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The mechanism of decoupling

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The mechanism of decoupling

P. N. R. STOYLE† and A. I. G. VARDULAKIS†

In the light of the concept of a decoupling vector (Stoyle and Vardulakis 1979) as a minimal algebraic parametrization of a controllability subspace (c.s.), the mechanism of decoupling control by state feedback is investigated. The essential equivalence of the Wonham and Morse (1970) and Falb and Wolovich (1969) solutions of the diagonal non-block restricted decoupling problem (RDP) might well be suspected and the method of decoupling vectors intertwines these so that each is seen to be crucial to the efficient formulation and solution of decoupling problems. A new formula for the classical necessary and sufficient condition given by Falb and Wolovich (1967) in terms of their B^* matrix emerges naturally from the analysis. The role of decoupling c.s.s in the guise of their associated and very easily obtained decoupling vectors in solving RDP for us becomes very clear. The solution given leads to the important definition of cyclic independence for c.s.s and this is relevant to reducing other types of decoupling problems to the basic case above.

1. Introduction

The modular concept of a decoupling vector as a polynomial parametrization of a controllability subspace (c.s.) and some of its nice algebraic properties were recorded and interpreted in system theoretic terms by Stoyle and Vardulakis (1979). In fact, we already have the machinery to look at simple decoupling problems and in this paper we give an illustrative application of some of the body of theory developed so far to the restricted (diagonal) decoupling problem in which we have each of m (non-block) inputs controlling m respective (non-block) outputs independently, i.e. a square system is to be decomposed by a constant square input transformation and state feedback into a set of independent s.i.s.o. subsystems on which we are to do what we can by way of allocating system zeros and placing poles. We will bring our new methodological angle on the problem into line with the well-known necessary and sufficient condition and solutions—both the polynomial matrix one given by Wolovich and Falb (1969) and Wolovich (1974) and the geometric one due to Wonham and Morse (1970)—and we also clarify the mechanism of decoupling. Only some slight familiarity with the basics of Wonham's (1974) type of formulation is assumed.

Our aim, roughly stated, is to arrive at a useful result about the compatibility of m independent c.s.s. These must be large enough to allow some system poles to be placed in each one of them outside their common area of overlap which is, as is well known, an (A, B)-invariant subspace, and small enough for each to have as intersection with $\mathscr{B}$ just its generator b in $\mathscr{B}$ (Stoyle and Vardulakis 1979). Thus each c.s. can in a sense be regarded as governing an independent single-input subsystem of its own, activated in the closed-loop situation, with compatible feedback F^* in operation, through its corresponding germ, the set of m linearly independent germs h_i forming a basis for

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the non-singular input transformation $H=(h_1, \ldots, h_m)$ which has to be applied along with the feedback F^*.

1.1. *Translation of geometric decoupling conditions into polynomial matrix terms*

We consider a system in discrete time form described by the difference equations

$$\left.\begin{aligned} x(t+1) &= \tilde{A}x(t)+\tilde{B}u(t) \\ y(t) &= \tilde{C}x(t) \end{aligned}\right\} \tag{1.1}$$

where $\tilde{A}\in\Re^{n\times n}$, $\tilde{B}\in\Re^{n\times m}$, $\tilde{C}\in\Re^{m\times n}$ and $(\tilde{A}, \tilde{B})$ are in the input transformed Luenberger canonical form (Stoyle and Vardulakis 1979). Using the linear constant state feedback law $\tilde{F}$ together with transformation of the input $\tilde{G}$

$$u(t)=\tilde{F}x(t)+\tilde{G}w(t) \tag{1.2}$$

the diagonal (non-block) restricted decoupling problem may be stated as follows. Find a pair $(\tilde{F}, \tilde{G})$ such that the closed-loop system

$$\left.\begin{aligned} x(t+1) &= (\tilde{A}+\tilde{B}\tilde{F})x(t)+\tilde{B}\tilde{G}w(t) \\ y(t) &= \tilde{C}x(t) \end{aligned}\right\} \tag{1.3}$$

is non-interacting in the sense that the ith scalar input w_i, $i\in\mathbf{m}$, affects only the ith scalar output y_i, $i\in\mathbf{m}$.

Let us now call the subsystem controlled by the ith input w_i, $i\in\mathbf{m}$, the ith subsystem, then in diagonal decoupling the design intention is that the ith feedback subsystem should comprise the dynamical connection between the ith input and the ith output only. Let $\tilde{C}_j$ be the matrix obtained from the $\tilde{C}$ matrix after the jth row $\tilde{c}_j$ of it has been omitted, i.e.

$$\tilde{C}_j=\begin{bmatrix} \tilde{c}_1 \\ \vdots \\ \tilde{c}_{j-1} \\ \\ \tilde{c}_{j+1} \\ \vdots \\ \tilde{c}_m \end{bmatrix}, \quad j\in\mathbf{m}$$

and let the subset of the state space reachable through the jth input w_j be denoted $\mathscr{R}_j=\{\tilde{A}+\tilde{B}\tilde{F}\,|\,\tilde{B}\tilde{g}_j\}$, the jth c.s. of $(\tilde{A}, \tilde{B})$, where $\tilde{G}=(\tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_m)$ is to be found. Then as is well known the geometrical decoupling conditions (GDC) which follow concisely set down the non-interaction requirements :

GDC I $\qquad \mathscr{R}_j\subseteq \operatorname{Ker} \tilde{C}_j, \quad j\in\mathbf{m}$

i.e. w_j does not affect y_i for $i\neq j$, $i\in\mathbf{m}$, $j\in\mathbf{m}$.

GDC II $\qquad \tilde{c}_j\mathscr{R}_j=\operatorname{Im}\tilde{c}_j, \quad j\in\mathbf{m}$

i.e. the jth output is pointwise output controllable (from the origin).

GDC III $\qquad \bigcap_{j\in\mathbf{m}} \mathbf{F}(\mathscr{R}_j)\neq\emptyset$

i.e. the compatibility condition, recalling that $\mathbf{F}(\mathscr{R}_j)$ is the class of all $\tilde{F}:\mathscr{X}\to\mathscr{U}$ achieving the c.s. $\mathscr{R}_j$ (Wonham and Morse 1970, Wonham 1974), and this is identical with the class of all $\tilde{F}$ attaining R_j through $\tilde{B}\tilde{g}_j$, where R_j is the cyclic listing of the c.s. $\mathscr{R}_j$, $j\in\mathbf{m}$ (Stoyle and Vardulakis 1979).

We now set about translating all these problem specifications in terms of subspaces into equivalent polynomial matrix statements by invoking the results of Stoyle and Vardulakis (1979), and the problem will become easier to handle than the geometrical formulation.

Let us make the initial assumption that $\tilde{C}(\lambda)=\tilde{C}S(\lambda)$, has full rank over $\mathfrak{R}(\lambda)$, then certainly $\tilde{C}_j(\lambda)=\tilde{C}_jS(\lambda)$, $j\in\mathbf{m}$, also has full rank. Let $D_C(\lambda)$ be the $m\times m$ non-singular, diagonal polynomial matrix diag $(d_i(\lambda))$, with each diagonal element $d_i(\lambda)$, $i\in\mathbf{m}$, equal to the greatest common (monic) divisor (g.c.d.) of the corresponding ith row $\tilde{c}_i(\lambda)$ of $\tilde{C}(\lambda)$, i.e. let

$$\tilde{C}(\lambda)=D_C(\lambda)\tilde{C}_R(\lambda) \tag{1.4}$$

then

$$\tilde{C}_j(\lambda)=D_{C_j}(\lambda)\tilde{C}_{Rj}(\lambda),\quad j\in\mathbf{m} \tag{1.5}$$

where

$$D_{C_j}(\lambda)=\operatorname*{diag}_{i\neq j}(d_i(\lambda)),\quad j\in\mathbf{m}$$

If $D_{Cj}(\lambda)$ is the $(m-1)\times(m-1)$ matrix obtained from $D_C(\lambda)$ after its jth row has been omitted we observe that

$$D_{Cj}(\lambda)=D_{C_j}(\lambda),\quad j\in\mathbf{m}$$

From (1.4) we have

$$\tilde{C}(\lambda)\text{ adj }\tilde{C}_R(\lambda)=\det\tilde{C}_R(\lambda)D_C(\lambda) \tag{1.6}$$

now if

$$\text{adj }\tilde{C}_R(\lambda)=(b_1(\lambda),b_2(\lambda),\ldots,b_m(\lambda))$$

then from (1.6)

$$\tilde{c}_j(\lambda)b_j(\lambda)=d_j(\lambda)\det\tilde{C}_R(\lambda),\quad j\in\mathbf{m} \tag{1.7}$$

$$\tilde{C}_j(\lambda)b_j(\lambda)=0,\quad j\in\mathbf{m} \tag{1.8}$$

Defining $\beta_j(\lambda)$ from $b_j(\lambda)=m_j(\lambda)\beta_j(\lambda)$, $j\in\mathbf{m}$, where

$$m_j(\lambda)=\text{g.c.d. }\{b_{1j}(\lambda),b_{2j}(\lambda),\ldots,b_{mj}(\lambda)\},\quad j\in\mathbf{m}$$

then (1.7) and (1.8) can be written respectively as

$$\tilde{c}_j(\lambda)\beta_j(\lambda)=d_j(\lambda)\det\tilde{C}_R(\lambda)/m_j(\lambda),\quad j\in\mathbf{m} \tag{1.9}$$

$$D_{C_j}(\lambda)m_j(\lambda)\tilde{C}_{Rj}(\lambda)\beta_j(\lambda)=0,\quad j\in\mathbf{m} \tag{1.10}$$

Remark 1

As $b_j(\lambda)$, $j\in\mathbf{m}$, is the jth column of adj $\tilde{C}_R(\lambda)$, it is evident that $m_j(\lambda)=$ g.c.d. {of all $(m-1)$-order minors of $\tilde{C}_{Rj}(\lambda)$}.

From Theorem 8(ii) in Stoyle and Vardulakis (1979) and (1.10) we have that $\beta_j(\lambda)$, $j\in\mathbf{m}$, is the unique (up to units in $\mathfrak{R}[\lambda]$) minimal degree polynomial vector solution of equation

$$\tilde{C}_j(\lambda)\beta_j(\lambda)=0,\quad j\in\mathbf{m} \tag{1.11}$$

or equally of the equation

$$\tilde{C}_{Rj}(\lambda)\beta_j(\lambda)=0, \quad j\in \mathbf{m} \tag{1.12}$$

Hence GDC I yields $\beta_j(\lambda)\sim R_j$, $j\in\mathbf{m}$, where R_j is the cyclic listing of the c.s. $\mathscr{R}_j$, $j\in\mathbf{m}$ (Stoyle and Vardulakis 1979).

Remark 2

As $\beta_j(\lambda)$ is the unique solution to (1.11) we note that $\mathscr{R}_j=\mathscr{R}_j^{\max}$, $j\in\mathbf{m}$, for this problem, so no complications arise from having sets of $\mathscr{R}_j$ not all of which yield a compatible set of m decoupling c.s.s.

Let $R_j=\langle x_{r_j-1}{}^j, x_{r_j-2}{}^j, \ldots, x_0{}^j\rangle$, $j\in\mathbf{m}$, be the cyclic listing of $\mathscr{R}_j$ corresponding to the generating function

$$x^j(\lambda)=x_0{}^j+x_1{}^j\,\lambda+\ldots+x_{r_j-1}{}^j\,\lambda^{r_j-1}, \quad j\in\mathbf{m}$$

then GDC II in this scalar output case is equivalent to

$$\begin{aligned}\tilde{c}_j\mathscr{R}_j\neq 0 &\leftrightarrow \tilde{c}_j \operatorname{span}\{x_0{}^j, x_1{}^j, \ldots, x_{r_j-1}{}^j\}\neq 0\\ &\leftrightarrow \tilde{c}_j S(\lambda)\beta_j(\lambda)\neq 0\\ &\leftrightarrow \tilde{c}_j(\lambda)\beta_j(\lambda)\neq 0, \quad j\in\mathbf{m}\end{aligned}$$

and from (1.9) we see that the rank assumption on $\tilde{C}(\lambda)$ is indispensable for the output pointwise controllability assumption to hold.

Finally, the compatibility condition GDC III is to be expressed giving an important sequel to the proof of Theorem 6 in Stoyle and Vardulakis (1979). From this theorem, we already know that there exists a compatible feedback $\hat{F}$ on our m decoupling c.s. $R_j\sim\beta_j(\lambda)$, $j\in\mathbf{m}$, iff we can produce a factorization :

$$\delta_F(\lambda)B(\lambda)=(h_1, h_2, \ldots, h_m)\begin{bmatrix} v_1(\lambda) & & \mathbf{0} \\ & v_2(\lambda) & \\ \mathbf{0} & & \ddots \\ & & & v_m(\lambda)\end{bmatrix} \tag{1.13}$$

where $v_i(\lambda)$, $i\in\mathbf{m}$, are polynomials of degree $\leqslant n-m+1$ (to be specified below) and $\beta(\lambda)$ is the matrix of decoupling vectors $\beta_j(\lambda)\sim R_j$ lumped together, i.e

$$B(\lambda)=(\beta_1(\lambda), \beta_2(\lambda), \ldots, \beta_m(\lambda))$$

We call $B(\lambda)$ the *(diagonal) decoupling matrix.*

The statement of Theorem 6 in Stoyle and Vardulakis (1979), also yields

$$(h_1, h_2, \ldots, h_m)=H=[[\lambda^v]B(\lambda)]_c{}^{\mathrm{h}} \tag{1.14}$$

and we make the further assumption that H is non-singular which as we will see ensures that the decoupling c.s.s are independent in the cyclic subspace or ordered list sense (Stoyle and Vardulakis 1979).

Proposition 1

The matrix H^{-1} is given by

$$H^{-1}=[\tilde{C}(\lambda)[\lambda^{v_m-v}]]_r{}^{\mathrm{h}} \tag{1.15}$$

where

$$v_m = \max_{i \in \mathbf{m}} \{v_i\} = \text{the controllability index of } (\tilde{A}, \tilde{B})$$

$$[\lambda^{v_m - v}] = \text{diag}\,(1, \lambda^{v_m - v_{m-1}}, \ldots, \lambda^{v_m - v_1})$$

and the symbol $[\;]_{\mathrm{r}}^{\mathrm{h}}$ denotes the highest row coefficient matrix of the expression inside the brackets.

Proof

From (1.4), (1.9) and (1.11) we have

$$\tilde{C}_R(\lambda) B(\lambda) = \det \tilde{C}_R(\lambda) M^{-1}(\lambda) \tag{1.16}$$

where $M(\lambda) = \text{diag}\,(m_i(\lambda))$, or equivalently

$$B(\lambda) = \text{adj}\; \tilde{C}_R(\lambda) M^{-1}(\lambda) \tag{1.17}$$

Now it may be checked without difficulty that if $P(\lambda)$ is any polynomial matrix, then

$$\text{adj}\; [P(\lambda)]_{\mathrm{c}}^{\mathrm{h}} = [\text{adj}\; P(\lambda)]_{\mathrm{r}}^{\mathrm{h}} \tag{1.18}$$

so from (1.14) we can express H directly in terms of $\tilde{C}_R(\lambda)$:

$$\text{adj}\; H = [\text{adj}\; B(\lambda)\; \text{adj}\; [\lambda^v]]_{\mathrm{r}}^{\mathrm{h}} \tag{1.19}$$

From (1.17)

$$\text{adj}\; B(\lambda) = \left\{ \frac{[\det \tilde{C}_R(\lambda)]^{m-2}}{\det M(\lambda)} I_m M(\lambda) \right\} \tilde{C}_R(\lambda) \tag{1.20}$$

and so from (1.19), (1.20) and the assumption of the invertibility of H we have

$$\begin{aligned} H^{-1} &= \frac{1}{\det H} [\text{adj}\; B(\lambda)\; \text{adj}\; [\lambda^v]]_{\mathrm{r}}^{\mathrm{h}} \\ &= \frac{1}{\det H} \left[\left\{ \frac{[\det \tilde{C}_R(\lambda)]^{m-2}}{\det M(\lambda)} I_m M(\lambda) \right\} \tilde{C}_R(\lambda)\; \text{adj}\; [\lambda^v] \right]_{\mathrm{r}}^{\mathrm{h}} \\ &= \frac{q}{\det H} [\tilde{C}_R(\lambda)[\lambda^{n-v}]]_{\mathrm{r}}^{\mathrm{h}} \end{aligned} \tag{1.21}$$

where if

$$\det \tilde{C}_R(\lambda) = a_k \lambda^k + a_{k-1} \lambda^{k-1} + \ldots + a_0$$

and

$$[\det \tilde{C}_R(\lambda)]^{m-2} = a_k^{m-2}\, \lambda^{k(m-2)} + \ldots$$

then $q = a_k^{m-2}$, and we have noted that pre-multiplication by a diagonal polynomial matrix of monic polynomials does not affect the operation of taking the highest row coefficient matrix.

Now as each column $\beta_j(\lambda)$, $j \in \mathbf{m}$, of $B(\lambda)$ is unique up to units in $\mathfrak{R}[\lambda]$ we can certainly multiply $B(\lambda)$ by a scalar $q' = \det H/q$ and use the resulting matrix as our $B(\lambda)$, i.e. we are normalizing the constant multiplier $q/\det H$ in (1.21) to be unity so that now

$$\begin{aligned} H^{-1} &= [\tilde{C}_R(\lambda)[\lambda^{n-v}]]_{\mathrm{r}}^{\mathrm{h}} \\ &= [D_C(\lambda) \tilde{C}_R(\lambda)[\lambda^{v_m - n}][\lambda^{n-v}]]_{\mathrm{r}}^{\mathrm{h}} \\ &= [\tilde{C}(\lambda)[\lambda^{v_m - v}]]_{\mathrm{r}}^{\mathrm{h}} \end{aligned}$$

Remark 3

From the Wolovich and Falb (1969) structure theorem, in our input-transformed Luenberger coordinates

$$[\lambda I - \tilde{A} - \tilde{B}\tilde{F} \vdots -\tilde{B}\tilde{G}] \begin{bmatrix} S(\lambda) \\ \cdots\cdots \\ \tilde{G}^{-1}\delta_F(\lambda) \end{bmatrix} B(\lambda) = 0$$

the cyclic listing R_j of the decoupling c.s. $\mathscr{R}_j$ is simply obtained as the listing of the jth column of $S(\lambda)B(\lambda)$ and the open-loop input sequence generating function achieving R_j is listed as the jth column of $\delta_0(\lambda)B(\lambda)$ (Stoyle and Vardulakis 1979). The actual listing of the decoupling c.s. is however unnecessary as a step towards solving the decoupling problem (cf. Wonham 1974) as we shall now show.

Theorem 1

A necessary and sufficient condition for the classical non-block RDP to have a solution via state feedback $\tilde{F}$ and input transformation $\tilde{G}$ is that

$$K = [\tilde{C}(\lambda)[\lambda^{v_m - v}]]_r^h \tag{1.22}$$

should be non-singular, and then on the ith s.i.s.o. subsystem of the decoupled system, up to $f_i + d_i + 1$ poles may be freely assigned, where

$$f_i = \min_{j \in \mathbf{m}} \{v_{m+1-j} - 1 - \deg \tilde{C}_{ij}(\lambda)\}, \quad i \in \mathbf{m}$$

($\tilde{C}_{ij}(\lambda)$ is the ijth element of $\tilde{C}(\lambda)$) and $d_i = \deg d_i(\lambda)$, $i \in \mathbf{m}$. Up to d_i fixed zeros of the ith diagonal element of the transfer function of the decoupled system may be present, these being the zeros of the g.c.d. of the ith row elements of $\tilde{C}(\lambda)$, i.e. the zeros of $d_i(\lambda)$. The decoupling pair $(\tilde{F}^*, \tilde{G}^*)$ is then given by $\tilde{G}^* = K^{-1} = H$ and $\tilde{F}^*$ is then calculated via the implicit equation

$$[\lambda^v] - (\tilde{A}_m + \tilde{F})S(\lambda) = K^{-1} \operatorname{diag}[\delta_i(\lambda)]\tilde{C}_R(\lambda) \tag{1.23}$$

once the poles of the ith s.i.s.o. subsystem have been specified as the zeros of the $\delta_i(\lambda)$, $i \in \mathbf{m}$. The relation between the given expression for K and the classical B^* matrix is simply

$$K = B^* \hat{B}_m^{-1} \tag{1.24}$$

(the matrix $\hat{B}_m$ is defined in Wolovich (1974) and Stoyle and Vardulakis (1979)).

Remark 4

Our decoupling condition in terms of $\tilde{C}(\lambda)$ has appeared in the more general context of the block restricted decoupling problem (cf. the work of Koussiouris (1977)).

Proof (Sufficiency)

From (1.13) there exists a set of m compatible decoupling c.s.s iff $H^{-1}\delta_F(\lambda)B(\lambda)$ is diagonal matrix. We note that $H^{-1}\delta_F(\lambda)$ is thus a left diagonalizer of $B(\lambda)$. Let $\Gamma(\lambda)$ be any left diagonalizer of $B(\lambda)$, then $\Gamma(\lambda)B(\lambda) =$ diagonal, implies

$$\gamma_i(\lambda)B_i(\lambda) = 0, \quad i \in \mathbf{m} \tag{1.25}$$

where $\gamma_i(\lambda)$ is the ith row of $\Gamma(\lambda)$ and $B_i(\lambda)$ is the matrix obtained from $B(\lambda)$ after the ith column has been omitted. Referring to (1.20) and noting by the definition of $\tilde{C}_R(\lambda)$ that

$$\underset{j\in\mathbf{m}}{\text{g.c.d.}}\ \{\tilde{C}_{Rij}(\lambda)\}=1, \quad i\in\mathbf{m}$$

(where $\tilde{C}_{Rij}(\lambda)$ is the ijth element of $\tilde{C}_R(\lambda)$) we invoke Theorem 8 (ii) in Stoyle and Vardulakis (1979) whose proof shows that the minimal degree polynomial vector solution of (1.25) is $\gamma_i^{\min}(\lambda)=\tilde{c}_{Ri}(\lambda)$ (= the ith row of $\tilde{C}_R(\lambda)$). This is saying that $\tilde{C}_R(\lambda)$ is the minimal order left diagonalizer of $B(\lambda)$. A general solution of (1.25) is then

$$\gamma_i(\lambda)=\delta_i(\lambda)\tilde{c}_{Ri}(\lambda)$$

So $H^{-1}\delta_F(\lambda)$ must factorize as

$$H^{-1}\delta_F(\lambda)=\Delta(\lambda)\tilde{C}_R(\lambda) \tag{1.26}$$

where $\Delta(\lambda)=\text{diag}\,[\delta_i(\lambda)]$, $i\in\mathbf{m}$, and the $\delta_i(\lambda)$ are monic polynomials of degree $\geqslant 1$, to be chosen below. We will see that eqn. (1.26) is always solvable for $\tilde{F}$ as the highest coefficient matrices agree on either side and deg {ith column of $\tilde{C}_R(\lambda)\}=k_i<v_{m+1-i}$, $i\in\mathbf{m}$.

It remains to select the $\delta_i(\lambda)$, $i\in\mathbf{m}$, specifying their degrees, and to calculate the $\tilde{F}^*$ which simultaneously attains all the decoupling c.s.s which ' germinate ' (Stoyle and Vardulakis 1979) in the linearly independent columns of the input transformation $\tilde{G}^*$ and which place the poles of the ith subsystem to be the zeros of $\delta_i(\lambda)$. Defining after Gilbert (1969) the decoupling invariants (Alonso-Concheiro 1973) f_i in a continuous time set-up :

$$\begin{aligned} f_i &= \min\{j : c_iA^jB\neq 0, \quad j=0,1,\ldots,n-1\} \\ &= \min\{j : \lim_{\lambda\to\infty} \lambda^{j+1}T_{F,G,i}(\lambda)\neq 0, \quad j=0,1,\ldots,n-1\}, \quad i\in\mathbf{m} \end{aligned}$$

where $T_{F,G,i}(\lambda)$ is the ith row of $T_{F,G}(\lambda)=C(\lambda I-A-BF)^{-1}BG$ and f_i is an (F,G) invariant, we have that

$$\begin{aligned} c_iA^{f_i}B\hat{B}_m^{-1} &= \lim_{\lambda\to\infty}\{\lambda^{f_i+1}\tilde{c}_i(\lambda)\delta_0^{-1}(\lambda)\} \\ &= \lim_{\lambda\to\infty}\{\lambda^{f_i+1}\tilde{c}_i(\lambda)[\lambda^v]^{-1}[I_m-\tilde{A}_mS(\lambda)[\lambda^v]^{-1}]^{-1}\} \\ &= \lim_{\lambda\to\infty}\{\lambda^{f_i+1-n}\tilde{c}_i(\lambda)[\lambda^{n-v}][I_m+O(\lambda^{-1})]\}\dagger \end{aligned} \tag{1.27}$$

(This last being a matrix power series expansion valid for $\lambda>\lambda_0$ for some suitably large positive λ_0, e.g. $\lambda_0=\max|\lambda_i|$, $\lambda_i\in$ spectrum of A.) So

$$c_iA^{f_i}B\hat{B}_m^{-1}=\lim_{\lambda\to\infty}\{\tilde{c}_i(\lambda)[\lambda^{f_i+1-v}]]^{\text{h}}[I_m+O(\lambda^{-1})]\} \tag{1.28}$$

From this expression f_i, $i\in\mathbf{m}$, is seen to be

$$f_i=\min_{j\in\mathbf{m}}\{v_{m+1-j}-1-\deg\tilde{C}_{ij}(\lambda)\} \tag{1.29}$$

† Where $O(\cdot)$ means ' order of '.

the ' degree deficiency ' of the ith row of $\tilde{C}(\lambda)$, which may conveniently be read off directly from $\tilde{C}(\lambda)$. With this alternative definition of f_i, we can now get the result that

$$B^* \triangleq \begin{bmatrix} c_1 A^{f_1} B \\ c_2 A^{f_2} B \\ \vdots \\ c_m A^{f_m} B \end{bmatrix} = [\tilde{C}(\lambda)[\lambda^{v_m - v}]]_r{}^h \hat{B}_m = K\hat{B}_m \tag{1.30}$$

That B^* should be non-singular is Morgan's classical condition for the solvability of the non-block RDP (Gilbert 1969).

We will now write down the implicit equation for the decoupling feedback $\tilde{F}^*$ and thus show that our assumption of K invertible is sufficient for the solvability of our problem. First we note that from the formula

$$K = [\tilde{C}(\lambda)[\lambda^{v_m - v}]]_r{}^h$$

it is only a short step to verify

$$K = [[\lambda^f]\tilde{C}(\lambda)[\lambda^{f_m - v}]]^h, \quad [\lambda^f] = \text{diag}\,[\lambda]^{f_i}, \quad i \in \mathbf{m}$$

and $[\;]^h$ stands for ' highest coefficient matrix '. Thus

$$K(\lambda) \triangleq [\lambda^f]\tilde{C}(\lambda)[\lambda^{v_m - v}] = K\lambda^\xi + K_1\lambda^{\xi - 1} + \dots$$

is a regular matrix (Gantmacher 1959) as $\det K \neq 0$. Now if $K(\lambda)$ is singular over $\mathfrak{R}(\lambda)$, there exists $K'(\lambda) \neq 0$ such that $L(\lambda) = K(\lambda)K'(\lambda) = 0$. By the additivity of the degrees property of a regular matrix (Gantmacher 1959) $\deg L(\lambda) = \deg K(\lambda) + \deg K'(\lambda) \geqslant 0$, but $\deg L(\lambda) = -\infty$ which is a contradiction. Thus K invertible over $\mathfrak{R}$ implies that $K(\lambda)$ is invertible over $\mathfrak{R}(\lambda)$, which is equivalent to our assumption made earlier that $\tilde{C}(\lambda)$ is invertible over $\mathfrak{R}(\lambda)$, so that we may now drop this latter condition.

From (1.28) and (1.29) by minor modifications

$$\lim_{\lambda \to \infty} \lambda^{f_i + d_i + 1} \tilde{c}_{Ri}(\lambda) \delta_F^{-1}(\lambda) = H^{-1} = K \tag{1.31}$$

recalling from (1.4) that $D_C(\lambda) = \text{diag}\,(d_i(\lambda))$, $i \in \mathbf{m}$, and we are letting $\deg d_i(\lambda) \triangleq d_i$. Hence we may equate coefficients on either side of (1.26), which is written in full as (1.23) for the corresponding columns if the ith diagonal element $\delta_i(\lambda)$ of the diagonal matrix is freely chosen to be a monic polynomial of degree $f_i + d_i + 1$, for example by allocating the poles of the ith subsystem to have the arbitrary symmetric spectrum $\Delta_i = \{\delta_i{}^1, \delta_i{}^2, \dots, \delta_i{}^{f_i + d_i + 1}\}$.

Let $\tilde{F}_0$ be the feedback so determined. Then the input transformed transfer function incorporating feedback $\tilde{F}_0$ and additional input transformation H is

$$\begin{aligned} T_{F_0, H}(\lambda) &= \tilde{C}(\lambda)\delta_{F_0, H}{}^{-1}(\lambda) = \tilde{C}(\lambda)[H^{-1}\delta_{F_0}(\lambda)]^{-1} \\ &= D_C(\lambda)\tilde{C}_R(\lambda)B(\lambda)[\Delta(\lambda)\tilde{C}_R(\lambda)B(\lambda)]^{-1} \end{aligned} \tag{1.32}$$

where $\delta_{F_0, H}(\lambda) = H^{-1}\delta_{F_0}(\lambda)$, $\Delta(\lambda) = \text{diag}\,(\delta_i(\lambda))$. But $B(\lambda) = \text{adj}\,\tilde{C}_R(\lambda)M^{-1}(\lambda)$ so putting the decoupling feedback $\tilde{F}^* = \tilde{F}_0$ and the input transformation

$\tilde{G}^* = H = K^{-1}$ we verify that the system is decoupled by this pair from the equation

$$T_{F^*,\tilde{G}^*}(\lambda) = D_C(\lambda) \det \tilde{C}_R(\lambda) M^{-1}(\lambda)[\Delta(\lambda) \det \tilde{C}_R(\lambda) M^{-1}(\lambda)]^{-1}$$
$$= D_C(\lambda)\Delta^{-1}(\lambda) \tag{1.33}$$

This establishes sufficiency.

Proof (Necessity)

Necessity follows if we note from eqn. (1.33) that the system can be decoupled by $(\tilde{F}^*, \tilde{G}^*)$

$$T_{F^*,\tilde{G}^*}(\lambda) = \tilde{C}(\lambda)\delta_{F^*}{}^{-1}(\lambda)\, \tilde{G}^* = D_C(\lambda)\Delta^{-1}(\lambda)$$

$D_C(\lambda)$, $\Delta(\lambda)$ are diagonal matrices of the zeros and the poles of the decoupled system. If $\tilde{G}^*$ is singular then at least one of the diagonal entries of $T_{F^*,\tilde{G}^*}(\lambda)$ will as a consequence be zero which contradicts the output controllability of each separate subsystem. This completes the proof.

Remark 5

From the c.s. point of view, $\tilde{G}^*$ or equivalently H, non-singular is an obvious condition in order to have m essentially distinct cyclic decoupling c.s.s, for if $h_i = \sum_{j<i} a_j h_j$ say, with not all $a_j = 0$ and the system is controllable, then the ith c.s. $\mathscr{R}_i$ is already determined on its generator by preceding ones $\mathscr{R}_j$, $j \in \mathbf{i-1}$, and hence cannot control an independent subsystem of its own, i.e. GDC (II) fails, contradicting the decouplability assumption.

Remark 6

Equation (1.32) shows that the mechanism of decoupling is that certain poles created on the ith subsystem governed by the ith decoupling c.s. $\mathscr{R}_i$ are brought over a certain subset of the system zeros (which are invariant under state feedback) namely the zeros of $(\det \tilde{C}_R(\lambda) m_i{}^{-1}(\lambda))$ so that cancellation occurs. The remaining degrees of freedom for pole placement in $\mathscr{R}_i$ assign the poles of the ith s.i.s.o. subsystem in the overall decoupled system. Further comments will be added in where appropriate in future papers to fill out this picture.

Remark 7

We mention that if we convert back to the original input from our transformed input equations, we should merely replace $\tilde{G}^*$ as calculated in the input transformed coordinates by $\hat{B}_m{}^{-1}\tilde{G}^*$. The decoupling pair in the original coordinates (A, B, C) are given by

$$F^* = \hat{B}_m{}^{-1}\tilde{F}^* T, \quad G^* = \hat{B}_m{}^{-1}\tilde{G}^*$$

Summarizing for clarity all the transformations made we have

$(A, B, C) \leftarrow$input transformation$\rightarrow (A, B\hat{B}_m{}^{-1}, C) \leftarrow$system similarity$\rightarrow (\hat{A}, \hat{B}\hat{B}_m{}^{-1}, \hat{C}) = (\tilde{A}, \tilde{B}, \tilde{C})$

(Where original coordinates$\xrightarrow{T}$Luenberger coordinates.)

Remark 8

We may now tie in the Wonham and Morse (1970) geometrical condition for compatibility of the maximal decoupling c.s. and hence for the solvability of the RDP. Under the standard assumption made here without loss of generality that rank $B=m$, the condition is stated as

$$\mathscr{B}=\sum_{i\in\mathbf{m}} \mathscr{B}\cap\mathscr{R}_i^{\max}$$

but we have seen that

$$\begin{aligned}\operatorname{Im}\tilde{B}=\mathscr{B}&=\{\tilde{B}h_1\}\oplus\{\tilde{B}h_2\}\oplus\ldots\oplus\{\tilde{B}h_m\}\\ &=\sum_{i\in\mathbf{m}}\mathscr{B}\cap\mathscr{R}_i^{\max}\end{aligned}$$

iff $H=\tilde{G}^*$, or equivalently K, is non-singular.

2. Compatibility of controllability subspaces with pole placement

Suppose we are given a general set of c.s. $\mathscr{R}_i$ of (A, B), $i\in\mathbf{m}$. We may ask how we may test whether they are compatible, and if so, whether each one of them governs an independent subsystem of its own, i.e. we enquire whether there exist certain system poles which can be placed through the ith input alone. Obviously this cannot be accomplished if one of the c.s.s is contained in the sum of the others. We formalize these two important properties in the following definition.

Definition 1

A set of m c.s.s $\mathscr{R}_j$, $j\in\mathbf{m}$, of (A, B) is called *cyclically independent* iff

(i) $\mathscr{B}=\sum_{j\in\mathbf{m}}\mathscr{B}\cap\mathscr{R}_j$ (where $\sum$ stands for direct sum of subspaces).

(ii) They are compatible.

Next, we state the following general theorem which answers the queries raised above concerning cyclic independence.

Theorem 2

Given a set of m c.s.s $\mathscr{R}_i$ of $(\tilde{A}, \tilde{B})$ with cyclic listings $R_i\sim\beta_i(\lambda)$, $i\in\mathbf{m}$, then the set is cyclically independent iff they qualify as decoupling c.s.s for some diagonal decoupling problem, i.e. there exists a $m\times n$ matrix $\tilde{C}$ such that the system $(\tilde{A}, \tilde{B}, \tilde{C})$ is decouplable and $\mathscr{R}_i$, $i\in\mathbf{m}$, are the decoupling c.s.s. $\tilde{C}$ is not necessarily unique. To test the cyclic independence of the R_i, stated as a polynomial matrix criterion in terms of the $\beta_i(\lambda)$, as usual we form $B(\lambda)=[\beta_1(\lambda), \ldots, \beta_m(\lambda)]$ and then we remove all polynomial factors of each row of adj $B(\lambda)$, and call the result $\tilde{N}(\lambda)$. Then the necessary and sufficient condition for cyclic independence is

(i) $[\tilde{N}(\lambda)[\lambda^{v_m-v}]]_{\mathrm{r}}^{\mathrm{h}}=K$ must be non-singular, or equivalently

(i)′ $[[\lambda^v]B(\lambda)]_{\mathrm{c}}^{\mathrm{h}}=H$ must be non-singular, and

(ii) $v_{m+1-i}-k_i\triangleq\eta_i>0$, $i\in\mathbf{m}$

where $k_i=\deg\{i\text{th column of }\tilde{N}(\lambda)\}$. The compatible feedback $\tilde{F}$ is then the one solving

$$K\delta_F(\lambda)=\Delta(\lambda)\tilde{N}(\lambda)$$

where $\Delta(\lambda)$ is chosen as in Theorem 1.

Proof

($R_i\sim\beta_i(\lambda)$, $i\in\mathbf{m}$, are cyclic listings of the decoupling c.s. $\mathscr{R}_i$ for the decouplable triple $(\tilde{A}, \tilde{B}, \tilde{C})=\tilde{S}\Rightarrow$polynomial conditions (i) or (i)′ and (ii).) We have seen in the course of the proof of the main Theorem 1 that when $\tilde{S}$ is decouplable, its decoupling c.s.s give rise to a matrix $B(\lambda)$ which in turn gives rise to an $\tilde{N}(\lambda)=\tilde{C}_R(\lambda)$ in our notation there, viz. eqn. (1.20) and this certainly satisfies conditions (i) and (ii). ($\tilde{S}$ decouplable $\Rightarrow$ its decoupling c.s.s $\mathscr{R}_i$ are cyclically independent.) Again this is trivial from the proof of Theorem 1 and from Remark 8 following it.

((i), (ii) $\Rightarrow$ there exists $\tilde{C}$ such that $\tilde{S}$ is decouplable and $R_i \sim \beta_i(\lambda)$ are cyclic listings of its decoupling c.s. $\mathscr{R}_i$.) By the manner of obtaining $\tilde{N}(\lambda)$ from $\tilde{B}(\lambda)$ we can say that $\tilde{N}(\lambda)$ is the minimal order left diagonalizer for $B(\lambda)$ and by reasoning similar to that at the beginning of the Proof of Theorem 1, we see that the overall system will split into subsystems with poles placed on each of them iff $K\delta_F(\lambda)$ splits as

$$K\delta_F(\lambda)=\Delta(\lambda)\tilde{N}(\lambda) \tag{2.1}$$

where $\Delta(\lambda)=\text{diag}\,[\delta_i(\lambda)], i\in\mathbf{m}$.

As already seen in the analysis following eqn. (1.26) above, this type of implicit equation for $\tilde{F}$ is uniquely solvable once the $\delta_i(\lambda)$ are assigned with degrees f_i+1, where $f_i=\min\limits_{j\in\mathbf{m}}\{v_{m+1-j}-1-\deg\tilde{n}_{ij}(\lambda)\}$, $i\in\mathbf{m}$, ($\tilde{n}_{ij}(\lambda)$ is the ijth element of $\tilde{N}(\lambda)$) provided we have $f_i\geqslant 0$, $i\in\mathbf{m}$. But the assumption (ii) guarantees this last condition. We are at liberty in this situation to take arbitrary numerator polynomials $\epsilon_i(\lambda)$ on the ith subsystem of degree up to f_i by defining, for example,

$$\tilde{N}'(\lambda)=\text{diag}\,[\epsilon_i(\lambda)]\tilde{N}(\lambda), \quad \deg\epsilon_i(\lambda)\leqslant f_i, \quad i\in\mathbf{m}$$

Then eqn. (2.1) above still remains solvable for some $\tilde{F}$ with $\tilde{N}'(\lambda)$ in place of $\tilde{N}(\lambda)$ and $\delta_i(\lambda)$ arbitrary but now of degree $f_i-\deg\epsilon_i(\lambda)+1$.

Finally define uniquely an output matrix $\tilde{N}$ in Luenberger coordinates via : $\tilde{N}(\lambda)=\tilde{N}S(\lambda)$ and then we can assert that $\beta_i(\lambda)$ is the decoupling vector of the ith decoupling c.s.s of the system $\tilde{S}=(\tilde{A},\tilde{B},\tilde{N})$ which is decouplable because of assumption (i). Thus $\beta_i(\lambda)$ is the minimal solution of

$$\tilde{N}_i(\lambda)\beta_i(\lambda)=0, \quad i\in\mathbf{m}$$

(where $\tilde{N}_i(\lambda)$ is the matrix obtained from $\tilde{N}(\lambda)$ after the ith row of it has been omitted), or equally of

$$\tilde{N}'_i(\lambda)\beta_i(\lambda)=0, \quad i\in\mathbf{m}$$

where $\tilde{N}'$ is similarly defined to $\tilde{N}$, so we could take our chosen output matrix $\tilde{C}$ to be $\tilde{N}'$, say.

($R_i\sim\beta_i(\lambda)$, $i\in\mathbf{m}$, cyclically independent $\Rightarrow$ polynomial conditions (i) and (ii).) R_i, $i\in\mathbf{m}$, cyclically independent implies from the definition that

$$\mathscr{B}=\sum_{i\in\mathbf{m}}\mathscr{B}\cap\mathscr{R}_i \tag{2.2}$$

where the sum is direct. Thus $\mathscr{B}\cap\mathscr{R}_i=\{h_i^e\}$: the subspace spanned by the generator of $\mathscr{R}_i$ in $\mathscr{B}$ which is unique up to a scalar multiplier in $\mathfrak{R}$, and the germ of R_i is denoted by h_i.

Condition (2.2) further shows that $\dim\mathscr{R}_i\leqslant n-m+1$. Form a matrix Γ^i whose $(m-1+l)$ rows, $l\geqslant 0$, span $\mathscr{R}_i^{\perp}$. If $\Gamma^i(\lambda)=\Gamma^iS(\lambda)$ then we note that rank $\Gamma^i(\lambda)=m-1$, since $\Gamma^i(\lambda)\beta(\lambda)=0$ has the unique solution $\beta_i(\lambda)$ of minimal degree. Let us strike out l linearly dependent (over $\mathfrak{R}(\lambda)$) rows of $\Gamma^i(\lambda)$

thereby obtaining some $(m-1)\times m$ matrix $\tilde{C}^i(\lambda)$ of rank $(m-1)$ and such that $\beta_i(\lambda)$ is the minimal degree solution of

$$\tilde{C}^i(\lambda)\beta_i(\lambda)=0$$

So for $i\in\mathbf{m}$, $\mathscr{R}_i\subseteq\operatorname{Ker}\tilde{C}^i$, where $\tilde{C}^i$ is the $(m-1)\times n$ matrix defined by $\tilde{C}^i S(\lambda)=\tilde{C}^i(\lambda)$.

Can we then conclude that there exists an $m\times n$ matrix $\tilde{N}$ such that $\mathscr{R}_i\subseteq\operatorname{Ker}\tilde{N}_i$? When $\mathscr{R}_i$ are general vector spaces with no special structure, such a result is not generally valid. However, here $R_i\sim\beta_i(\lambda)$ and $\tilde{C}^i(\lambda)\beta_i(\lambda)=0$, $i\in\mathbf{m}$, and in essence we exploit the module-theoretic duality existing between the decoupling vectors $\beta_i(\lambda)$ and the polynomial module ' hyperplanes ' $\tilde{C}^i(\lambda)$.

Let us form $B(\lambda)=[\beta_1(\lambda),\ldots,\beta_m(\lambda)]$ and reduce adj $B(\lambda)$ to $\tilde{N}(\lambda)$ as in the statement of the theorem. Then certainly

$$\tilde{N}_i(\lambda)\beta_i(\lambda)=0$$

and as before

$$[\tilde{N}(\lambda)[\lambda^{v_m-v}]]_{\mathrm{r}}^{\mathrm{h}}=kH^{-1}$$

for some normalizing constant k. It remains to show that $\tilde{N}(\lambda)$ qualifies as numerator matrix, i.e. we must have

$$\deg\{j\text{th column of }\tilde{N}(\lambda)\}<v_{m+1-j},\quad j\in\mathbf{m}$$

Our assumption of compatibility implies that $H^{-1}\delta_F(\lambda)$ is a left diagonalizer for $B(\lambda)$, but we have arranged that $\tilde{N}(\lambda)$ is the minimal order left diagonalizer for $B(\lambda)$, so $H^{-1}\delta_F(\lambda)$ must have some expression as

$$H^{-1}\delta_F(\lambda)=\Delta(\lambda)\tilde{N}(\lambda)\tag{2.3}$$

where $\Delta(\lambda)=\operatorname{diag}[\delta_i(\lambda)]$ and $\delta_i(\lambda)$ are polynomials (possibly constants). From (2.2) there exists a vector $\tilde{n}'_i$ say such that $\tilde{n}'_i\perp\mathscr{R}_j$, $j\neq i$, hence $\tilde{n}'_i(\lambda)\beta_j(\lambda)=0$, $i\neq j$ or $N'_i(\lambda)\beta_j(\lambda)=0$. Removing row factors of numerator matrix $N'(\lambda)$ clearly yields $N(\lambda)$, hence $\delta_i(\lambda)$ cannot be constant and condition (ii) is satisfied.

Example

Consider the controllable and observable system in Luenberger controllable canonical form (for notation see also Stoyle and Vardulakis (1979))

$$\hat{A}=\tilde{A}=\left[\begin{array}{ccc|ccc|cc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ -1 & 0 & 0 & -4 & 0 & 1 & 0 & 0\\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],\quad \hat{B}=\left[\begin{array}{ccc} 0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 0 & 3\\ \hline 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 1 & -2\\ \hline 0 & 0 & 0\\ 0 & 0 & 1 \end{array}\right]$$

$$\hat{C}=\tilde{C}=\begin{bmatrix} 3 & 1 & 0 & 0 & 0 & 0 & 1 & 1\\ -2 & -2 & 0 & 1 & 2 & 1 & 0 & 0\\ -3 & -4 & -1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

So that $v_3=3$, $v_2=3$, $v_1=2$, $n=8$, $m=3$ and

$$\hat{B}_m=\begin{bmatrix}1 & 0 & 3\\ 0 & 1 & -2\\ 0 & 0 & 1\end{bmatrix},\quad \hat{B}_m{}^{-1}=\begin{bmatrix}1 & 0 & -3\\ 0 & 1 & 2\\ 0 & 0 & 1\end{bmatrix}$$

$$\tilde{B}=\begin{bmatrix}0&0&0\\0&0&0\\1&0&0\\0&0&0\\0&0&0\\0&1&0\\0&0&0\\0&0&1\end{bmatrix},\quad S(\lambda)=\begin{bmatrix}1&0&0\\\lambda&0&0\\\lambda^2&0&0\\0&1&0\\0&\lambda&0\\0&\lambda^2&0\\0&0&1\\0&0&\lambda\end{bmatrix}$$

$$\tilde{C}(\lambda)=\tilde{C}S(\lambda)=\begin{bmatrix}\lambda+3 & 0 & \lambda+1\\ -2(\lambda+1) & (\lambda+1)^2 & 0\\ -(\lambda+1)(\lambda+3) & 0 & \lambda+1\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0 & 0\\ 0 & \lambda+1 & 0\\ 0 & 0 & \lambda+1\end{bmatrix}\begin{bmatrix}\lambda+3 & 0 & \lambda+1\\ -2 & \lambda+1 & 0\\ -(\lambda+3) & 0 & 1\end{bmatrix}$$

$$=D_C(\lambda)\tilde{C}_R(\lambda)$$

From $\det\tilde{C}(\lambda)=(\lambda+1)^3(\lambda+3)(\lambda+2)$, the zeros of the system are -1, -1, -1, -3, -2. From $D_C(\lambda)$, $d_1(\lambda)=1$, $d_2(\lambda)=(\lambda+1)$, $d_3(\lambda)=(\lambda+1)$, so that $d_1=0$, $d_2=1$, $d_3=1$.

$$\operatorname{adj}\tilde{C}_R(\lambda)=\begin{bmatrix}\lambda+1 & 0 & -(\lambda+1)^2\\ 2 & (\lambda+3)(\lambda+2) & -2(\lambda+1)\\ (\lambda+1)(\lambda+3) & 0 & (\lambda+3)(\lambda+1)\end{bmatrix}$$

$$=\begin{bmatrix}\lambda+1 & 0 & -(\lambda+1)\\ 2 & 1 & -2\\ (\lambda+1)(\lambda+3) & 0 & \lambda+3\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & (\lambda+3)(\lambda+2) & 0\\ 0 & 0 & \lambda+1\end{bmatrix}$$

$$=B(\lambda)M(\lambda)$$

Hence

$$B(\lambda)=[\beta_1(\lambda),\ \beta_2(\lambda),\ \beta_3(\lambda)]=\begin{bmatrix}\lambda+1 & 0 & -(\lambda+1)\\ 2 & 1 & -2\\ (\lambda+1)(\lambda+3) & 0 & \lambda+3\end{bmatrix}$$

$$K=[\tilde{C}(\lambda)[\lambda^{v_m-v}]]_r{}^{h}=\left[\begin{bmatrix}\lambda+3 & 0 & \lambda+1\\ -2(\lambda+1) & (\lambda+1)^2 & 0\\ -(\lambda+1)(\lambda+3) & 0 & \lambda+1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&\lambda\end{bmatrix}\right]_r^{h}$$

$$=\begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ -1 & 0 & 1\end{bmatrix},\quad \det K=1$$

hence the system can be decoupled by state feedback. So take

$$\tilde{G}^* = K^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$f_1 = \min\{3\text{–}1\text{–}1,\ 3\text{–}1\text{–}0,\ 2\text{–}1\text{–}1\} = \min\{1, 2, 0\} = 0$$

$$f_2 = \min\{3\text{–}1\text{–}1,\ 3\text{–}1\text{–}2,\ 2\text{–}1\text{–}0\} = \min\{1, 0, 1\} = 0$$

$$f_3 = \min\{3\text{–}1\text{–}2,\ 3\text{–}1\text{–}0,\ 2\text{–}1\text{–}1\} = \min\{0, 2, 0\} = 0$$

$$\delta_1 = f_1 + d_1 + 1 = 1, \quad \delta_2 = f_2 + d_2 + 1 = 2, \quad \delta_3 = f_3 + d_3 + 1 = 2$$

Let all the closed loop poles be equal to -2, i.e. let us take :

$$\delta_1(\lambda) = (\lambda+2), \quad \delta_2(\lambda) = (\lambda+2)^2, \quad \delta_3(\lambda) = (\lambda+2)^2$$

(so that no further cancellation occurs in the transfer function of the decoupled system). From $\tilde{A}$ by inspection we have

$$\tilde{A}_m = \begin{bmatrix} -1 & 0 & 0 & -4 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So that

$$[\lambda^v] - \tilde{A}_m S(\lambda) = \begin{bmatrix} \lambda^3+1 & 4-\lambda^2 & 0 \\ 0 & \lambda^3-1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$K^{-1}\Delta(\lambda)\tilde{C}_R(\lambda) = \begin{bmatrix} (\lambda+2)(\lambda+3)^2 & 0 & -(\lambda+2) \\ -2(\lambda+2)^2 & (\lambda+2)^2(\lambda+1) & 0 \\ (\lambda+2)(\lambda+3) & 0 & (\lambda+2)(\lambda+1) \end{bmatrix}$$

From eqn. (1.23)

$$[\lambda^v] - \tilde{A}_m S(\lambda) - K^{-1}\Delta(\lambda)\tilde{C}_R(\lambda) =$$

$$\begin{bmatrix} -17-21\lambda-8\lambda^2 & 4-\lambda^2 & \lambda+2 \\ 8+8\lambda+2\lambda^2 & -5-8\lambda-5\lambda^2 & 0 \\ -6-5\lambda-\lambda^2 & 0 & -2-3\lambda \end{bmatrix}$$

$$= \tilde{F}^* S(\lambda)$$

And by inspection :

$$\tilde{F}^* = \begin{bmatrix} -17 & -21 & -8 & 4 & 0 & -1 & 2 & 1 \\ 8 & 8 & 2 & -5 & -8 & -5 & 0 & 0 \\ -6 & -5 & -1 & 0 & 0 & 0 & -2 & -3 \end{bmatrix}$$

Although as we saw the determination of the decoupling c.s. $\mathscr{R}_j^{\max}$, $j = 1, 2, 3$ is not necessary for the solution of the RDP (see Remark 3) we can very easily obtain the c.s. $\mathscr{R}_j^{\max}$ by considering the minimal degree generating functions $\tilde{x}_j(\lambda)$, $j=1, 2, 3$ which correspond to the 'decoupling vectors' $\beta_j(\lambda)$, $j=1, 2, 3$ that are the columns of the (diagonal) decoupling matrix $B(\lambda)$. So from $\tilde{x}_j(\lambda)=S(\lambda)\beta_j(\lambda)$, $j=1, 2, 3$ we obtain

$$\tilde{x}_1(\lambda)=\begin{bmatrix}\lambda+1\\ \lambda^2+\lambda\\ \lambda^3+\lambda^2\\ 2\\ 2\lambda\\ 2\lambda^2\\ \lambda^2+4\lambda+3\\ \lambda^3+4\lambda^2+3\lambda\end{bmatrix}=\begin{bmatrix}1\\0\\0\\2\\0\\0\\3\\0\end{bmatrix}+\begin{bmatrix}1\\1\\0\\0\\2\\0\\4\\3\end{bmatrix}\lambda+\begin{bmatrix}0\\1\\1\\0\\0\\2\\1\\4\end{bmatrix}\lambda^2+\begin{bmatrix}0\\0\\1\\0\\0\\0\\0\\1\end{bmatrix}\lambda^3$$

$$\mathscr{R}_1^{\max}=\text{span}\left[\begin{bmatrix}1\\0\\0\\2\\0\\0\\3\\0\end{bmatrix}\begin{bmatrix}1\\1\\0\\0\\2\\0\\4\\3\end{bmatrix}\begin{bmatrix}0\\1\\1\\0\\0\\2\\1\\4\end{bmatrix}\begin{bmatrix}0\\0\\1\\0\\0\\0\\0\\1\end{bmatrix}\right]$$

$$\tilde{x}_2(\lambda)=\begin{bmatrix}0\\0\\0\\1\\ \lambda\\ \lambda^2\\0\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\\1\\0\\0\\0\\0\end{bmatrix}+\begin{bmatrix}0\\0\\0\\0\\1\\0\\0\\0\end{bmatrix}\lambda+\begin{bmatrix}0\\0\\0\\0\\0\\1\\0\\0\end{bmatrix}\lambda^2,\quad \mathscr{R}_2^{\max}=\text{span}\left[\begin{bmatrix}0\\0\\0\\1\\0\\0\\0\\0\end{bmatrix}\begin{bmatrix}0\\0\\0\\0\\1\\0\\0\\0\end{bmatrix}\begin{bmatrix}0\\0\\0\\0\\0\\1\\0\\0\end{bmatrix}\right]$$

$$\tilde{x}_3(\lambda)=\begin{bmatrix}-\lambda-1\\ -\lambda^2-\lambda\\ -\lambda^3-\lambda^2\\ -2\\ -2\lambda\\ -2\lambda^2\\ \lambda+3\\ \lambda^2+3\lambda\end{bmatrix}=\begin{bmatrix}-1\\0\\0\\-2\\0\\0\\3\\0\end{bmatrix}+\begin{bmatrix}-1\\-1\\0\\0\\-2\\0\\1\\3\end{bmatrix}\lambda+\begin{bmatrix}0\\-1\\-1\\0\\0\\-2\\0\\1\end{bmatrix}\lambda^2+\begin{bmatrix}0\\0\\-1\\0\\0\\0\\0\\0\end{bmatrix}\lambda^3$$

$$\mathscr{R}_3^{\max}=\text{span}\left[\begin{bmatrix}-1\\0\\0\\-2\\0\\0\\3\\0\end{bmatrix}\begin{bmatrix}-1\\-1\\0\\0\\-2\\0\\1\\3\end{bmatrix}\begin{bmatrix}0\\-1\\-1\\0\\0\\-2\\0\\1\end{bmatrix}\begin{bmatrix}0\\0\\-1\\0\\0\\0\\0\\0\end{bmatrix}\right]$$

Finally the transfer function matrix of the decoupled system will be :

$$T_{F^*\bar{G}^*}(\lambda)=\begin{bmatrix}\dfrac{1}{(\lambda+2)} & 0 & 0\\ 0 & \dfrac{(\lambda+1)}{(\lambda+2)^2} & 0\\ 0 & 0 & \dfrac{(\lambda+1)}{(\lambda+2)^2}\end{bmatrix}$$

3. Conclusion

We have seen that the compatibility and simultaneous pole placement problem for m c.s.s is always equivalent to a suitable (diagonal) RDP. This result will be our starting point, in a forthcoming paper, for developing necessary and sufficient conditions for the solvability of a large class of other decoupling problems and in providing constructive general solutions. Classically, it will be recalled, the essential difficulty in such problems was the lack of systematic method for determining a compatible set of decoupling c.s.s, when the maximal ones would not do. The question of minimal order compensation for cases of weak inherent coupling (Gilbert 1969) (G^* above singular but $\bar{C}(\lambda)$ still non-singular), will be dealt with at the same time. In every case, the basic procedure is that decoupling requirements are naturally and concisely expressed in geometrical terms first and then translated into equally compact polynomial matrix language and solved in this, yielding virtually a plug-in approach to solving problems in non-interacting control. In the case of the RDP above, such an analysis led finally to a solution close in spirit to the invertible decoupling algorithm of Wolovich (1974). His method amounts to spotting how to factorize the canonical controllable matrix fraction decomposition of a transfer function $T_{F,G}(\lambda)$ as a diagonal matrix in which of course the poles and zeros are exposed. However, in other types of decoupling problem, more generally in model-matching, it is not always so easy to see by inspection how the requisite factorizations would go, or where poles and zeros on the subsystems would then lie. This presents no problem when using the method proposed here, as the intervention of a suitable $B(\lambda)$ matrix, embodying just the c.s. information provided by the geometrical statement of our problem, acts as a template for the factorization. Thus the method above gives satisfying answers to a number of the important queries raised by Wonham and Morse (1971) concerning the limitations of the geometric method, in the conclusion of their notable survey of non-interacting control.

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