interest and we believe will find many applications. Thus, apart from the intrinsic interest of the long standing and until now, largely unsatisfactory solved problems of non-interacting control, its importance will have been vindicated in the context of highlighting and understanding the essential structure of linear multivariable systems in general with a view to the design of practical control systems.

Section 2 provides some background theory. Section 3 sets up the basic machinery with, as starting point, the Warren and Eckberg (1975) lemma on the characterization of controllability subspaces. Their result is slightly refined and rendered in a form suitable for establishing the crucial connection with the Wolovich and Falb (1969), Wolovich (1974) structure theorem for linear time-invariant multivariable systems, by introducing the fundamental notion of a 'decoupling vector' as a polynomial vector parametrization of controllability subspaces. The class of feedbacks figuring in the Warren and

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Eckberg paper (1975) and causing a given controllability subspace to be reached is enlarged suitably to allow an illuminating new proof of the well known pole placing properties of controllability subspaces. Section 4 examines the important notion of compatibility of a family of controllability subspaces and establishes a necessary and sufficient condition for such a family to be compatible. Section 5 finally deals with the conditions under which a controllability subspace is to be contained in a fixed vector subspace of the state space and the set of all controllability subspaces (in their polynomial vector parametrization) contained in the kernel of some map is derived.

Notation. Upper case italic letters A, B, ... refer to linear transformations between vector spaces, or their associated matrices. Script letters $\mathscr{R}$, $\mathscr{U}$, ... denote vector spaces or subspaces over $\mathfrak{R}$, the field of reals. By a, b, x, ... we denote elements of vector spaces and the image of a map (e.g. B) we denote by the corresponding script capital (e.g. $\mathscr{B}$). $\mathfrak{R}(s)$ is the field of rational functions in s with coefficients in $\mathfrak{R}$ and $\mathfrak{R}[s]$ is the ring of polynomials in s with coefficients in $\mathfrak{R}$. The set of all $p \times q$ matrices with elements in $\mathfrak{R}$ will be denoted by $\mathfrak{R}^{p \times q}$; the space of polynomial vectors with coefficients in $\mathfrak{R}^q$ is denoted by $\mathfrak{R}^q[s]$ and this is an $\mathfrak{R}[s]$-module. The set of integers $\{1, 2, ..., m\}$ is denoted by $\mathbf{m}$. If $u(s) = u_0 + u_1 + ... + u_k s^k \in \mathfrak{R}^q[s]$ then the notation span $\{u(s)\}$ is used as a shorthand for the vector space span of the coefficients of the powers of s in $u(s)$, e.g. span $\{u(s)\} =$ span $\{u_0, u_1, ..., u_k\}$ and the 'listing of $u(s)$', denoted by L $\{u(s)\}$ is defined as the ordered list of vectors $\langle u_k, u_{k-1}, ..., u_0 \rangle$. We say that the list L $\{u(s)\}$ above has length $k+1$. If $A : \mathfrak{R}^n \to \mathfrak{R}^n$ is a linear map and $\mathscr{V}$ is an A-invariant subspace of $\mathfrak{R}^n$ then we write $A|\mathscr{V}$ to indicate the restriction of A to the subdomain $\mathscr{V}$.

2. Background

We consider a linear, time invariant, finite dimensional multivariable system described by a set of differential equations and an output expression in state space form:

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}\right\} \tag{1}$$

or equally by the analogous set of difference equations:

$$\left.\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}\right\} \tag{2}$$

where $A \in \mathfrak{R}^{n \times n}$, $B \in \mathfrak{R}^{n \times m}$, $C \in \mathfrak{R}^{p \times n}$. We assume that the pair (A, B) is completely controllable and that B is of full rank $m \leqslant n$. It is then well known that there exists a non-singular coordinate transformation $\tilde{x}(t) = Tx(t)$ which transforms system (1) to the system

$$\left.\begin{aligned} \dot{\tilde{x}}(t) &= \hat{A}\tilde{x}(t) + \hat{B}u(t) \\ y(t) &= \hat{C}\tilde{x}(t) \end{aligned}\right\} \tag{3}$$

where $\hat{A}=TAT^{-1}$, $\hat{B}=TB$, $\hat{C}=CT^{-1}$ and the triplet of matrices $(\hat{A}, \hat{B}, \hat{C})$ is in the Luenberger (1967) controllable canonical form (Wolovich and Falb 1969). If by

$$1 \leqslant v_1 \leqslant v_2 \leqslant \ldots \leqslant v_m$$

we denote the so called Kronecker (or controllability) indices of the pair (A, B) then $\sum_{i=1}^{m} v_i = n$ and as described by Denham (1974) and Dickinson (1976)

$$\hat{A}=\begin{bmatrix} A_{11} & \ldots & A_{1m} \\ \vdots & & \vdots \\ A_{m1} & \ldots & A_{mm} \end{bmatrix}, \quad \hat{B}=\begin{bmatrix} B_1 \\ \vdots \\ B_m \end{bmatrix} .$$

and the blocks $A_{ii} \in \Re^{v_{m+1-i} \times v_{m+1-i}}$, $A_{ij} \in \Re^{v_{m+1-i} \times v_{m+1-i}}$, $i \in \mathbf{m}$, $j \in \mathbf{m}$ have the form

$$A_{ii}=\begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & & & \vdots \\ + & + & + & \ldots & + \end{bmatrix}, \quad A_{ij}=\begin{bmatrix} 0 & 0 & 0 & & \ldots & & 0 \\ \\ 0 & 0 & 0 & & \ldots & & 0 \\ \vdots & & & & & & \vdots \\ + & + & \ldots & + & 0 & \ldots & 0 \end{bmatrix}$$

where + denotes a possibly non-zero entry. That is for $i \neq j$, the last row of A_{ij} can have possibly non-zero entries in the first min $\{v_{m+1-i}, v_{m+1-j}\}$ columns and the remaining (if any) columns are occupied by zeros. The block $B_i \in \Re^{v_{m+1-i} \times m}$, $i \in \mathbf{m}$ has zero entries except that in row v_{m+1-i} it has a ' 1 ' in column i and a ' + ' in every column $j > i$ for which $v_{m+1-j} < v_{m+1-i}$. The matrix $\hat{C}$ has no special form and it contains only possibly non-zero entries ' + '. If we set

$$v_i^* = \sum_{j=m}^{m+1-i} v_j, \quad i \in \mathbf{m}$$

then by collecting the m v_i^*th rows of $\hat{B}$ into an $m \times m$ non-singular matrix $\hat{B}_m$ and performing in (3) an input transformation

$$u(t) = \hat{B}_m^{-1} u'(t) \tag{4}$$

we arrive at the input-transformed Luenberger coordinates where the system is described now by the triplet $\tilde{A}=\hat{A}$, $\tilde{B}=(e_{v_1^*}, e_{v_2^*}, \ldots, e_{v_m^*})$ (e_i denotes the ith standard unit vector in $\Re^n$) and $\tilde{C}=\hat{C}$.

3. Algebraic characterization of some controllability subspace properties

3.1.

Let us consider now the class of controllability subspaces (c.s.) $\mathscr{R}$ of the pair (A, B) and denote it by $\mathscr{C}(A, B, \Re^n)$ (Wonham and Morse 1970, Morse and Wonham 1971, Wonham 1974). Warren and Eckberg (1975) gave a characterization of the elements $\mathscr{R}$ of $\mathscr{C}(A, B, \Re^n)$ in terms of the elements of the kernel of the polynomial matrix $(sI-A, -B)$ which is important enough to be restated here in full.

Theorem 1 (Warren and Eckberg 1975)

A subspace $\mathscr{R} \subset \mathfrak{R}^n$ of dimension r is a c.s. of (A, B) if and only if there exist $x(s) \in \mathfrak{R}^n[s]$ and $u(s) \in \mathfrak{R}^m[s]$ such that

(1) $\deg u(s) = k$ and $\deg x(s) = k-1$, for $k \geqslant r$

(2) $(sI - A)x(s) = Bu(s)$

(3) if $x(s) = x_0 + x_1 s + \ldots + x_{k-1} s^{k-1}$, then $\mathscr{R} = \operatorname{span}\{x_{i-1}, i \in \mathbf{k}\}$

We now make the following definition.

Definition 1

If $x(s) = x_0 + x_1 s + \ldots + x_{k-1} s^{k-1} \in \mathfrak{R}^n[s]$ is such that there exist a $u(s) = u_0 + u_1 s + \ldots + u_{k-1} s^{k-1} + u_k s^k \in \mathfrak{R}^m[s]$ so that $(x^T(s),\ u^T(s)) \in \ker(sI - A,\ -B)$ then $x(s)$ is called a 'generating function' of the c.s. $\mathscr{R} = \operatorname{span}\{x_{i-1}, i \in \mathbf{k}\}$, and $u(s)$ is called an (open-loop) c.s. input sequence generating function.

Consider the polynomial matrix equation

$$(sI - A,\ -B)\begin{bmatrix} x(s) \\ u(s) \end{bmatrix} = 0 \tag{5}$$

then again from the results of Warren and Eckberg (1975) we know that a set of free generators for $\ker(sI - A,\ -B)$ is given by a 'fundamental series' of solutions to eqn. (5). A fundamental series of solutions to (5) is defined by Gantmacher (1955), Rosenbrock (1970), Warren and Eckberg (1975). Having chosen a fundamental series of solutions to (5) let it be denoted by $z_1(s), z_2(s), \ldots, z_m(s)$ where $z_i(s) = (x_i^T(s),\ u_i^T(s))^T$, $i \in \mathbf{m}$. Then it is known that $\deg z_i(s) = v_i$, $i \in \mathbf{m}$ (Rosenbrock 1970) and any $z(s) \in \ker(sI - A,\ -B)$ can be uniquely written as

$$z(s) = \sum_{i\,:\,v_i \leqslant \deg z(s)} z_i(s) a_i(s) \tag{6}$$

for appropriate $a_i(s) \in \mathfrak{R}[s]$ satisfying the degree condition $\deg a_i(s) \leqslant \deg z(s) - v_i$. As noted by Warren and Eckberg (1975), this is a consequence of Forney's (1975) fundamental work on the minimal bases of rational vector spaces. The fundamental series $z_i(s)$, $i \in \mathbf{m}$ constitutes such a minimal polynomial basis for $\ker(sI - A,\ -B)$ considered as embedded in the natural way in a vector space $\mathfrak{R}^{n+m}(s)$ over the field of rational functions in s, $\mathfrak{R}(s)$. Any such basis is clearly not unique. However, if we happen to choose a basis for the state space so that the pair (A, B) is in the Luenberger (1967) controllable canonical form $(\hat{A}, \hat{B})$ then Vardulakis (1978) has noted that in this basis the elements of the fundamental series $z_i(s)$, $i \in \mathbf{m}$ are seen to have the particularly simple form

$$\tilde{z}_{m+1-i}(s) = [\underbrace{0\ 0\ \ldots\ 0}_{v_{i-1}{}^*}\ 1\ s\ \ldots\ s^{v_{m+1-i}-1}\ \underbrace{0\ 0\ \ldots\ 0}_{n - v_i{}^*}\ u_{1i}(s), \ldots, u_{mi}(s)]^T \tag{7}$$

$i \in \mathbf{m}$. This observation is a simple consequence of the Wolovich and Falb (1969) and Wolovich (1974) structure theorem for linear multivariable systems and gives rise to the following theorem.

Theorem 2

Let $\tilde{x}(s)=\tilde{x}_0+\tilde{x}_1 s+\ldots+\tilde{x}_{k-1}s^{k-1}\in\Re^n[s]$ be a generating function of a c.s. $\mathscr{R}$ of $(\hat{A},\hat{B})$ with corresponding c.s. input sequence generating function $u(s)=u_0+u_1 s+\ldots+u_{k-1}s^{k-1}+u_k s^k\in\Re^m[s]$. Then there exists a polynomial vector $\beta(s)\in\Re^m[s]:\beta(s)=(\beta_m(s),\beta_{m-1}(s),\ldots,\beta_1(s))^{\mathrm{T}}$ such that

$$\tilde{x}(s)=S(s)\beta(s) \tag{8}$$

$$u(s)=\hat{B}_m{}^{-1}\,\delta_0(s)\beta(s) \tag{9}$$

where $S(s)=\text{block diag}\,(\hat{s}_1,\hat{s}_2,\ldots,\hat{s}_m)$, $\hat{s}_i{}^{\mathrm{T}}=(1\; s\; s^2\;\ldots\; s^{v_{m+1-i}-1})$, $i\in\mathbf{m}$, $\delta_0(s)=[s^v]-\hat{A}_m S(s)$, $[s^v]=\text{diag}\,(s^{v_m},s^{v_{m-1}},\ldots,s^{v_1})$, $\hat{A}_m$ the $m\times n$ matrix consisting of the m ordered $v_i{}^*$, $i\in\mathbf{m}$ rows of $\hat{A}=\tilde{A}$ and $\hat{B}_m$ the $m\times m$ matrix consisting of the m ordered $v_i{}^*$, $i\in\mathbf{m}$ rows of $\hat{B}$. (We use the same notation as the one used by Wolovich (1974).)

Proof

From the Wolovich and Falb (1969) structure theorem we have that

$$(sI-\hat{A},\;-\hat{B})\begin{bmatrix} S(s) \\ \cdots\cdots \\ \hat{B}_m{}^{-1}\,\delta_0(s)\end{bmatrix}=0 \tag{10}$$

Now the columns $\tilde{z}_1(s),\tilde{z}_2(s),\ldots,\tilde{z}_m(s)$ $(\tilde{z}_i{}^{\mathrm{T}}(s)=(\tilde{x}_i{}^{\mathrm{T}}(s),\,u_i{}^{\mathrm{T}}(s),\,i\in\mathbf{m})$ of the polynomial matrix

$$\tilde{Z}(s)=[\tilde{z}_m(s),\tilde{z}_{m-1}(s),\ldots,\tilde{z}_1(s)]=\begin{bmatrix} S(s) \\ \cdots\cdots \\ \hat{B}_m{}^{-1}\,\delta_0(s)\end{bmatrix} \tag{11}$$

constitute a fundamental series of solutions to the polynomial matrix equation

$$(sI-\hat{A},\;-\hat{B})\begin{bmatrix}\tilde{x}(s)\\ u(s)\end{bmatrix}=0 \tag{12}$$

Furthermore, due to its special structure the polynomial matrix $\tilde{Z}(s)$ is a minimal basis in echelon form for the ker $(sI-\hat{A},\;-\hat{B})$ (Forney 1975) and according to Forney's (1975) main theorem any polynomial vector $\tilde{z}(s)\in$ ker $(sI-\hat{A},\;-\hat{B})$ of degree deg $\tilde{z}(s)=k$ can be expressed as

$$\tilde{z}(s)=\begin{bmatrix}\tilde{x}(s)\\ u(s)\end{bmatrix}=\sum_{i\,:\,v_i\leqslant \deg \tilde{z}(s)}\tilde{z}_i(s)\beta_i(s)=\begin{bmatrix} S(s) \\ \cdots\cdots \\ \hat{B}_m{}^{-1}\,\delta_0(s)\end{bmatrix}\begin{bmatrix}\beta_m(s)\\ \beta_{m-1}(s)\\ \vdots\\ \beta_1(s)\end{bmatrix} \tag{13}$$

for appropriate $\beta_i(s)\in\Re[s]$, $i\in\mathbf{m}$ satisfying the degree restrictions

$$\deg \beta_i(s)\leqslant k-v_i,\quad i\in\mathbf{m} \tag{14}$$

Remark 1

Warren and Eckberg (1975) correctly note that if $x(s)$ is a generating function of a c.s. $\mathscr{R}$ of (A, B) of dimension r and deg $x(s)=k-1$ then in general $k \geqslant r$. However, the possibility that $k>r$ is a rather awkward condition which in our analysis in this section we find useful to dispense with, by effectively getting rid of the redundancy which always exists in such a polynomial vector representation of a c.s. We observe that in fact it is already implicit in the proof of Warren and Eckberg Lemma 2, that we can always pick $k=r$ by Wonham's definition (Wonham and Morse 1970, Wonham 1974) and subsequent characterization of a c.s. of (A, B).

For the sake mainly of keeping our notation as uncomplicated as possible in the sequel, we are using the input transformed Luenberger controllable canonical form $(\tilde{A}, \tilde{B}, \tilde{C})$ so that the Wolovich and Falb (1969) structure theorem is modified to

$$(sI-\tilde{A}, \; -\tilde{B}) \begin{bmatrix} S(s) \\ \cdots\cdots \\ \delta_0(s) \end{bmatrix} = 0 \tag{15}$$

and we henceforth drop the prime on the new input $u'(t)$ of eqn. (4). In view of Remark 1 above we make the following definition.

Definition 2

If $\tilde{x}(s)=\tilde{x}_0+\tilde{x}_1 s+\ldots+\tilde{x}_{r-1}s^{r-1}$ is a generating function of a c.s. $\mathscr{R}$ of $(\tilde{A}, \tilde{B})$ and the vectors $\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_{r-1}$ are linearly independent then $\tilde{x}(s)$ is called a proper minimal degree generating function of the c.s. $\mathscr{R}$.

Theorem 3

Any c.s. $\mathscr{R}^r$ of $(\tilde{A}, \tilde{B})$ of dimension r can be written as

$$\mathscr{R}^r = \text{span}\,\{S(s)\beta(s)\} \tag{16}$$

where $\beta(s) \in \mathfrak{R}^m[s]$, $\beta^{\mathrm{T}}(s)=(\beta_m(s), \beta_{m-1}(s), \ldots, \beta_1(s))$ and the $\beta_i(s)$, $i \in \mathbf{m}$ satisfy the degree conditions deg $\beta_i(s) \leqslant r-v_i$, $i \in \mathbf{m}$ with equality holding for at least one i.

Proof

From Theorem 1 and Remark 1, if r is a possible dimension of a c.s. $\mathscr{R}^r$ of $(\tilde{A}, \tilde{B})$ then there must exist a proper minimal degree generating function $\tilde{x}(s)$ and an input sequence generating function $u(s)$ of degrees $r-1$ and r respectively such that

$$(sI-\tilde{A}, \; -\tilde{B}) \begin{bmatrix} \tilde{x}(s) \\ \\ u(s) \end{bmatrix} = 0 \tag{17}$$

and $\mathscr{R}^r = \text{span}\,\{\tilde{x}(s)\}$. In view of Theorem 2

$$\tilde{x}(s) = S(s)\beta(s) \tag{18}$$

for unique $\beta^{\mathrm{T}}(s)=(\beta_m(s), \beta_{m-1}(s), \ldots, \beta_1(s))$ where the $\beta_i(s)$, $i\in\mathbf{m}$ satisfy the degree conditions of the theorem, i.e. $\tilde{x}(s)$ is being composed along the polynomial basis vectors consisting the columns of $S(s)$.

Thus the c.s. $\mathscr{R}^r$ of $(\tilde{A}, \tilde{B})$ is now expressed in shorthand form as the equivalent polynomial vector parametrization of the last theorem, the charting of some of whose properties will be our theme in this paper, shifting the focus of attention away from the generating function $\tilde{x}(s)$ of Warren and Eckberg (1975). What is more the c.s. $\mathscr{R}^r$ of $(\tilde{A}, \tilde{B})$ is now exhibited via the associated $\beta(s)$ as a cyclic subspace on one generator $b\in\mathscr{B}$. We remark that it is very important for our development to make the distinction between $\mathscr{R}^r$ as a vector subspace and $\mathscr{R}^r$ in its capacity as a cyclic subspace.

If a c.s. $\mathscr{R}^r$ has more than one generator in $\mathscr{B}$, i.e. $\dim(\mathscr{B}\cap\mathscr{R}^r)=r_{\mathscr{B}}>1$ then it will generally have an infinite number of different representations as a cyclic subspace, of which only $r_{\mathscr{B}}$ are essentially independent corresponding to some basis for $\mathscr{R}^r\cap\mathscr{B}$.

Definition 3

If $\tilde{x}(s)=\tilde{x}_0+\tilde{x}_1s+\ldots+\tilde{x}_{r-1}s^{r-1}$ is a proper minimal degree generating function of a c.s. $\mathscr{R}^r$ of $(\tilde{A}, \tilde{B})$ then the list of $\tilde{x}(s)$

$$\mathrm{L}\,\{\tilde{x}(s)\}=\langle\tilde{x}_{r-1}, \tilde{x}_{r-2}, \ldots, \tilde{x}_0\rangle\triangleq R^r \tag{19}$$

is called a 'cyclic listing' of the c.s. $\mathscr{R}^r$ of $(\tilde{A}, \tilde{B})$ with cyclic generator $b=\tilde{x}_{r-1}$.

From Theorem 3 we see that the polynomial vector $\beta(s)$ determines and is uniquely determined by R^r. If $\beta_i(s)=\beta_{i0}+\beta_{i1}s+\ldots+\beta_{i,r-v_i}s^{r-v_i}$, $i\in\mathbf{m}$

$$\tilde{x}(s)=S(s)\beta(s)=\begin{bmatrix}\beta_m(s)\\ s\beta_m(s)\\ \vdots\\ s^{v_m-1}\beta_m(s)\\ \cdots\cdots\\ \vdots\\ \cdots\cdots\\ \beta_1(s)\\ s\beta_1(s)\\ s^{v_1-1}\beta_1(s)\end{bmatrix}=\tilde{x}_0+\tilde{x}_1s+\ldots+\tilde{x}_{r-1}s^{r-1}$$

$$=(\tilde{x}_{r-1}, \tilde{x}_{r-2}, \ldots, \tilde{x}_1, \tilde{x}_0)\begin{bmatrix}s^{r-1}\\ s^{r-2}\\ \vdots\\ s\\ 1\end{bmatrix}=R^r\hat{s}$$

it is easily verified that the general form of R^r is

$$\left[\begin{array}{cccccccccc}
0 & 0 & \cdots & 0 & \beta_{m,r-v_m} & \beta_{m,r-v_m-1} & \cdots & \beta_{m1} & & \beta_{m0} \\
0 & 0 & \cdots & \beta_{m,r-v_m} & \beta_{m,r-v_m-1} & & \cdots & \beta_{m0} & & 0 \\
\vdots & \vdots & & & & & & & & \\
0 & \beta_{m,r-v_m} & & \cdots & & \beta_{m1} & \beta_{m0} & 0 & \cdots & 0 \\
\beta_{m,r-v_m} & \beta_{m,r-v_m-1} & & \cdots & \beta_{m1} & \beta_{m0} & 0 & 0 & \cdots & 0 \\
\hdashline
\vdots & & & & & & & & & \\
\hdashline
0 & 0 & \cdots & 0 & \beta_{1,r-v_1} & \beta_{1,r-v_1-1} & \cdots & \beta_{11} & & \beta_{10} \\
0 & 0 & \cdots & \beta_{1,r-v_1} & \beta_{1,r-v_1-1} & & \cdots & \beta_{10} & & 0 \\
\vdots & \vdots & & & & & & & & \\
0 & \beta_{1,r-v_1} & & \cdots & & \beta_{11} & \beta_{10} & 0 & \cdots & 0 \\
\beta_{1,r-v_1} & \beta_{1,r-v_1-1} & & \cdots & \beta_{11} & \beta_{10} & 0 & 0 & \cdots & 0
\end{array}\right] \tag{20}$$

Definition 4

A polynomial vector $\beta(s) \in \mathfrak{R}^m[s]$ giving rise via eqn. (18) to a proper minimal degree generating function $\tilde{x}(s)$ of a c.s. $\mathscr{R}^r$ of $(\tilde{A}, \tilde{B})$ and hence to a cyclic listing R^r of $\mathscr{R}^r$, is called a 'decoupling vector'. We write this geometric/algebraic equivalence as

$$R^r \sim \beta(s) \tag{21}$$

Remark 2

Given an arbitrary polynomial vector $\beta(s) \in \mathfrak{R}^m[s]$ if we write $\tilde{x}(s) = S(s)\beta(s)$ then obviously the listing $\mathrm{L}\{\tilde{x}(s)\} = \langle \tilde{x}_{r-1}, \tilde{x}_{r-2}, \ldots, \tilde{x}_0 \rangle$ is not in general a proper cyclic listing, but nevertheless span $\{\tilde{x}(s)\}$ represents some c.s. $\mathscr{R}$ of $(\tilde{A}, \tilde{B})$. In such a case $\beta(s)$ is not necessarily a decoupling vector and we write $\beta(s) \approx \mathscr{R}$. We say that $\mathscr{R}$ is the 'underlying c.s.' of $\beta(s)$.

From Definition 4 and the previous arguments it becomes clear that every cyclic listing R^r of a c.s. $\mathscr{R}^r$ of $(\tilde{A}, \tilde{B})$ has an associated algebraic object, its decoupling vector satisfying the degree conditions of Theorem 3. It is also clear that to every c.s. $\mathscr{R}^r$ of $(\tilde{A}, \tilde{B})$ there corresponds an infinite family of cyclic listings R^r and hence an infinite family of decoupling vectors $\beta(s)$. It is now natural to ask the converse question as to whether any polynomial vector $\beta(s) \in \mathfrak{R}^m[s]$ satisfying the degree conditions gives rise to a listing $\mathrm{L}\{S(s)\beta(s)\}$ which is cyclic. This somewhat delicate question is dealt with in the Appendix; the answer is nearly always yes, meaning that the condition is indeed sufficient generically in the vector space of all the coefficients of the $\beta_i(s)$, $i \in \mathbf{m}$.

Proposition 1

A necessary condition for $\beta(s) \in \mathfrak{R}^m[s]$ to be a decoupling vector corresponding to a cyclic listing R^r of the c.s. $\mathscr{R}^r$ of $(\tilde{A}, \tilde{B})$ is that g.c.d. $\{\beta_i(s), i \in \mathbf{m}\} = 1$.

In order to prove the proposition we need the following lemma.

Lemma

For every $x(s) \in \Re^n[s]$ and $a(s) \in \Re[s]$

$$\text{span}\,\{x(s)\} = \text{span}\,\{a(s)x(s)\}$$

Proof

If $x(s) = x_0 + x_1 s + \ldots + x_{r-1}s^{r-1}$, $a(s) = a_0 + a_1 s + \ldots + a_q s^q$ let

$$a(s)x(s) = x'_0 + x'_1 s + \ldots + x'_{k-1}\, s^{k-1} = x'(s) \tag{22}$$

where $k = r + q$. Then by equating similar powers of s in (22) we obtain that the vectors x'_j, $j = 0, 1, \ldots, k-1$ are linear combinations of the vectors $x_0, x_1, \ldots, x_{r-1}$, hence span $\{x'_0, x'_1, \ldots, x'_{k-1}\} =$ span $\{x_0, x_1, \ldots, x_{r-1}\}$. Q.E.D.

Proof of Proposition 1

Suppose $\beta(s) \sim R^r$, but that $\beta(s) = a(s)\hat{\beta}(s)$ where $a(s) =$ g.c.d. $\{\beta_i(s), i \in \mathbf{m}\}$ and deg $a(s) = q \geqslant 1$. Accordingly

$$\deg\,\{S(s)\hat{\beta}(s)\} = r - 1 - q < \deg\,\{S(s)\beta(s)\} = r - 1$$

From the preceding lemma, span $\{S(s)\hat{\beta}(s)\} =$ span $\{S(s)\beta(s)\}$ and therefore dim (span $\{S(s)\beta(s)\}) =$ dim (span $\{S(s)\hat{\beta}(s)\}) \leqslant r - q$. Thus L $\{S(s)\beta(s)\}$ has length r whereas the subspace formed by the span of its elements has dimension $< r - q < r$, so these elements cannot be independent and form a listing which is cyclic. Thus finally $\beta(s)$ is not a decoupling vector, contrary to the assumption.

Note

A polynomial representation $\beta(s)$ of a c.s. $\mathscr{R}^r$ with cyclic listing $R^r \sim \hat{\beta}(s)$, where $\beta(s)$ has the form $a(s)\hat{\beta}(s)$ is one very important case of redundancy which occurs later in this paper. The other type of redundancy is closely related and is discussed in the Appendix, where a procedure is given for reducing a redundant $\beta(s) \approx \mathscr{R}^r$ to a decoupling vector of some cyclic listing of $\mathscr{R}^r$.

3.2.

Let $\mathscr{R}^r$ be a c.s. of $(\tilde{A}, \tilde{B})$ and denote by $\tilde{\mathbf{F}}(\mathscr{R}^r)$ the class of all matrices $\tilde{F} \in \Re^{m \times n}$ such that $(\tilde{A} + \tilde{B}\tilde{F})\mathscr{R}^r \subset \mathscr{R}^r$ (Wonham and Morse 1970, Morse and Wonham 1971, Wonham 1974). Let $\tilde{x}(s) = \tilde{x}_0 + \tilde{x}_1 s + \ldots + \tilde{x}_{r-1}s^{r-1}$ be a proper minimal degree generating function of $\mathscr{R}^r$ with (open-loop) input sequence generating function $u(s) = u_0 + u_1 s + \ldots + u_r s^r$. Considering eqn. (17) it then follows by equating coefficients of similar powers of s that

$$\tilde{x}_{r-1} = \tilde{B}u_r = b, \quad -\tilde{A}\tilde{x}_j = -\tilde{x}_{j-1} + \tilde{B}u_j, \quad j \in \mathbf{r-1}, \quad -\tilde{A}\tilde{x}_0 = \tilde{B}u_0 \tag{23}$$

Define now state feedback $\tilde{F}_0$ via

$$u(s) = \tilde{F}_0\tilde{x}(s) + u_r s^r \tag{24}$$

or equivalently via

$$(u_{r-1}, u_{r-2}, \ldots, u_0) = \tilde{F}_0(\tilde{x}_{r-1}, \tilde{x}_{r-2}, \ldots, \tilde{x}_0) \tag{25}$$

Then combining eqns. (23) and (25) we obtain the equations :

$$\left.\begin{array}{c} b=\tilde{B}u_r=\tilde{x}_{r-1}, \quad (\tilde{A}+\tilde{B}\tilde{F}_0)\tilde{x}_{r-1}=\tilde{x}_{r-2}, \ldots, (\tilde{A}+\tilde{B}\tilde{F}_0)\tilde{x}_1=\tilde{x}_0 \\ (\tilde{A}+\tilde{B}\tilde{F}_0)\tilde{x}_0=0 \end{array}\right\} \tag{26}$$

which can be written in matrix form as

$$(\tilde{A}+\tilde{B}\tilde{F}_0)(\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_{r-1})=(\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_{r-1})\begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}$$

The above equations graphically illustrate that :

(1) $\mathscr{R}^r=\text{span}\,\{R^r\}=\text{span}\,\{\langle\tilde{x}_{r-1}, \tilde{x}_{r-2}, \ldots, \tilde{x}_0\rangle\}$ is a cyclic subspace with generator $\tilde{x}_{r-1}=b$.

(2) $\tilde{F}_0\in\mathbf{F}(\mathscr{R}^r)$ and that

$$\tilde{A}+\tilde{B}\tilde{F}_0|\mathscr{R}^r=\begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}$$

For notational reference we write

$$\langle\tilde{A}+\tilde{B}\tilde{F}_0|b\rangle=R^r=\langle b, (\tilde{A}+\tilde{B}\tilde{F}_0)b, \ldots, (\tilde{A}+\tilde{B}\tilde{F}_0)^{r-1}b\rangle$$

and for the coarser vector subspace concept

$$\{\tilde{A}+\tilde{B}\tilde{F}_0|b\}=\mathscr{R}^r=\text{span}\,\{\langle\tilde{A}+\tilde{B}\tilde{F}_0|b\rangle\}$$

Definition 5

The listing $\text{L}\,\{u(s)\}=\langle u_r, u_{r-1}, \ldots, u_0\rangle$ of open loop inputs figuring in eqns. (23) is called the input generating sequence of R^r and it is said to 'achieve' R^r. The state feedback matrix $\tilde{F}_0$ defined by eqn. (24) is said to 'attain R^r through $b=\tilde{x}_{r-1}$'.

We now spell out the essence of the preceding discussion in more formal Nerode-equivalence type of language. The consideration of discrete time case entails no loss of generality, and as it is well known the arrival of an anti-causal input sequence at a set of shift registers corresponds to an injection of delta functions and derivatives into an analogous bank of integrators at time 0^-.

Lemma 1

(1) The (discrete-time) input generating sequence of R^r, $U_{R^r}=\text{L}\,\{u(s)\}=\langle u_r, u_{r-1}, \ldots, u_0\rangle$, written in order of forward time sequence starting at time $t=-r$, which achieves the cyclic listing $R^r\sim\beta(s)$, $R^r=\langle\tilde{x}_{r-1}, \tilde{x}_{r-2}, \ldots, \tilde{x}_0\rangle$ displayed in ordered time profile starting at $t=-(r-1)$, is given by $U_{R^r}=\text{L}\,\{\delta_0(s)\beta(s)\}$.

(2) The class of state feedback matrices $\tilde{F}_0$ which attain R^r through $b=\tilde{x}_{r-1}$ and cause simultaneous dead-beat state regulation in r steps is given by the $\tilde{F}_0$'s which solve eqn. (25). The only closed-loop input needed to 'activate' the c.s. $\mathscr{R}^r$ is the simple input $u_r \neq 0$ at time $t=-r$ followed by a string of $(r-1)$ zeros.

(3) If $R^r \sim \beta(s)$, then

$$u_r = [[s^v]\beta(s)]^h = (\beta_{m,r-v_m}, \beta_{m-1,r-v_{m-1}}, \ldots, \beta_{1,r-v_1})^T = h$$

where $[\ ]^h$ means the vector coefficient of the highest degree term occurring inside the brackets when it is expanded in powers of s.

(4) $u_r = h$ and $x_{r-1} = (0, 0, \ldots, \beta_{m,r-v_m}, 0, 0, \ldots, \beta_{m-1,r-v_{m-1}}, \ldots, 0, 0, \ldots, \beta_{1,r-v_1})^T = b$ are feedback invariants.

Proof

(1) The result is an easy consequence of Theorems 2 and 3, and eqn. (23).

(2) Follows from eqns. (23), (25) and (26).

(3) $u_r = [u(s)]^h = [\delta_0(s)\beta(s)]^h$

$$= \left[\{[s^v] - \tilde{A}_m S(s)\} \begin{bmatrix} \beta_{m,r-v_m} s^{r-v_m} + \ldots + \beta_{m0} \\ \beta_{m-1,r-v_{m-1}} s^{r-v_{m-1}} + \ldots + \beta_{m-1,0} \\ \vdots \\ \beta_{1,r-v_1} s^{r-v_1} + \ldots + \beta_{1,0} \end{bmatrix} \right]^h$$

$$= [[s^v]\beta(s)]^h = (\beta_{m,r-v_m}, \beta_{m-1,r-v_{m-1}}, \ldots, \beta_{1,r-v_1})^T$$

where the highest degree term $[s^v]$ is here picking out the highest coefficient term, while the lower degree $S(s)$ does not contribute.

(4) That $u_r = h$ is a feedback invariant can be appreciated from the state feedback version of the Wolovich–Falb (1969), Wolovich (1974) structure theorem which in our coordinates can be stated as :

$$(sI - \tilde{A} - \tilde{B}\tilde{F}_0, -\tilde{B}) \begin{bmatrix} S(s) \\ \delta_{F_0}(s) \end{bmatrix} = 0$$

where

$$h = u_r = [[s^v]\beta(s)]^h = [\{[s^v] - (\tilde{A}_m + \tilde{F}_0)S(s)\}\beta(s)]^h$$
$$= [\delta_{F_0}(s)\beta(s)]^h = [w(s)]^h = w_r$$

where $w(s)$ is the closed-loop input sequence, analogous to $u(s)$, after the loop is closed by linear state feedback $\tilde{F}_0$. Lastly, the form of the first state set up by the entry of the first input u_r (or w_r) into the system, after unit delay, is clear from $\tilde{x}_{r-1} = \tilde{B}u_r = b$.

Definition 6

We call u_r in (3) above the input generator or the 'germ' of the cyclic listing R^r corresponding to the c.s. $\mathscr{R}^r$ of $(\tilde{A}, \tilde{B})$.

Remark 3

The choice of feedback via eqn. (25), in general non-unique except when $r = n$ and useful for obtaining the cyclic listing R^r of a c.s. $\mathscr{R}^r$ of $(\tilde{A}, \tilde{B})$, is of course unduly restrictive for our algebraic theory, leading as it does to an operator $\tilde{A} + \tilde{B}\tilde{F}_0$ whose minimal polynomial on $\mathscr{R}^r$ is s^r. We shall presently broaden the class of feedbacks which attain R^r through $b = \tilde{x}_{r-1}$ and incidentally lead to arbitrary pole placement on $\mathscr{R}^r$. Also since we have characterized the geometric concept of a c.s. algebraically and in the process constructively exhibited all c.s.'s of a possible dimension r and isolated the whole information of any c.s. $\mathscr{R}^r$ of $(\tilde{A}, \tilde{B})$ with a cyclic listing R^r, in a decoupling vector $\beta(s) \sim R^r$, we would then like to be able to obtain the class of state feedback matrices $\tilde{F}_0$ attaining R^r in terms of $\beta(s)$.

Theorem 4

Given a c.s. $\mathscr{R}^r$ of $(\tilde{A}, \tilde{B})$ with cyclic listing $R^r \sim \beta(s)$ then (1) the class of all state feedback matrices $\tilde{F}_0$ which attain R^r with simultaneous state regulation in r steps is given by solving for $\tilde{F}_0$ the polynomial matrix equation

$$\delta_{F_0}(s)\beta(s) = hs^r \tag{27}$$

where $\delta_{F_0}(s) = [s^v] - (\tilde{A}_m + \tilde{F}_0)S(s)$ and $h = [[s^v]\beta(s)]^h$.

(2) If instead of the simple input $u_r s^r$ of Lemma 1 we allow the more general one $u_r v(s)$ where $v(s)$ arbitrary monic polynomial of $\deg v(s) = r$ then we no longer have in general the r-step regulation property described by eqns. (26) but still $\tilde{x}_{i-1} \in \mathscr{R}^r$, $i \in \mathbf{r}$. The most general class of state feedback matrices $\tilde{F}$ causing $\mathscr{R}^r$ to be reached through the single input port $h = u_r$ belongs to $\mathbf{F}(\mathscr{R}^r)$ and is obtained by solving for $\tilde{F}$ the more general polynomial matrix equation

$$\delta_F(s)\beta(s) = hv(s) \tag{28}$$

Remark 4

It is noted that eqn. (28) establishes a major and fundamental direct link between the Wonham and Morse theories and the Rosenbrock and Wolovich polynomial matrix methods of system analysis.

Proof

(1) Let $\tilde{x}(s) = \tilde{x}_0 + \tilde{x}_1 s + \ldots + \tilde{x}_{r-1}s^{r-1}$ be a proper minimal degree generating function of the c.s. $\mathscr{R}^r$ of $(\tilde{A}, \tilde{B})$ with corresponding input sequence generating function $u(s) = u_0 + u_1 + \ldots + u_r s^r$, and define $\tilde{F}_0$ via eqn. (24). Then from Theorems 2 and 3 we have that $\tilde{x}(s) = S(s)\beta(s)$ and $u(s) = \delta_0(s)\beta(s)$ where $\beta(s)$ is the decoupling vector corresponding to the cyclic listing $R^r = \langle \tilde{x}_{r-1}, \ldots, \tilde{x}_0 \rangle$ of the c.s. $\mathscr{R}^r$, and eqn. (24) can be written as

$$\delta_0(s)\beta(s) = \tilde{F}_0 S(s)\beta(s) + u_r s^r$$
$$\delta_{F_0}(s)\beta(s) = hs^r$$

where $\delta_{F_0}(s) = \delta_0(s) - \tilde{F}_0 S(s)$ (see structure theorem under state feedback in Wolovich (1974)). The input to the closed-loop system $u_{F_0}(s)$ is given by :

$$u_{F_0}(s) = \delta_{F_0}(s)\beta(s) = hs^r$$

The argument works entirely in reverse, i.e. assuming eqn. (27) we derive (24).

(2) Let $v(s) = v_0 + v_1 s + \ldots + v_{r-1}s^{r-1} + s^r \in \Re[s]$ and define state feedback $\tilde{F}$ via

$$u(s) = \tilde{F}\tilde{x}(s) + u_r v(s) \tag{29}$$

or equivalently via the r equations :

$$u_i = \tilde{F}\tilde{x}_i + u_r v_i, \quad i = 0, 1, \ldots, r-1 \tag{30}$$

which can be written in matrix form as

$$(u_0, u_1, \ldots, u_{r-1}) = \tilde{F}(\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_{r-1}) + u_r(v_0, v_1, \ldots, v_{r-1}) \tag{31}$$

Combining eqns. (30) and (23) (which is again valid since $\tilde{x}(s)$ is by assumption a proper minimal degree generating function) we obtain :

$$\begin{aligned} \tilde{B}u_r &= \tilde{B}h = \tilde{x}_{r-1} = b \\ (\tilde{A} + \tilde{B}\tilde{F})\tilde{x}_{r-1} &= \tilde{x}_{r-2} - \tilde{B}u_r v_{r-1} = \tilde{x}_{r-2} - \tilde{x}_{r-1}v_{r-1} \\ &\vdots \\ (\tilde{A} + \tilde{B}\tilde{F})\tilde{x}_1 &= \tilde{x}_0 - \tilde{B}u_r v_1 = \tilde{x}_0 - \tilde{x}_{r-1}v_1 \\ (\tilde{A} + \tilde{B}\tilde{F})\tilde{x}_0 &= -\tilde{B}u_r v_0 = -\tilde{x}_{r-1}v_0 \end{aligned}$$

The above equations can be written in matrix form as

$$(\tilde{A} + \tilde{B}\tilde{F})(\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_{r-1}) = (\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_{r-1}) \times \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & & & \vdots \\ -v_0 & -v_1 & -v_2 & \ldots & -v_{r-1} \end{bmatrix} \tag{32}$$

Equation (32) again graphically illustrates that $\tilde{F} \in \mathbf{F}(\mathscr{R}^r)$ and that

$$\tilde{A} + \tilde{B}\tilde{F}\,|\,\mathscr{R}^r = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & & & \\ -v_0 & -v_1 & -v_2 & \ldots & -v_{r-1} \end{bmatrix}$$

Generality of $\tilde{F}$ follows as this is a controllable canonical form of general form. As in (1) from eqn. (29) we obtain

$$\begin{aligned} \delta_0(s)\beta(s) &= \tilde{F}S(s)\beta(s) + u_r v(s) \\ \delta_F(s)\beta(s) &= hv(s) \end{aligned}$$

where $\delta_F(s) = \delta_0(s) - \tilde{F}S(s)$. Furthermore from

$$(sI - \tilde{A})\tilde{x}(s) = \tilde{B}u(s) = \tilde{B}\tilde{F}\tilde{x}(s) + \tilde{B}u_r v(s)$$

we obtain :

$$(sI - (\tilde{A} + \tilde{B}\tilde{F}),\ -\tilde{B}) \begin{bmatrix} \tilde{x}(s) \\ u_r v(s) \end{bmatrix} = (sI - (\tilde{A} + \tilde{B}\tilde{F}),\ -\tilde{B}) \begin{bmatrix} S(s) \\ \delta_F(s) \end{bmatrix} \beta(s)$$

implying that the input $u_F(s)$ to the closed-loop system is $u_F(s) = \delta_F(s)\beta(s) = hv(s)$. Conversely if we equate coefficients of similar powers of s in (28) we obtain (30) and hence the theorem is proved.

Clearly this polynomial characterization of state feedback $\tilde{F}$ inducing a given c.s. $\mathscr{R}^r$ is the right manner in which to generalize the regulating state feedback $\tilde{F}_0$. But eqn. (28) as it stands is not a sensible method of computing these state feedback matrices $\tilde{F}$, as there is no simplification over the equivalent eqn. (31). We note from the last theorem that $\mathscr{R}^r$ is entered by a scalar input filtered through the vector h before entering the system with appropriate state feedback, $(\tilde{A}+\tilde{B}\tilde{F}, \tilde{B})$, in other words any c.s. arises by looking at the state space through a single element $h \in \mathscr{U}$ (the input space) with a suitable state feedback closing the loop, i.e. $x(t+1)=(A+BF)x(t)+Bhv(t)$, $x(t_0)\in\mathscr{R}^r \rightarrow x(t)\in\mathscr{R}^r$ for every $t \geqslant t_0$, where $v(t)$ is the scalar control.

Theorem 5

Given a c.s. $\mathscr{R}^r$ of $(\tilde{A}, \tilde{B})$ of dimension r and a symmetric set of r complex numbers $\Lambda_r=\{s_1, s_2, \ldots, s_f, s_{f+1}, s_{f+2}, s_{f+3}, \ldots, s_r\}$ where $s_i \in \mathfrak{R}$, $i \in \mathbf{f-1}$, $s_i \in \mathfrak{C}$, $i=f, f+1, \ldots, r$ and $s_f=\bar{s}_{f+1}$, $s_{f+2}=\bar{s}_{f+3}$ etc., then there exists a class of state feedback matrices $\tilde{F}$, (a unique matrix if $r=n$) such that $\{\tilde{A}+\tilde{B}\tilde{F}|b\}=\mathscr{R}^r$, where $b=\tilde{x}_{r-1}\in\mathscr{B}$ is the generator of a cyclic listing $R^r \sim \beta(s)$ of the c.s. $\mathscr{R}^r$ of $(\tilde{A}, \tilde{B})$ and spectrum $(\tilde{A}+\tilde{B}\tilde{F}|\mathscr{R}^r)=\Lambda_r$.

Proof

Again let $\tilde{x}(s)$ be a proper minimal degree generating function of the c.s. $\mathscr{R}^r$ with input sequence generating function $u(s)$ and let $\tilde{F}$ solve $\delta_F(s)\beta(s)=hv(s)$ where $v(s)=\prod_{i\in\mathbf{r}}(s-s_i)$. Then $v(s_i)=0$, $i\in\mathbf{r}$ and

$$(sI-(\tilde{A}+\tilde{B}\tilde{F}),\ -\tilde{B})\begin{bmatrix} S(s) \\ \delta_F(s)\end{bmatrix}\beta(s)=(sI-(\tilde{A}+\tilde{B}\tilde{F}),\ -\tilde{B})\begin{bmatrix}\tilde{x}(s) \\ hv(s)\end{bmatrix}$$

for $s=s_i\in\Lambda_r$ implies that

$$(s_iI-(\tilde{A}+\tilde{B}\tilde{F}))\tilde{x}(s_i)=0, \quad i\in\mathbf{r}$$

i.e.

$$\Lambda_r=\text{spectrum }(\tilde{A}+\tilde{B}\tilde{F}|\mathscr{R}^r) \quad \text{and} \quad \tilde{x}(s_i)=(\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_{r-1})\begin{bmatrix}1 \\ s_i \\ \vdots \\ s_i^{r-1}\end{bmatrix}$$

$$=S(s_i)\beta(s_i)$$

is the formula for the eigenvector of $\tilde{A}+\tilde{B}\tilde{F}$ corresponding to the eigenvalue s_i. If s_i is a repeated zero of $v(s)$ of multiplicity p_i, then because of the simple structure of the c.s. $\mathscr{R}^r$ with respect to $\tilde{A}+\tilde{B}\tilde{F}$ the eigenvector and generalized eigenvectors corresponding to s_i will be :

$$\tilde{x}(s_i), \left.\frac{d\tilde{x}(s)}{ds}\right|_{s=s_i}, \left.\frac{1}{2}\frac{d^2\tilde{x}(s)}{ds^2}\right|_{s=s_i}, \ldots, \left.\frac{1}{(p_i-1)}\frac{d^{p_i-1}\tilde{x}(s)}{ds^{p_i-1}}\right|_{s=s_i}$$

This follows by differentiating

$$(sI-(\tilde{A}+\tilde{B}\tilde{F}),\ -\tilde{B})\begin{bmatrix}\tilde{x}(s) \\ hv(s)\end{bmatrix}=0$$

with respect to s $(p_i - 1)$ times and evaluating the resulting equations at $s = s_i$. This ends the proof.

Remark 5

Strictly speaking the above is not a 'proof' of the theorem describing the pole-placement properties of c.s.'s because Warren and Eckberg's (1975) paper implicitly assumes this property proved by Wonham and Morse (1970). However, if we wish to stick to the polynomial vector method and render the above an actual theorem, we could easily do so by adopting the Warren and Eckberg characterization as fundamental.

Remark 6

In the classic proof of Hayman's lemma (1968) the multi-input controllability case is reduced to controllability through a single port $b \in \mathscr{B}$. Using our approach and taking $r = n$ it is not a difficult matter to check that, generically by choosing an arbitrary b we get a c.s. of dimension n and the above almost trivially shows that we can still place all the poles where we like. In effect then c.s.'s can be regarded as the single input reachable subsystems on which arbitrary pole-placement is possible.

4. Compatibility of controllability subspaces

Let $\mathscr{R}^{r_j}$, $j \in \mathbf{m}$ be a set of m c.s. of $(\tilde{A}, \tilde{B})$ and $\tilde{F}_j \in \Re^{m \times n}$, $j \in \mathbf{m}$ be such that $(\tilde{A} + \tilde{B}\tilde{F}_j)\mathscr{R}^{r_j} \subset \mathscr{R}^{r_j}$, $j \in \mathbf{m}$, i.e. let $\tilde{F}_j \in \tilde{\mathbf{F}}(\mathscr{R}^{r_j})$, $j \in \mathbf{m}$. If $\bigcap_{j \in \mathbf{m}} \tilde{\mathbf{F}}(\mathscr{R}^{r_j}) \neq \emptyset$ then the set of the c.s. $\mathscr{R}^{r_j}$ is called a compatible set (Wonham and Morse 1970, Morse and Wonham 1971, Wonham 1974). We shall give now a condition for the compatibility of the above set of m c.s. in terms of their decoupling vectors $\beta^j(s) \in \Re^m[s]$, $j \in \mathbf{m}$. The c.s. $\mathscr{R}^{r_j}$ may not all be distinct in the vector subspace sense (indeed they might all be the same, although not in cases of interest). Also there is no loss of generality in assuming that we have m c.s. each one 'germinating' through a different basis element u_{r_j}, $j \in \mathbf{m}$ (a 'germ' of R^{r_j} by Definition 6) of the input space $\mathscr{U}$. With this observation we can now state the partial compatibility result, which relies on Theorem 4 whose power will soon be appreciated and refined as necessary.

Theorem 6

Given m c.s. $\mathscr{R}^{r_1}, \mathscr{R}^{r_2}, \ldots, \mathscr{R}^{r_m}$ of $(\tilde{A}, \tilde{B})$ of dimensions $r_1, r_2, \ldots, r_m$ respectively with corresponding cyclic listings $R^{r_1} \sim \beta^1(s)$, $R^{r_2} \sim \beta^2(s)$, ..., $R^{r_m} \sim \beta^m(s)$ then they are compatible if and only if there exists an $\tilde{F}$ satisfying the polynomial matrix equation

$$\delta_F(s)B(s) = (h_1, h_2, \ldots, h_m)\begin{bmatrix} v_1(s) & 0 & \cdots & 0 \\ 0 & v_2(s) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & v_m(s) \end{bmatrix} \tag{33}$$

where $B(s) = (\beta^1(s), \beta^2(s), \ldots, \beta^m(s))$ is the $m \times m$ polynomial matrix formed by 'lumping' the m c.s. (represented by their associated decoupling vectors

$\beta^j(s)$) together and $v_j(s)$, $j \in \mathbf{m}$ are arbitrary monic polynomials of deg $v_j(s) = r_j$, $j \in \mathbf{m}$. In addition $H = (h_1, h_2, \ldots, h_m) = [[s^v]B(s)]_c^h$ where $[\;]_c^h$ denotes the highest column coefficient matrix of the polynomial matrix inside the brackets.

Proof

The theorem is an immediate corollary of Theorem 4 (2) which says that R^{r_j}, $j \in \mathbf{m}$ is attained through $h_j = u_{r_j}$ if and only if there exists a matrix $\tilde{F}$ solving

$$\delta_F(s)\beta^j(s) = h_j v_j(s) \tag{34}$$

Therefore there exists an $\tilde{F}$ attaining all R^{r_j}, $j \in \mathbf{m}$ simultaneously if and only if there exists an $\tilde{F}$ such that satisfies eqn. (34) for every $j \in \mathbf{m}$, i.e. if and only if there exists an $\tilde{F}$ solving eqn. (33).

If the c.s. $\mathscr{R}^{r_j}$ are independent then eqns. (34) for $j \in \mathbf{m}$ are easily solved for $\tilde{F}$ if we assign arbitrary values to the $v_j(s)$, but from Wonham's work (1974) compatibility in this case is trivial anyway. In general a set of m c.s. that we may be interested in, e.g. those typically arising in decoupling theory, will be highly overlapping and will of course have a special structure associated with the problem in which they are embedded. This in turn implies relations holding among the parameters of the $\beta_i(s)$ and consequent restrictions on the form of possible $v_j(s)$. We will see how this works out in a future paper with the apparatus we set in the next section.

5. Controllability subspaces contained in a given fixed subspace

If $\mathscr{S}$ is a given fixed subspace of the state space consider the class $\mathscr{C}(\tilde{A}, \tilde{B}, \mathscr{S})$ of c.s. $\mathscr{R}$ of $(\tilde{A}, \tilde{B})$ contained in $\mathscr{S}$. Then it is known (Wonham and Morse 1970, Morse and Wonham 1971, Wonham 1974) that $\mathscr{S}$ contains a unique supremal c.s. which we denote by $\mathscr{R}^{\max}$. A very important problem which arises in the Wonham and Morse geometric approach to the solution of the decoupling problem is that of computing $\mathscr{R}^{\max}$. Wonham and Morse (1970), Morse and Wonham (1971), Wonham (1974) describe two ways of computing $\mathscr{R}^{\max}$ (other subspaces than the maximal one being harder to get at). Both of these require a prior computation of a subspace $\mathscr{V}^{\max}$ which is the maximal $(\tilde{A}, \tilde{B})$-invariant subspace contained in $\mathscr{S}$. Further the two schemes work differently. The first necessitates the computation of a matrix $\tilde{F}$ such that $(\tilde{A} + \tilde{B}\tilde{F})\mathscr{V}^{\max} \subset \mathscr{V}^{\max}$, then $\mathscr{R}^{\max} = \sum_{j=1}^{n} (\tilde{A} + \tilde{B}\tilde{F})^{j-1}(\mathscr{B} \cap \mathscr{V}^{\max})$; the second scheme is algorithmic and makes use of the 'controllability subspace algorithm' (Wonham 1974). However, these algorithms are somewhat lengthy, slow to converge at times and cumbersome to implement (see e.g. Bengtsson 1975). Utilizing the more fruitful Warren and Eckberg (1975) angle of attack, the problem has received the attention of Kimura (1977) who listed the c.s. of interest to his output feedback pole assignment problem in parametric form (similar to that given by eqn. (20)) and then expressed the orthogonality between $\tilde{C}$ and the vectors spanning the c.s., resulting in a system of linear equations for the parameters β_{ij}. Stoyle and Vardulakis (1977) gave a set of state space conditions of independent interest, and Denham (1976) approached close to the type of solution given here. Keeping in with the spirit of this paper we will stick to the decoupling vector parametrization of c.s., which renders an appealing simple solution and which lends itself to a compact insight.

First we note the equivalence of the problem to find all c.s. contained in $\mathscr{S}$, to an equivalent kernel problem. If $S=(s_1, s_2, \ldots, s_r)$ is a basis for $\mathscr{S}$ then it is more convenient to treat $\mathscr{S}$ as the kernel of a map $\tilde{C}$ in matrix representation. Then we are looking for all c.s. $\mathscr{R}$ that $\tilde{C}$ annihilates. Obviously the largest possible such c.s. $\mathscr{R}^{\max}$ has dimension r or less, and is unique in the vector subspace sense.

Theorem 7

Let $\mathscr{R}^r$ be a c.s. of $(\tilde{A}, \tilde{B})$ and let $\beta(s)\in\mathfrak{R}^m[s]$ such that $\beta(s)\approx\mathscr{R}^r$, i.e. if $\tilde{x}(s)\triangleq S(s)\beta(s)$ let span $\{\tilde{x}(s)\}=\mathscr{R}^r$ (see Remark 2 in § 3.1). Then $R^r\subseteq\ker\tilde{C}$ if and only if $\beta(s)\in\ker\tilde{C}(s)$, where $\tilde{C}(s)=\tilde{C}S(s)$, an $\mathfrak{R}[s]$-module morphism $\mathfrak{R}^m[s]\rightarrow\mathfrak{R}^p[s]$.

Proof

Let $\tilde{x}(s)=\tilde{x}_0+\tilde{x}_1 s+\ldots+\tilde{x}_{k-1}s^{k-1}$, be a generating function of the c.s. $\mathscr{R}^r$. Then from Theorem 2 there exists a $\beta(s)\in\mathfrak{R}^m[s]$ such that $\tilde{x}(s)=S(s)\beta(s)$. Now $\mathscr{R}^r\subseteq\ker\tilde{C}$ or $\tilde{C}\mathscr{R}^r=0$ if and only if $\tilde{C}\tilde{x}_i=0$, $i=0, 1, \ldots, k-1$ or

$$\tilde{C}\tilde{x}(s)=\tilde{C}S(s)\beta(s)=\tilde{C}(s)\beta(s)=0 \tag{35}$$

Thus the original vector space kernel problem has been transformed into a generally much lower dimensional module kernel problem which in principle is much easier to solve. In the case where $\tilde{C}$ is the output matrix we recognize $\tilde{C}(s)$ as the 'numerator' matrix in the Wolovich–Falb (1969) (1974) system decomposition

$$T(s)=C(sI-A)^{-1}B=\tilde{C}S(s)\delta_0^{-1}(s)\hat{B}_m$$

Before proving the theorem of this section we need some slight generalizations of Proposition 1 in § 3.1.

Proposition 2

If $\beta_1(s)\approx\mathscr{R}_1, \ldots, \beta_k(s)\approx\mathscr{R}_k$ and $\beta(s)=\sum_{j=1}^{k}a_j(s)\beta_j(s)$, $a_j(s)\in\mathfrak{R}[s]$, $j\in\mathbf{k}$ then $\mathscr{R}\subseteq\sum_{j\in\mathbf{k}}\mathscr{R}_j$ where $\beta(s)\approx\mathscr{R}$.

Proof

If we write $\tilde{x}_j(s)=S(s)\beta_j(s)$, $j\in\mathbf{k}$ and $\tilde{x}(s)=S(s)\beta(s)$ then $\tilde{x}(s)=\sum_{j\in\mathbf{k}}a_j(s)\tilde{x}_j(s)$ and as in the proof of Proposition 1, equating coefficients of like powers of s exhibits the coefficients of $\tilde{x}(s)$ as linear combinations of elements in the $\mathscr{R}_j$, $j\in\mathbf{m}$, which is the result.

Proposition 3

If $\beta_j(s)\approx\mathscr{R}_j$, $j\in\mathbf{k}$, $B(s)=(\beta_1(s), \beta_2(s), \ldots, \beta_k(s))$, $k\leqslant m$ and we apply arbitrary Forney operations (Forney 1975, § 4) to $B(s)$ to reduce it to $B'(s)=(\beta'_1(s), \beta'_2(s), \ldots, \beta'_k(s))$ then $\sum_{j\in\mathbf{k}}\mathscr{R}_j=\sum_{j\in\mathbf{k}}\mathscr{R}'_j$ where $\beta'_j(s)\approx\mathscr{R}'_j$.

Proof

Forney operations are of two distinct types. Type 1 can be applied if $B(s)$ does not have full rank mod $p(s)$, $\deg p(s) \geqslant 1$. In such a case Forney shows in step 2 of his reduction algorithm that one column of $B(s)$, without loss of generality $\beta_1(s)$ say, can be replaced by a polynomial vector $\beta'_1(s) = (\sum_{i\in\mathbf{k}} a_i(s)\beta_i(s))/p(s)$, $a_1(s) \neq 0$ of lower degree. If $p(s)\beta'_1(s) \approx \mathscr{R}'_1$ say, by Proposition 2, $\mathscr{R}'_1 \subseteq \sum_{i\in\mathbf{k}} \mathscr{R}_i$ but by Proposition 1, $\beta'_1(s) \approx \mathscr{R}'_1$. Together with $\beta_i(s) = \beta'_i(s)$, $i \geqslant 2$, implying that $\mathscr{R}_i = \mathscr{R}'_i$, $i \geqslant 2$, this yields $\sum_{i\in\mathbf{k}} \mathscr{R}'_i \subseteq \sum_{i\in\mathbf{k}} \mathscr{R}_i$. The opposite inclusion also holds because of the analogous relation to the above equation

$$a_1(s)\beta_1(s) = p(s)\beta'_1(s) - \sum_{i=2}^{k} a_i(s)\beta'_i(s)$$

The proof regarding the second type of Forney operations (step 3 of his reduction algorithm and applicable when the highest column coefficient matrix of $B(s)$ is singular) is in fact easier and will not be written out. All Forney-type operations are compounded of these two applied in some order and the result follows.

Note that the proposition can yield some interesting and intuitive interpretations of Forney operations on a polynomial matrix $X(s) = S(s)B(s)$ in terms of the underlying c.s. of its columns. In this direction we now state the main theorem of this section, which classifies the set of all c.s. in ker $\tilde{C}$ in terms of the rank of $\tilde{C}(s)$. In the following by rank $\tilde{C}(s)$ we mean the rank over $\mathfrak{R}(s)$.

Theorem 8

Consider the system $(\tilde{A}, \tilde{B})$ in Luenberger form and let $\tilde{C}(s) = \tilde{C}S(s)$. Then (1) there is no c.s. $\mathscr{R}$ of $(\tilde{A}, \tilde{B})$ in ker $\tilde{C}$ if and only if rank $\tilde{C}(s) \geqslant m$, (2) there is a unique c.s. $\mathscr{R}^{\max}$ of $(\tilde{A}, \tilde{B})$ in ker $\tilde{C}$ if and only if rank $\tilde{C}(s) = m-1$, then dim $(\mathscr{R}^{\max} \cap \mathscr{B}) = 1$ and the decoupling vector $\beta(s) \sim R^{\max}$ (where $R^{\max}$ is some cyclic listing of $\mathscr{R}^{\max}$) is the unique minimal degree polynomial vector solution of the equation

$$\tilde{C}(s)\beta(s) = 0 \tag{35}$$

(3) there is a family $\mathscr{C}(\tilde{A}, \tilde{B}, \ker \tilde{C})$ (generally infinite when we are working over the ground field $\mathfrak{R}$) of c.s. $\mathscr{R}$ of $(\tilde{A}, \tilde{B})$ in ker $\tilde{C}$ if and only if rank $\tilde{C}(s) = m-k$, $1 < k \leqslant m-1$. If $N(s) = [n_1(s), n_2(s), \ldots, n_k(s)]$ is a minimal polynomial basis in echelon form for ker $\tilde{C}(s)$ then an expression for the generating function $\tilde{x}(s)$ of a general member $\mathscr{R}$ of $\mathscr{C}(\tilde{A}, \tilde{B}, \ker \tilde{C})$ may be written as

$$\tilde{x}(s) = S(s) \sum_{i=1}^{k} a_i(s)n_i(s) = S(s)N(s)a(s)$$

$a(s) = (a_1(s), a_2(s), \ldots, a_k(s))^{\mathrm{T}}$, $a_i(s) \in \mathfrak{R}[s]$, $i \in \mathbf{k}$. If $X(s) = S(s)N(s)$ is reduced if necessary (i.e. if its highest coefficient matrix is singular) to a minimal basis in echelon form $X^*(s) = S(s)N^*(s)$, then the columns of this latter matrix are proper minimal degree generating functions of the independent c.s. $\mathscr{R}_i^*$, $i \in \mathbf{k}$ (with dim $(\mathscr{R}_i^* \cap \mathscr{B}) = 1$) whose direct sum is $\mathscr{R}^{\max} \subseteq \ker \tilde{C}$, hence dim $(\mathscr{R}^{\max} \cap \mathscr{B}) = k$.

Proof

(1) It is clear that the normal rank conditions for the existence of solutions to linear equations over $\mathfrak{R}$ carry over to vector spaces over $\mathfrak{R}(s)$. (1) follows because these rank conditions furthermore carry over without change to the simple type of modules under consideration here, since we have a module defined over a principal ideal domain (Hartley and Hawkes 1970). However, this fact may be pieced together in an elementary way. Consider $\tilde{C}(s)$ as an $\mathfrak{R}[s]$-module morphism $\tilde{C}(s) : \mathfrak{R}^m[s] \to \mathfrak{R}^p[s]$, then

$$\ker \tilde{C}(s) = \{\beta(s) \mid \beta(s) \in \mathfrak{R}^m[s] \text{ such that } \tilde{C}(s)\beta(s) = 0\}$$

and there exists a polynomial vector solution $\beta(s)$ of eqn. (35) if and only if there exists a rational vector solution to this equation. We observe that we may pass trivially from one solution to the other on the one hand by straightforward embedding and allowing scalar multiplication by rational functions and in reverse by simple clearing of denominators by multiplication by least common denominators.

(2) From (1) and the fact that dim $\{\ker \tilde{C}(s)\} = 1$ we see that a solution to eqn. (35) exists and is unique up to multiplicative polynomial ' scalars ', i.e. if $\beta(s) \in \mathfrak{R}^m[s]$ solves (35) and is the minimum degree polynomial vector to do so, then for every $a(s) \in \mathfrak{R}[s]$, $a(s)\beta(s)$ is also a solution. That is to say that the polynomial vector solution set of (35) forms a singly-generated submodule of the $\mathfrak{R}[s]$-module $\mathfrak{R}^m[s]$. By Theorem 7 there exists at least one c.s. of $(\tilde{A}, \tilde{B})$, $\mathscr{R} = \mathscr{R}^{\max} \subseteq \ker \tilde{C}$. To show that $\beta(s)$ is actually the unique decoupling vector solution of (35), we know from § 3 that there exists at least one decoupling vector $\beta'(s)$ for $\mathscr{R}^{\max}$ which by Theorem 7 will solve (35). Then for all $a(s) \in \mathfrak{R}[s]$, $\deg a(s) \geqslant 1$, $a(s)\beta'(s) \approx \mathscr{R}^{\max}$ but $a(s)\beta'(s)$ is not a decoupling vector corresponding to a cyclic listing $R^{\max}$ of $\mathscr{R}^{\max}$, by Proposition 1, as $a(s)$ is not a unit of $\mathfrak{R}[s]$. Thus we conclude that $\beta'(s) = \beta(s)$ and is the unique decoupling vector solution of eqn. (35), hence $\mathscr{R}^{\max} = \text{span}\,\{S(s)\beta(s)\}$ is the unique (maximal) c.s. in ker $\tilde{C}$. If $\gamma(s) \in R^m[s]$ is any solution of eqn. (35) then we have also that $\mathscr{R}^{\max} = \text{span}\,\{S(s)\gamma(s)\}$. To demonstrate that $\dim(\mathscr{R}^{\max} \cap \mathscr{B}) = 1$ let us suppose that $\dim(\mathscr{R}^{\max} \cap \mathscr{B}) = q > 1$ and let $\tilde{F} \in \tilde{F}(\mathscr{R}^{\max})$. Then the controllable system pair $(\tilde{A} + \tilde{B}\tilde{F} \mid \mathscr{R}^{\max}, \mathscr{R}^{\max} \cap \mathscr{B})$ will have (say) controllability indices $k_1 \leqslant k_2 \leqslant \ldots \leqslant k_q$ independent of the choice of $\tilde{F}$ and there will exist ' elementary ' c.s. $\mathscr{R}'_i \subset \mathscr{R}^{\max}$, $i \in \mathbf{q}$, $\dim \mathscr{R}'_i = k_{q+1-i}$, $i \in \mathbf{q}$ which will be independent subspaces decomposing $\mathscr{R}^{\max}$ under the map $\tilde{A} + \tilde{B}\tilde{F}$. (By ' elementary ' c.s. of $(\tilde{A}, \tilde{B})$ we mean the ones that have as minimal degree generating functions the columns $s_i(s)$, $i \in \mathbf{m}$, of $S(s) = (s_1(s), s_2(s), \ldots, s_m(s))$.) Let $n'_i(s)$, $i \in \mathbf{q}$ be a decoupling vector for some cyclic listing R'_i of $\mathscr{R}'_i$. Then clearly $n'_i(s) \neq n'_j(s)$, $i \neq j$, $i, j \in \mathbf{q}$ (nor any polynomial multiple of $n'_j(s)$) otherwise $\mathscr{R}'_i$ and $\mathscr{R}'_j$ would coincide again by Proposition 1. From the above argument and the fact that in this case eqn. (35) has a unique decoupling vector solution we are forced to conclude that $\dim(\mathscr{R}^{\max} \cap \mathscr{B}) = 1$. We have incidentally proved that $\dim(\mathscr{R}^{\max} \cap \mathscr{B}) = 1$ is also a sufficient condition for $\tilde{C}(s)$ to have rank $m-1$ or equally for $\mathscr{R}^{\max}$ to have a unique associated decoupling vector. The latter is anyhow obvious from considerations of the generator of $\mathscr{R}^{\max}$ in $\mathscr{B}$.

(3) If rank $\tilde{C}(s) = m-k$, $1<k\leqslant m-1$, then there exists a polynomial vector submodule solution $\mathcal{N}[s]$ of eqn. (35) with rank $\mathcal{N}[s]=k$. Moreover this submodule can be regarded in the sense of Forney (1975) as a vector space over $\mathfrak{R}(s)$ with associated minimal polynomial basis in echelon form say $[n_1(s), n_2(s), \ldots, n_k(s)] = N(s)$, forming a basis for ker $\tilde{C}(s)$ and with associated dynamical indices $n_1 \leqslant n_2 \leqslant \ldots \leqslant n_k$, $(n_i = \deg n_i(s),\ i\in\mathbf{k})$. Then any solution $\beta(s)$ of eqn. (35) can be written in a unique way as

$$\beta(s) = \sum_{i=1}^{k} a_i(s) n_i(s) = N(s)a(s) \tag{36}$$

$a(s) = (a_1(s), a_2(s), \ldots, a_k(s))^{\mathrm{T}}$, $a_i(s)\in\mathfrak{R}[s]$, $i\in\mathbf{k}$ and the expression for the generating function $\tilde{x}(s)$ of a general c.s. $\mathscr{R}\in\mathscr{C}(\tilde{A}, \tilde{B}, \ker \tilde{C})$ will be $\tilde{x}(s) = S(s)\beta(s) = S(s)N(s)a(s)$, and so $X(s) = S(s)N(s)$ is a basis with (say) ordered dynamical indices (Forney 1975): $\gamma_1 \leqslant \gamma_2 \leqslant \ldots \leqslant \gamma_k$ for the rational vector space of all generating functions $\tilde{x}(s)$ which correspond to c.s. $\mathscr{R} = \text{span}\ \{\tilde{x}(s)\} \in \mathscr{C}(\tilde{A}, \tilde{B}, \ker \tilde{C})$. Now note that in general if $N(s)$ is a minimal basis in echelon form for ker $\tilde{C}(s)$ then $X(s) = S(s)N(s)$ is not necessarily a minimal basis in echelon form of its respective rational vector space. In such a case we may by further Forney (1975) operations (of the type used in step 3 of his reduction algorithm) reduce $X(s) = S(s)N(s)$ to a minimal basis in echelon form $[x_1^*(s), x_2^*(s), \ldots, x_k^*(s)] = X^*(s) = S(s)N^*(s)$ where $N^*(s) = [n_1^*(s), n_2^*(s), \ldots, n_k^*(s)]$ with dynamical indices $\gamma_1 \leqslant \gamma_2 \leqslant \ldots \leqslant \gamma_k$ $(\gamma_i = \deg x_{k+1-i}^*(s),\ i\in\mathbf{k})$. Let us now refer back to the 'elementary c.s.' $\mathscr{R}'_i$, $i\in\mathbf{q}$, dim $\mathscr{R}'_i = k_{q+1-i}$, having some cyclic listing $R'_i \sim n'_i(s)$ introduced in the proof of part (2) above. These c.s. will always decompose $\mathscr{R}^{\max}$ under the map $\tilde{A} + \tilde{B}\tilde{F}$ for all $\tilde{F}\in\tilde{\mathbf{F}}(\mathscr{R}^{\max})$. It is straightforward to check the details of showing that $X'(s) = S(s)N'(s) = S(s)[n'_1(s), n'_2(s), \ldots, n'_q(s)]$ is a minimal basis with ordered indices for the rational vector space spanned by the columns of $X(s)$ (Stoyle 1978) hence $k=q$. These ordered indices are known to be unique and in fact they are $\gamma_i = k_i - 1$, $i\in\mathbf{k}$. Now dim $\mathscr{R}_i^* \leqslant \gamma_{k+1-i} + 1 = k_{k+1-i}$. To see that equality holds, note that $S(s)N'(s)$ and $S(s)N^*(s)$ are obtainable one from the other by Forney column operations and Proposition 3 thereby yields that

$$\sum_{i=1}^{k} \mathscr{R}_i^* = \sum_{i=1}^{k} \mathscr{R}'_i = \mathscr{R}^{\max}, \quad \text{where dim } \mathscr{R}^{\max} = \sum_{i=1}^{k} k_i$$

Then a simple dimensionality argument proves that dim $\mathscr{R}_i^* = \gamma_{k+1-i} + 1 = k_{k+1-i}$, $i\in\mathbf{k}$, and that the $\mathscr{R}_i^*$ are independent subspaces. Also we then get immediately that dim $(\mathscr{R}_i^* \cap \mathscr{B}) = 1$ and $n_i^*(s)$ is the unique decoupling vector for $\mathscr{R}_i^*$. This ends the proof.†

Note that for the above proof to go through, $N(s)$ and $X^*(s)$ need only to be minimal bases of their respective rational vector spaces; the echelon forms employed are not essential. Finally, degree restrictions may be imposed upon the polynomials $a_i(s)$ in eqn. (36) in order not to have c.s. listings which are in general linearly dependent, i.e. $\beta(s)$ in eqn. (36) is a redundant polynomial

† $N^*(s)$ may be obtained in one step from $C(s)$ as a minimal basis of ker $C(s)[s^{v_m - v}]$. Verify this observing that $C(s)\beta(s) = 0 \Leftrightarrow C(s)[s^{v_m - v}][s^{v-1}]\beta(s) = 0$.

representation of some underlying c.s. As indicated in the Appendix, the sensible restriction to make is

$$\deg a_i(s) \leqslant \sum_{\substack{j=1 \\ j \neq i}}^{k} (\gamma_j + 1)$$

We now provide an easy method of computing a solution to eqn. (35) for the case of rank $\tilde{C}(s) = m-1$ also giving the minimal degree solution. For the purpose of computing the unique c.s. $\mathscr{R}^{\max} \subseteq \ker \tilde{C}$ the former is enough. Without loss of generality we assume that $\tilde{C}(s)$ is a $(m-1) \times m$ polynomial matrix, as we can always single out $m-1$ linearly dependent (over $\mathfrak{R}(s)$) rows of $\tilde{C}(s)$. Let the m distinct $(m-1)$-order minors of $\tilde{C}(s)$ be denoted by

$$M_i(s) = \tilde{C}(s) \begin{cases} 1, 2, \ldots, m-1, \\ \\ 1, 2, \ldots, \hat{\imath}, \ldots, m, \end{cases} \qquad i \in \mathbf{m}$$

where $\hat{\imath}$ denotes that the ith and only the ith column of $\tilde{C}(s)$ is omitted, $i \in \mathbf{m}$. Let row i of $\tilde{C}(s)$ be $\tilde{c}_i(s) = [\tilde{c}_{i1}(s), \tilde{c}_{i2}(s), \ldots, \tilde{c}_{im}(s)]$ and let $\alpha_i(s) = (-1)^{1+i} M_i(s)$. Then the Laplace determinantal expansion of the $m \times m$ polynomial matrix

$$\tilde{C}_E{}^i(s) = \begin{bmatrix} \tilde{c}_i(s) \\ \cdots\cdots\cdots \\ \tilde{C}(s) \end{bmatrix} \begin{matrix} \}1 \\ \\ \}m-1 \end{matrix}$$

by its top row for each $i \in \mathbf{m}$ in turn gives that

$$\det \tilde{C}_E{}^i(s) = \tilde{c}_{i1}(s)\alpha_1(s) + \tilde{c}_{i2}(s)\alpha_2(s) + \ldots + \tilde{c}_{im}(s)\alpha_m(s) = \tilde{c}_i(s)\alpha(s) = 0, \quad i \in \mathbf{m}$$

where $\alpha(s) = (\alpha_1(s), \alpha_2(s), \ldots, \alpha_m(s))^{\mathrm{T}}$, i.e. that

$$\tilde{C}(s)\alpha(s) = 0$$

If by $\alpha_R(s)$ we denote the polynomial vector obtained from $\alpha(s)$ after the removal of any non-unit common factor of all its components $\alpha_i(s)$, $i \in \mathbf{m}$, then we can take our irreducible decoupling vector $\beta(s)$ to be $\alpha_R(s)$. In fact if $\tilde{C}(s)$ has any non-trivial polynomial common factors among the elements of any of its rows, these may all be removed as a diagonal left factor of $\tilde{C}(s)$, i.e. $\tilde{C}(s) = D_C(s)\tilde{C}_R(s)$ where $D_C(s)$ is diagonal and now

$$\text{g.c.d. } \{\tilde{C}_{Rij}(s),\ j \in \mathbf{m}\} = 1, \quad i \in \mathbf{m-1}$$

If we then calculate the associated reduced $\alpha(s)$, denoted by $\alpha^R(s)$, from $\tilde{C}_R(s)$ by computing cofactors the result is a strictly lower degree solution for eqn. (35), or equally of $\tilde{C}_R(s)\beta(s) = 0$, than $\alpha(s)$ itself. However, unfortunately, we cannot always conclude that $\alpha_R(s) = \alpha^R(s)$, although it is generically unlikely that this is not so in the parameter space of $\tilde{C}$. Now the main theorem by Forney (1975) gives the equivalence of the following two conditions :

(1) $\tilde{C}(s)$ has full rank modulo $p(s)$, for all irreducible $p(s) \in \mathfrak{R}[s]$.

(2) The g.c.d. of all $(m-1)$-order minors of $\tilde{C}(s)$ is 1.

So our procedure for computing $\alpha(s)$ from $\tilde{C}(s)$ will only give a decoupling vector solution of eqn. (35) in one step if and only if $\tilde{C}(s)$ is first reduced to a

$\tilde{C}_R(s)$ by Forney operations of the type employed in step 2 of his reduction algorithm.

Example

As an illustration of the preceding observations, consider the case where $p=2$, $m=3$, $n=7$, $v_1=v_2=2$, $v_3=3$ and

$$\tilde{C}=\begin{bmatrix} -1 & 0 & 1 & 0 & 1 & -1 & 2 \\ -1 & -1 & 0 & -1 & 0 & -2 & 1 \end{bmatrix}$$

So that

$$\tilde{C}S(s)=\tilde{C}(s)=\begin{bmatrix} (s^2-1) & s & (2s-1) \\ -(s+1) & -1 & (s-2) \end{bmatrix}=\begin{bmatrix} \tilde{c}_1(s) \\ \tilde{c}_2(s) \end{bmatrix}$$

We cannot extract any row factor from $\tilde{C}_1(s)$ or $\tilde{C}_2(s)$. But

$$\alpha(s)=[s^2-1 \quad -(s^3+1) \quad s+1]^{\mathrm{T}}=(s+1)\alpha_R(s)$$

and the minimal order solution of eqn. (35) is

$$\alpha_R(s)=[s-1 \quad -s^2+s-1 \quad 1]^{\mathrm{T}}=\beta(s)$$

up to scalar multipliers in $\mathfrak{R}$. The further reduction of $\tilde{C}(s)$ by Forney operations is necessary in this case because rank $\tilde{C}(s)=1$ mod $(s+1)$, as it is evident by forming

$$\tilde{C}'(s)=\begin{bmatrix} \tilde{c}_1(s)-\tilde{c}_2(s) \\ \tilde{c}_2(s) \end{bmatrix}=\begin{bmatrix} (s+1)s & s+1 & s+1 \\ -(s+1) & -1 & s-2 \end{bmatrix}$$

Then

$$\tilde{C}_R(s)=\begin{bmatrix} s & 1 & 1 \\ -(s+1) & -1 & s-2 \end{bmatrix}$$

and

$$\alpha^R(s)=[s-1 \quad -s^2+s-1 \quad 1]^{\mathrm{T}}$$

An analogous computational procedure for case (3) of the last theorem is included for completeness but which we have space only to sketch. Let us assume that $\tilde{C}(s)$ is $(m-k)\times m$ of rank $m-k$. Suppose that the minor

$$\tilde{C}(s)\begin{Bmatrix} 1, 2, \ldots, m-k \\ 1, 2, \ldots, m-k \end{Bmatrix}$$

is non-zero, otherwise by trivial changes of indexing the argument may readily be adapted. Consider the $(m-k+1)\times m$ matrix $\tilde{C}_E{}^i(s)$ and expand the matrix

$$\tilde{C}_E{}^i(s)\begin{Bmatrix} 1, 2, \ldots, m-k+1, \\ j, j+1, \ldots, m-k+j, \end{Bmatrix} \qquad j\in\mathbf{k}$$

by its top row as before. Let the $(m-k+1)$-vector of signed minors in this expansion be denoted by $\hat{\alpha}_j(s)$, then if we insert $(j-1)$-zeros before the first element of $\hat{\alpha}_j(s)$ and also zeros after the $(m-k+1)$th element to convert it to an m-polynomial vector $\beta_j(s)$, then it may easily be checked that

$$\tilde{c}_i(s)\beta_j(s)=0, \quad i\in \mathbf{m}-\mathbf{k}, \quad j\in\mathbf{k}$$

The matrix $B(s)=(\beta_1(s), \beta_2(s), \ldots, \beta_k(s))$ will have full rank as the last k rows of it will by construction form a non-zero diagonal matrix. Thus it is again possible to write down some basis for ker $\tilde{C}(s)$, in general non-minimal, simply by inspection. $B(s)$ could subsequently be reduced to some suitable minimal basis (Sain 1975), but for the purpose of computing $\mathscr{R}^{\max}$, it suffices by Proposition 3 to form $\mathscr{R}_i$, $i\in\mathbf{k}$, where $\beta_i(s)\approx\mathscr{R}_i$ and then find that

$$\mathscr{R}^{\max}=\sum_{i=1}^{k}\mathscr{R}_i$$

Appendix

We are given a polynomial m-vector $\beta(s)=[\beta_m(s), \beta_{m-1}(s), \ldots, \beta_1(s)]$ deg $\beta_i(s)\leqslant r-v_i$, $i\in\mathbf{m}$ with equality holding for at least one i and we should then enquire whether it is a decoupling vector for the given set of controllability indices v_i. We can always list $\tilde{x}(s)=S(s)\beta(s)$ and obtain a stripe matrix as the one in eqn. (20). First we comment that if $\beta(s)$ is chosen at random to satisfy the degree condition we almost always get an independent set of vectors in the list (provided of course that $r\leqslant n$, otherwise the listing will certainly be dependent) as may be verified by testing all the r-order minors of the stripe matrix. The conditions of them all being simultaneously zero yields a hypersurface in the space of all coefficients β_{ij} of the $\beta_i(s)$.

Theorem 8 shows that $\mathscr{R}^{\max}$ is being decomposed into an internal direct sum for the k c.s. $\mathscr{R}_i^*$ or $\mathscr{R}'_i$, $i\in\mathbf{k}$ which are analogues of the elementary c.s. $\mathscr{R}_i$, $i\in\mathbf{m}$ which correspond to the columns $s_i(s)$ of $S(s)$. From that theorem the generating function of a general c.s. $\mathscr{R}\subseteq\ker\tilde{C}$ may be written as

$$\tilde{x}(s)=X^*(s)a(s) \tag{A 1}$$

where $X^*(s)$ is precisely analogous to $S(s)$ and the k-vector $a(s)$ to the m-vector $\beta(s)$. Thus continuing the analogy we see that we can always take

$$\deg a_i(s)\leqslant\sum_{\substack{j=1\\ j\neq i}}^{k}(\gamma_j+1) \tag{A 2}$$

where

$$\sum_{j=1}^{k}(\gamma_j+1)=\dim\mathscr{R}^{\max}$$

Returning to the first question raised in this Appendix, let us suppose that L $\{\tilde{x}(s)\}$ is redundant, i.e. there exists some linear dependency among its elements. Let $\beta(s)\approx\mathscr{R}$, i.e. span $\{\text{L}\{\tilde{x}(s)\}\}=\mathscr{R}$ and let $\tilde{C}$ be a matrix in our coordinates representing the subspace $\mathscr{R}^\perp$. Then $\mathscr{R}$ is the maximal c.s. in ker $\tilde{C}$, in fact plainly $\mathscr{R}=\ker\tilde{C}$. If $x'(s)$ denotes any generating function of $\mathscr{R}^{\max}$ there exists k, $X^*(s)$ such that $\tilde{x}'(s)$ satisfies (A 1) for some $a(s)$, with

$\tilde{x}'(s)$ in place of $\tilde{x}(s)$. Letting $a(s)$ vary freely subject to the degree restriction (A 2) with equality holding for at least one i as we have seen will give us generically a minimal degree (i.e. non-redundant) generating function $\tilde{x}'(s)$ for $\mathscr{R}^{\max}$. Moreover this set of generating functions will always yield an underlying c.s. $\mathscr{R} \subset \mathscr{R}^{\max}$. Thus the original representation $\tilde{x}(s) = S(s)\beta(s)$ to have been redundant, condition (A 2) with equality holding for at least one i must have been violated for some such sum decomposition (A 1). This is then a polynomial vector test for redundancy, but one of purely theoretical interest as it would generally be much more difficult to use this test than simply to list and check the rank of the listing.

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