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On infinite zeros

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On infinite zeros

ANTONIS I. G. VARDULAKIS†

The infinite zero structure of completely controllable and observable linear multivariable systems which give rise to square, non-singular, strictly proper transfer function matrices is investigated via the polynomial matrix approach to the solution of the decoupling problem. In the process various connections between infinite zeros, their degrees and geometric and polynomial matrix ideas are demonstrated. The asymptotic behaviour of closed loop eigenvectors under high gain output feedback is also examined.

1. Introduction

The zeros at infinity (or infinite zeros) of an $r \times m$ proper rational matrix $T(s)$ have been studied by circuit theorists ever since the work of McMillan (1952). Rosenbrock (1970, 1974) originally gave a definition of the infinite zeros of a rational matrix and during the past few years infinite zeros have been studied by several authors (Verghese 1978, Pugh and Ratcliffe 1979). The concept of zeros at infinity appears also in the root locus theory (Kouvaritakis and Shaked 1976, Postlethwaite and MacFarlane 1979) where they are the infinite terminal locations of the root loci as a scalar gain $k \to \infty$ (Kouvaritakis and Edmunds 1979).

It has been pointed out by Owens (1978) that under certain conditions there is a relation between the ' orders ' of infinite zeros of the root locus theory and certain structural invariants defined by Morse (1973). Verghese and Kailath (1979) commented recently on these results and using Verghese's (1978) ' diagonal decomposition ' of a rational matrix $T(s)$, they argued that generically the degrees† η_i of (the root locus theory) infinite zeros as given in Owens (1978) are in fact the degrees of the zeros of $T(s)$ at $s = \infty$ in the Smith–McMillan sense. These results are somewhat obscure mainly for two reasons : Firstly, the terms ' infinite zeros ' and their ' degrees ' in association with the root locus theory are not clearly defined. Secondly it is not very clear through Owens' paper what the ' structural invariants ' η_i represent with respect to a given system.

In a recent paper, Pugh and Ratcliffe (1979) showed that Rosenbrock's (1970, p. 132) original definition of a zero of $T(s)$ at infinity is unsatisfactory because apart from the fact that it does not involve the concept of degree of an infinite zero, it only constitutes a sufficient condition for the existence of a zero of a rational matrix at infinity. By giving a new definition of a finite zero of a rational matrix which also involved the concept of degree, Pugh and Ratcliffe (1979) were able to define infinite zeros and their degrees.

The main objective of this paper is to investigate the infinite zero structure of square, non-singular, strictly proper transfer function matrices in the light of the theory of decoupling.

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† In the sequel the terms ' order ' and ' degree ' are used interchangeably.

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By adopting the Pugh–Ratcliffe definition of infinite zeros and their degrees, and using certain results from the polynomial matrix approach to the solution of the decoupling problem, the number and the degrees of infinite zeros are determined. Specifically it is shown that in the case of a decouplable (see Definition 4, § 3) system Σ the degrees η_i of the infinite zeros are given by $\eta_i = f_i + 1$ where f_i are the so-called decoupling invariants of Σ. This idea is then extended to non-decouplable systems, and in the process various connections between geometric and polynomial matrix ideas are demonstrated.

Our approach is in fact a generalization to the multivariable case of the simple idea that, given a scalar transfer function

$$t(s) = \frac{c(s)}{\delta(s)} = (c_0 + c_1 s + \dots + c_q s^q)/(a_0 + a_1 s + \dots + a_{n-1}s^{n-1} = s^n)$$

where $c(s)$ and $\delta(s)$ are relatively prime and $c_q > 0$ (i.e. with n poles and $q \leqslant n-1$ finite zeros) then if one considers the minimal realization $(\tilde{A}, \tilde{b}, \tilde{c}^{\mathrm{T}})$ of $t(s)$ given by

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 \dots\dots 0 & \\ 0 & 0 & 1 & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \dots\dots & -a_{n-1} \end{bmatrix}, \quad \tilde{b} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$\tilde{c}^{\mathrm{T}} = [\underbrace{c_0, c_1, \dots, c_q}_{\leftarrow q+1 \rightarrow}, \underbrace{0, 0, \dots, 0}_{\leftarrow f \rightarrow}]$$

the integer number f of the zeros appearing after c_q in $\tilde{c}^{\mathrm{T}}$ is such that: $f+1 = n-q$ gives the ' degree ' of the unique zero at infinity of $t(s)$.

We start by considering a linear, time-invariant, finite-dimensional multivariable system Σ described by the set of state space equations

$$\dot{x} = Ax + Bu \tag{1 a}$$

$$y = Cx \tag{1 b}$$

where $A : \mathscr{X} \to \mathscr{X}$, $B : \mathscr{U} \to \mathscr{X}$, $C : \mathscr{X} \to \mathscr{Y}$ are the associated maps with $\dim \mathscr{X} = n$, $\dim \mathscr{U} = \dim \mathscr{Y} = m$ and A, B, C are also the matrix representations of these maps. We assume that $\Sigma = (A, B, C)$ is completely controllable and observable, that B and C have full rank $m \leqslant n$ and that Σ gives rise to a square $(m \times m)$ non-singular (over the field of rational functions $\mathbb{R}(s)$) transfer function matrix: $T(s) = C(sI - A)^{-1}B$.

It is then well known that $T(s)$ can always be factored (in a non-unique way) as a product

$$T(s) = C(s)D(s)^{-1} \tag{2}$$

where $C(s)$ and $D(s)$ are $m \times m$ relatively right prime (r.r.p.) polynomial matrices which have full rank m (over $\mathbb{R}(s)$), and that all irreducible (right) matrix fraction descriptions (MFD's) of $T(s)$ are unimodular (column) equivalent (Wolovich 1973 a), i.e. if $C_1(s)$, $D_1(s)$ is any other irreducible (right) MFD of $T(s)$ then

$$C_1(s) = C(s)Q(s), \quad D_1(s) = D(s)Q(s) \tag{3}$$

for some $(m \times m)$ unimodular polynomial matrix $Q(s)$. Polynomial matrices $C(s)$ and $D(s)$ satisfying (2) are called respectively numerators and denominators of $T(s)$.

Pugh and Ratcliffe (1979) gave the following definitions of finite and infinite zeros of $T(s)$ and also (since Σ is assumed completely controllable and observable) of Σ, which are important enough to be restated here in full.

Definition 1 (Pugh and Ratcliffe)

Given an $m \times m$ polynomial matrix $C(s)$ then $s_0 \in \mathbb{C}$ is a *finite zero of degree k of* $C(s)$ when $(s - s_0)$ is an elementary divisor of $C(s)$. The set of finite zeros of $C(s)$ is the set of all such numbers s_0, a zero of degree k being included k times.

Definition 2 (Pugh and Ratcliffe)

$s_0 \in \mathbb{C}$ is a *finite zero of degree k of* $T(s)$ if it is a zero of degree k of any numerator $C(s)$ of $T(s)$.

Definition 3 (Pugh and Ratcliffe)

$T(s)$ is said to have an *infinite zero of degree k* when $w = 0$ is a finite zero of degree k for the rational matrix $T(1/w)$.

2. Notation and preliminary results

Under the controllability assumption of (A, B) it is known (Luenberger 1967) that there always exists a non-singular coordinate transformation : $\tilde{x} = Tx$ such that $TAT^{-1} = \hat{A}$, $TB = \hat{B}$, $CT^{-1} = \hat{C}$ and the pair $(\hat{A}, \hat{B})$ is in the Luenberger controllable canonical form. If by $v_m \geqslant v_{m-1} \geqslant \dots \geqslant v_1 \geqslant 1$ we denote the controllability indices of $(\hat{A}, \hat{B})$ and $\hat{B}_m$ is the $m \times m$ (non-singular) matrix consisting of the $p_i = \sum_{j=1}^{i} v_{m+1-j}$, $i \in \mathbf{m}$ rows of $\hat{B}$ then $\hat{B} = \tilde{B}\hat{B}_m$ where $\tilde{B} = \text{block diag}\,[\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_m]$, $\tilde{b}_j = (0, 0, \dots, 0, 1)^{\mathrm{T}} \in \mathbb{R}^{v_{m+1-j} \times 1}$, $j \in \mathbf{m}$ and according to the Wolovich and Falb (1969) structure theorem the transfer function matrix $T(s)$ of Σ can be written as

$$T(s) = C(sI - A)^{-1}B = \hat{C}(sI - \hat{A})^{-1}\hat{B} = \tilde{C}(s)\tilde{D}_0(s)^{-1} \tag{4}$$

where

$$\tilde{C}(s) = \tilde{C}S(s), \quad \tilde{C} \equiv \hat{C}, \quad S(s) = \text{block diag}\,[\hat{s}_1(s), \hat{s}_2(s), \dots, \hat{s}_m(s)]$$

$$\hat{s}_j(s) = (1, s, s^2, \dots, s^{v_{m+1-j}-1})^{\mathrm{T}}, \quad j \in \mathbf{m}$$

and

$$\tilde{D}_0(s) = \hat{B}_m^{-1}\,\delta_0(s) \tag{5}$$

where $\delta_0(s) = [s^v] - \tilde{A}_m S(s)$, $[s^v] = \text{diag}\,(s^{v_m}, s^{v_{m-1}}, \dots, s^{v_1})$ and $\tilde{A}_m$ the $m \times n$ matrix consisting of the p_i, $i \in \mathbf{m}$ ordered rows of $\tilde{A} \equiv \hat{A}$.

Consider now eqns. (1) together with a linear state variable feedback control law (l.s.v.f.) given by

$$u = Fx + Gv \tag{6}$$

where $F \in \mathbb{R}^{m \times n}$, $G \in \mathbb{R}^{m \times m}$, det $G \neq 0$ and take

$$F = \hat{B}_m^{-1} \tilde{F} T \tag{7}$$

$$G = \hat{B}_m^{-1} \tilde{G} \tag{8}$$

then it can be easily verified that the transfer function matrix $T_{F,G}(s)$ of the closed loop system $\Sigma_{F,G} = (A + BF, BG, C)$ is given by (Wolovich 1974):

$$T_{F,G}(s) = C(sI - A - BF)^{-1} BG = \tilde{C}(sI - \tilde{A} - \tilde{B}\tilde{F})^{-1} \tilde{B}\tilde{G}$$

$$= \tilde{C}(s) \delta_{\tilde{F}}^{-1}(s) \tilde{G} = T_{\tilde{F},\tilde{G}}(s) \tag{9}$$

where

$$\delta_{\tilde{F}}(s) = [s^v] - (\tilde{A}_m + \tilde{F}) S(s) = \delta_0(s) - \tilde{F} S(s)$$

3. Finite and infinite zero structure of $T(s)$; decouplable systems

Having the irreducible right MFD of $T(s)$ given by (4) then from the numerator matrix $\hat{C}(s) = \tilde{C}S(s)$, we can define a set of integers (Vardulakis 1979) f_i, $i \in \mathbf{m}$ and an $m \times m$ polynomial matrix $K(s)$ as follows:

(i)
$$f_i \triangleq \min_{j \in \mathbf{m}} \{v_{m+1-j} - 1 - \deg \tilde{c}_{ij}(s)\}, \quad i \in \mathbf{m} \tag{10}$$

where $\tilde{c}_{ij}(s)$ is the ijth polynomial element of $\tilde{C}(s)$†

(ii)
$$K(s) = \tilde{C}(s)[s^{v_m - v}] \tag{11}$$

where

$$[s^{v_m - v}] = \operatorname{diag}(1, s^{v_m - v_{m-1}}, \ldots, s^{v_m - v_1}) \tag{12}$$

Let $k_i^{\mathrm{T}}(s)$, $i \in \mathbf{m}$ be the ith row of $K(s)$ and let $g_i = \deg k_i^{\mathrm{T}}(s)$, then it can be proved (Vardulakis 1979) that

$$g_i = v_m - 1 - f_i, \quad i \in \mathbf{m} \tag{13}$$

and if we write: $k_i^{\mathrm{T}}(s) = \sum_{j=0}^{g_i} k_{ij}^{\mathrm{T}} s^j$, $i \in \mathbf{m}$, then the matrix $K(s)$ can be written as

$$K(s) = [s^{v_m - 1 - f_i}] \begin{bmatrix} k^{\mathrm{T}}_{1g_1} \\ k^{\mathrm{T}}_{2g_2} \\ \vdots \\ k^{\mathrm{T}}_{mg_m} \end{bmatrix} + K_R(s) \tag{14}$$

† It turns out that the integers f_i are the so-called 'decoupling invariants' of Σ which can be defined alternatively by

(a)
$$\left.\begin{array}{l} f_i = \min\{k : c_i^{\mathrm{T}} A^k B \neq 0, \quad k = 0, 1, 2, \ldots, n-1\} \\ \text{or } f_i = n-1 \quad \text{if } c_i^{\mathrm{T}} A^k B = 0 \text{ for every } k \end{array}\right\} i \in \mathbf{m}$$

(b)
$$f_i = \min\{k : \lim_{s \to \infty} s^{k+1} T_i(s) \neq 0, \quad k = 0, 1, 2, \ldots, n-1\}$$

where c_i^{T}, $T_i(s)$ represent the ith rows of C and $T(s)$ respectively.

where $[s^{v_m-1-f_i}] = \text{diag}\,(s^{v_m-1-f_1}, s^{v_m-1-f_2}, \ldots, s^{v_m-1-f_m})$ and $K_R(s)$ is an $m \times m$ polynomial matrix whose ith row has degree less than $g_i = v_m - 1 - f_i$, $i \in \mathbf{m}$.

Let

$$K = \begin{bmatrix} k^{\mathrm{T}}{}_{1g^1} \\ \vdots \\ k^{\mathrm{T}}{}_{mg^m} \end{bmatrix} \tag{15}$$

then $K(s)$ is said to be *row proper* (Wolovich 1974) iff rank $K = m$.

The importance of the matrix $K(s)$ and the indices f_i, $i \in \mathbf{m}$ becomes clear from the following two propositions whose proof is given in Vardulakis and Stoyle (1979).

Proposition 1

There exist a l.s.v.f. control law $(\tilde{F}^*, \tilde{G}^*)$ such that $T_{\tilde{F}^*, \tilde{G}^*}(s)$ is diagonal and non-singular iff $K(s)$ is row proper, i.e. iff rank $K = m$.

In view of the above proposition we can make the following

Definition 4

A system $\Sigma = (A, B, C)$ is said to be *decouplable* (by the use of l.s.v.f.) iff $K(s) = \tilde{C}(s)[s^{v_m - v}]$ is row proper.

Proposition 2

If $\Sigma = (A, B, C)$ is decouplable then the number q of the finite zeros of Σ, and also (since Σ is assumed completely controllable and observable) of $T(s)$, is given by

$$q = n - m - \sum_{i=1}^{m} f_i \tag{16}$$

Now it can be formally established (see Proposition 4 below) that if Σ is decouplable and $q > 0$ then among the state feedback matrices $\tilde{F}^*$ which belong to the family of l.s.v.f. control laws $(\tilde{F}^*, \tilde{G}^*)$ that decouple Σ, there is a particular state feedback matrix : (say) $\tilde{F}^*{}_1$ which is such that it also assigns q eigenvalues of $\tilde{A} + \tilde{B}\tilde{F}^*{}_1$ to be equal to the q finite zeros of Σ (in such a way† so that when $T_{\tilde{F}^*{}_1, \tilde{G}^*}(s)$ is formed cancellations occur between coincident eigenvalues of $\tilde{A} + \tilde{B}\tilde{F}^*{}_1$ and the q finite zeros of Σ) while the remaining $n - q$ eigenvalues of $\tilde{A} + \tilde{B}\tilde{F}^*{}_1$ are shifted to zero.

It is evident that a state feedback matrix $\tilde{F}$ that will also cause cancellations of infinite zeros of $T(s)$ must have an unbounded norm because only then the eigenvalues of $\tilde{A} + \tilde{B}\tilde{F}$ become arbitrarily large in magnitude and so to speak may ' cancel ' the infinite zeros. From this argument it is evident that we can state the following.

Proposition 3

The infinite zero structure of Σ (or equivalently of $T(s)$) is invariant under any (finite) l.s.v.f. control law (F, G).

† i.e. so that the corresponding eigenvectors $\tilde{x}_i \subset \ker \tilde{C}$, $i \in \mathbf{q}$.

Proof

It is well known (e.g. see theorem 4.1 in Rosenbrock (1970)) that the finite zeros of Σ (or equivalently of $T(s)$) are given by the zeros of the invariant polynomials of the Smith form of the $(n+m)\times(n+m)$ system matrix (Rosenbrock 1970)

$$P(s)=\begin{bmatrix} sI-A & B \\ -C & 0 \end{bmatrix}$$

From Definitions 1, 2 and 3 it follows now that the infinite zeros of $P(s)$ are the finite zeros of $P(1/w)$ at $w=0$. Now

$$P\left(\frac{1}{w}\right)=\begin{bmatrix} \frac{1}{w}I-A & B \\ -C & 0 \end{bmatrix}$$

and a (left) prime factorization of $P(1/w)$ is given by

$$P\left(\frac{1}{w}\right)=\begin{bmatrix} wI & 0 \\ 0 & I \end{bmatrix}^{-1}\begin{bmatrix} I-Aw & Bw \\ -C & 0 \end{bmatrix}=D(w)^{-1}N(w)$$

So the infinite zeros of $P(s)$ are the finite zeros of

$$N(w)=\begin{bmatrix} I-Aw & Bw \\ -C & 0 \end{bmatrix} \quad \text{at } w=0$$

Let us consider now l.s.v.f. (F, G). Then the system matrix $P_{F,G}(s)$ of $\Sigma_{F,G}$ is

$$P_{F,G}(s)=\begin{bmatrix} sI-A-BF & BG \\ -C & 0 \end{bmatrix}$$

and the infinite zeros of $P_{F,G}(s)$ are the finite zeros of $P_{F,G}(1/w)$ at $w=0$.

A (left) prime factorization of $P_{F,G}(1/w)$ is

$$P_{F,G}\left(\frac{1}{w}\right)=\begin{bmatrix} wI & 0 \\ 0 & I \end{bmatrix}^{-1}\begin{bmatrix} I-(A+BF)w & BGw \\ -C & 0 \end{bmatrix}=D(w)^{-1}N_{F,G}(w)$$

and the infinite zeros of $P_{F,G}(s)$ are the finite zeros of

$$N_{F,G}(w)=\begin{bmatrix} I-(A+BF)w & BGw \\ -C & 0 \end{bmatrix}$$

at $w=0$. But

$$N_{F,G}(w)=N(w)\begin{bmatrix} I & 0 \\ -F & G \end{bmatrix}$$

Hence $N_{F,G}(w)$ and $N(w)$ have the same zero structure and in particular they have the same zero structure at $w=0$. Hence $P(s)$ and $P_{F,G}(s)$ have the same infinite zeros. Now both $P(s)$ and $P_{F,G}(s)$ are what Verghese *et al.* (1979) term 'strongly irreducible'. Hence by Theorem 1 in the above reference the infinite zeros of $T(s)$ are those of $P(s)$ and the infinite zeros of $T_{F,G}(s)$ are those of $P_{F,G}(s)$. □

From Proposition 3 we see that instead of examining the infinite zero structure of $T(s)$ we may we well examine the infinite zero structure of $T_{F^*_1, G^*_1}(s)$ where $(\tilde{F}^*_1, \tilde{G}^*_1)$ has been chosen so that it decouples Σ and (i) all q finite zeros of $T(s)$ are cancelled by eigenvalues of $\tilde{A}+\tilde{B}\tilde{F}^*_1$ and (ii) all the remaining $n-q$ eigenvalues are shifted to zero.

The existence of such a state feedback matrix $\tilde{F}^*_1$ is formally established in the following.

Proposition 4

There always exists a state feedback matrix $\tilde{F}^*_1$ such that

$$K\delta_{F^*_1}(s) = \operatorname{diag}\,(s^{f_1+1}, \ldots, s^{f_m+1})\tilde{C}(s) \tag{17}$$

iff $K(s)$ is row proper, i.e. iff rank $K=m$.

Proof

The proof of a more general form of the above proposition is given in Vardulakis (1979). □

It is also shown by Vardulakis (1979) that the *unique* state feedback matrix $\tilde{F}^*_1$ that satisfies (17) is given by

$$\tilde{F}^*_1 = -K^{-1}M - \tilde{A}_m \tag{18}$$

where M is the $m \times n$ matrix determined uniquely via the equation

$$\operatorname{diag}\,[s^{f_1+1}, \ldots, s^{f_m+1}]K_R(s) = M \begin{bmatrix} 1 & & & \bigcirc \\ s & & & \\ \vdots & & & \\ s^{v_m-1} & & & \\ & s^{v_m-v_{m-1}} & & \\ & \vdots & & \\ & s^{v_m-1} & & \\ & & \ddots & \\ \bigcirc & & & s^{v_m-v_1} \\ & & & \vdots \\ & & & s^{v_m-1} \end{bmatrix} \tag{19}$$

and $K_R(s)$ is defined in (14). We can now state

Theorem 1

If $K(s)$ is row proper (i.e. Σ is decouplable), then with $\tilde{F}=\tilde{F}^*_1$ that satisfies (17) and $\tilde{G}=\tilde{G}^*_1=K^{-1}$ the transfer function matrix $T_{\tilde{F}^*_1,\tilde{G}^*_1}(s)$ of $\Sigma_{\tilde{F}^*_1,\tilde{G}^*_1}=(\tilde{A}+\tilde{B}\tilde{F}^*_1,\ \tilde{B}\tilde{G}^*_1,\ \tilde{C})$ is given by :

$$T_{\tilde{F}^*_1,\tilde{G}^*_1}(s)=\begin{bmatrix} \frac{1}{s^{f_1+1}} & & & \bigcirc \\ & \frac{1}{s^{f_2+1}} & & \\ & & \ddots & \\ \bigcirc & & & \frac{1}{s^{f_m+1}} \end{bmatrix} \tag{20}$$

Proof

The proof follows directly from (17) and the Wolovich–Falb structure theorem under l.s.v.f. expressed by (9). □

Now according to Definition 3 the *infinite zeros* of $T_{\tilde{F}^*_1,\tilde{G}^*_1}(s)$ are equal to the *finite zeros* of $T_{\tilde{F}^*_1,\tilde{G}^*_1}(1/w)$ (obtained from (20) by making the substitution $s=1/w$) that are equal to zero. From (20)

$$T_{\tilde{F}^*_1,\tilde{G}^*_1}\left(\frac{1}{w}\right)=\operatorname{diag}\left(w^{f_1+1},\ w^{f_2+1},\ \ldots,\ w^{f_m+1}\right) \tag{21}$$

So in view of (21), Definition 3 and Proposition 3 we have

Proposition 5

If Σ is decouplable then $T(s)$ (and also $T_{\tilde{F},\tilde{G}}(s)$ for any finite l.s.v.f. control law $(\tilde{F},\ \tilde{G})$) has m infinite zeros, each infinite zero having degree $\eta_i=f_i+1$, $i\in\mathbf{m}$. The total number of infinite zeros (multiplicities or degrees accounted for) is $\sum_{i=1}^{m}(f_i+1)=f+m$, $(f=\sum_{i=1}^{m}f_i)$.

Remark

In view of (16) the total number of zeros (finite and infinite ones and multiplicities accounted for) is equal to n as expected.

The importance of the indices f_i, $i\in\mathbf{m}$, which may also be read off directly from $\tilde{C}$ by inspection (Vardulakis 1979), and their association with Morse's (1973) results becomes also clearer from the following considerations.

Define the $(f+m)\times m$ polynomial matrix

$$Z(s)=\begin{bmatrix} 1 & & \bigcirc \\ s & & \\ \vdots & & \\ s^{f_1} & & \\ & \ddots & \\ & & 1 \\ & & s \\ & & \vdots \\ \bigcirc & & s^{f_m} \end{bmatrix} \tag{22}$$

and through $Z(s)$ and $\tilde{C}(s)$ the $(f+m)\times m$ polynomial matrix

$$\tilde{C}_E(s) = Z(s)\tilde{C}(s) \tag{23}$$

and finally let $\tilde{C}_E$ be the unique $(f+m)\times n$ real matrix satisfying†

$$\tilde{C}_E(s) = \tilde{C}_E S(s) \tag{24}$$

Then it can be easily verified (Vardulakis 1979) that rank $\tilde{C}_E = f+m$ and if we consider the q eigenvectors of $\tilde{A}+\tilde{B}\tilde{F}^*_1$ (where $\tilde{F}^*_1$ is the state feedback matrix of Proposition 3) that correspond to its q eigenvalues that are equal to the q finite zeros of Σ, then these eigenvectors span $\mathscr{V}^{\max}$: the maximal $(\tilde{A}, \tilde{B})$-invariant subspace in ker $\tilde{C}$, and we have in fact that (Vardulakis 1979),

$$\mathscr{V}^{\max} = \ker \tilde{C}_E, \quad \dim \mathscr{V}^{\max} = n-m-f = q$$

Now clearly $(\tilde{A}+\tilde{B}\tilde{F}^*_1)\mathscr{V}^{\max} \subset \mathscr{V}^{\max}$. Let $\tilde{x}\in\mathscr{X}$, $\bar{\mathscr{X}} = \mathscr{X}/\mathscr{V}^{\max}$ and $\bar{x}\in\bar{\mathscr{X}}$; then dim $\bar{\mathscr{X}} = \dim \mathscr{X} - \dim \mathscr{V}^{\max} = f+m$ and the function $\tilde{x}\mapsto\bar{x}$ is the map $\tilde{C}_E : \mathscr{X}\to\bar{\mathscr{X}}$ introduced by eqns. (22), (23), (24) and called the canonical projection of $\mathscr{X}$ on $\bar{\mathscr{X}}$ (Wonham 1974), i.e. $\bar{x} = \tilde{C}_E\tilde{x}$. Let $\bar{A} : \bar{\mathscr{X}}\to\bar{\mathscr{X}}$ the map induced in $\bar{\mathscr{X}}$ by $\tilde{A}+\tilde{B}\tilde{F}^*_1$, then $\bar{A}$ is a $(f+m)\times(f+m)$ matrix and

$$\bar{A}\tilde{C}_E = \tilde{C}_E(\tilde{A}+\tilde{B}\tilde{F}^*_1) \tag{25}$$

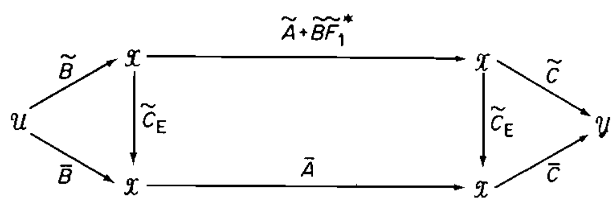
Also there exist a map $\bar{B} : \mathscr{U}\to\bar{\mathscr{X}}$ ($\bar{B}$ is a $(f+m)\times m$ matrix) such that

$$\bar{B} = \tilde{C}_E\tilde{B} \tag{26}$$

and a map $\bar{C} : \bar{\mathscr{X}}\to\mathscr{Y}$ (C is a $m\times(f+m)$ matrix) such that

$$\bar{C}\tilde{C}_E = \tilde{C} \tag{27}$$

(see diagram)



From eqns. (25), (26), (27) and the special 'stripe' structure of $\tilde{C}_E$ (see Vardulakis 1979) we may easily verify that the matrices $\bar{A}$, $\bar{B}$, $\bar{C}$ have the form

$$\bar{A} = \text{block diag}\,[\bar{A}_1, \bar{A}_2, \ldots, \bar{A}_m] \tag{28}$$

$$\bar{A}_i = \begin{bmatrix} 0 & 1 & 0\ldots0 \\ 0 & 0 & 1 & 0 \\ \vdots & \vdots & & \ddots \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0\ldots0 \end{bmatrix}_{(f_i+1)\times(f_i+1)}, \qquad i\in\mathbf{m} \tag{29}$$

† As it is shown by Vardulakis (1979) $\tilde{C}_E$ can be obtained directly from $\tilde{C}$ by inspection.

$$\bar{B} = [\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_m]_{(f+m)\times m} \tag{30}$$

where $\bar{b}_i$ are the p_ith ($p_i = \sum_{j=1}^{i} v_{m+1-j}$, $i \in \mathbf{m}$) columns of $\tilde{C}_E$, and

$$\bar{C} = \left[\begin{array}{cccc:cccc:c:cccc} 1 & 0 & 0 \ldots & 0 & & & & & & & & & \\ & & & & 1 & 0 & 0 \ldots & 0 & \ldots & & \bigcirc & & \\ & \bigcirc & & & & & & & & & & & \\ & & & & & & & & & 1 & 0 & 0 \ldots & 0 \end{array}\right] \begin{matrix} \uparrow \\ m \\ \downarrow \end{matrix} \tag{31}$$

$$\longleftarrow f_1+1 \longrightarrow \quad \longleftarrow f_2+1 \longrightarrow \qquad \longleftarrow f_m+1 \longrightarrow$$

Moreover if $\bar{B}_m$ is the $m \times m$ matrix consisting of the σ_ith ($\sigma_i = \sum_{j=1}^{i} (f_j+1)$, $i \in \mathbf{m}$) rows of $\bar{B}$ then $\bar{B} = B^* \bar{B}_m$ where B^* is a $(f+m)\times m$ matrix having the form

$$B^* = \left[\begin{array}{ccc} 0 & & \\ 0 & \bigcirc & \\ \vdots & & \\ 1 & & \\ \hdashline & 0 & \\ & 0 & \\ & \vdots & \\ & 1 & \\ \hdashline & & \ddots \\ \hdashline & & 0 \\ & & 0 \\ \bigcirc & & \vdots \\ & & 1 \end{array}\right] \begin{matrix} \uparrow \\ f_1+1 \\ \downarrow \\ \uparrow \\ f_2+1 \\ \downarrow \\ \\ \uparrow \\ f_m+1 \\ \downarrow \end{matrix} \tag{32}$$

Now the triple $(\bar{A}, \bar{B}, \bar{C})$ is controllable and observable (Wonham 1974, lemma 3.2) and the Wolovich–Falb (1969, 1974) structure theorem can be used in order to express the transfer function matrix $T_f(s)$ of the factor system $\Sigma_f = (\bar{A}, \bar{B}, \bar{C})$ as

$$T_f(s) = \bar{C} Z(s) \, [\operatorname{diag}(s^{f_1+1}, \ldots, s^{f_m+1})]^{-1} \bar{B}_m \tag{33}$$

From the structure of $\bar{C}$ and $Z(s)$ we have $\bar{C}Z(s) = I_m$, i.e. Σ_f has no finite zeros. Also from (17)

$$T_{F^*_1}(s) = \tilde{C}(s)\delta_{F^*_1}(s)^{-1} = [\operatorname{diag}(s^{f_1+1}, \ldots, s^{f_m+1})]^{-1} K = T_f(s)$$

we obtain that

$$\bar{B}_m = K \tag{34}$$

Equations (28)–(32) indicate that the indices f_i+1, $i \in \mathbf{m}$ are the controllability indices of the factor system pair $(\bar{A}, \bar{B})$ and that the triple $(\bar{A}, \bar{B}, \bar{C})$ is what Morse (1973) defined as a *prime system* (see also Morse's remarks at the end of his paragraph 3).

Proposition 6

If $f_i = 0$, $i \in \mathbf{m}$ then

$$K = \bar{B}_m = \tilde{C}\tilde{B} \tag{35}$$

Proof

If $f_i = 0$ for every $i \in \mathbf{m}$ then $Z(s) \equiv I_m$ implying $\tilde{C}_E(s) = \tilde{C}(s)$, i.e. $\tilde{C}_E = \tilde{C}$. Also $B^* \equiv I_m$, i.e. $\bar{B}_m \equiv \bar{B} = \tilde{C}_E \tilde{B} = \tilde{C}\tilde{B}$. □

If the ith row $\tilde{c}_i^{\mathrm{T}}$ of $\tilde{C}$ is partitioned into m blocks of sizes v_{m+1-j}, $j \in \mathbf{m}$ as

$$\tilde{c}_i^{\mathrm{T}} = [c^1{}_{i0}, c^1{}_{i1}, \ldots, c^1{}_{iq_{i1}}, 0, \ldots, 0 | c^2{}_{i0}, c^2{}_{i1}, \ldots, c^2{}_{iq_{i2}}, \\ 0, \ldots, 0| \ldots |c^m{}_{i0}, c^m{}_{i1}, \ldots, c^m{}_{iq_{im}}, 0, \ldots, 0]$$

(and $c_{iq_{ij}} \neq 0$ for at least one $j \in \mathbf{m}$) then from the definition of the f_i's in (10) it follows that f_i is equal to the minimum number of zeros appearing in the 'trailing edge' of the blocks (of size v_{m+1-j}, $j \in \mathbf{m}$) of $\tilde{c}_i^{\mathrm{T}}$. From this fact we deduce the following.

Proposition 7

If $f_i \neq 0$ for at least one $i \in \mathbf{m}$ then rank $CB < m$.

Proof

rank $CB =$ rank $\tilde{C}\tilde{B}$. Also $f_i \neq 0$ for some $i \in \mathbf{m}$ implies that all the i, p_jth, $i \in \mathbf{m}$, $j \in \mathbf{m}$ $\left(p_j = \sum_{i=1}^{j} v_{m+1-i}, j \in \mathbf{m}\right)$ entries of $\tilde{c}_i^{\mathrm{T}}$ are equal to zero, and because of the special canonical structure of $\tilde{B}$ we will have that $\tilde{c}_i^{\mathrm{T}} \tilde{B} = 0$, i.e. that the ith row of $\tilde{C}\tilde{B}$ is equal to zero. □

As a consequence we have

Corollary

rank $CB = m$ implies $f_i = 0$ for every $i \in \mathbf{m}$ and therefore

(i) from Propositions 1 and 6 the system $\Sigma = (A, B, C)$ is decouplable†;

(ii) the system Σ (or $T(s)$) has $n - m$ finite zeros and m infinite zeros each of degree equal to 1;

(iii) $\mathscr{V}^{\max} = \ker \tilde{C}$ and therefore $\dim \mathscr{V}^{\max} = n - m$.

Example

Consider the system $\Sigma = (\hat{A}, \hat{B}, \hat{C})$ in the Luenberger controllable canonical form, with $n = 5$, $m = 2$, $v_2 = 3$, $v_1 = 2$ and

$$\hat{A} = \tilde{A} = \left[\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -9 & -6 \end{array}\right], \quad \hat{B} = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 1 & 2 \\ \hline 0 & 0 \\ 0 & 1 \end{array}\right], \quad \hat{C} = \tilde{C} = \left[\begin{array}{ccc|cc} 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right]$$

† This is the classical sufficient condition for decouplability obtained by Morgan (1964).

$$\tilde{C}(s)=\tilde{C}S(s)=\begin{bmatrix}2&1&0&0&0\\0&0&1&1&1\end{bmatrix}\begin{bmatrix}1&0\\s&0\\s^2&0\\0&1\\0&s\end{bmatrix}=\begin{bmatrix}s+2&0\\s^2&s+1\end{bmatrix}$$

$$K(s)=\tilde{C}(s)\text{ diag }(1,\,s^{v_2-v_1})=\begin{bmatrix}s+2&0\\s^2&s+1\end{bmatrix}\begin{bmatrix}1&0\\0&s\end{bmatrix}=\begin{bmatrix}s+2&0\\s^2&s^2+s\end{bmatrix}$$

and

$$K=\begin{bmatrix}1&0\\1&1\end{bmatrix},\quad \text{rank } K=2$$

i.e. Σ is decouplable.

$$\begin{aligned}f_1&=\min\{v_2-1-\deg(s+2),\,v_1-1-\deg(0)\}\\&=\min\{3-1-1,\,2-1-(-\infty)\}=1\\f_2&=\min\{v_2-1-\deg s^2,\,v_1-1-\deg(s+1)\}\\&=\min\{3-1-2,\,2-1-1\}=0\end{aligned}$$

and Σ has $q=n-m-(f_1+f_2)=5-2-1=2$ finite zeros at -2 and -1 and $m=2$ infinite zeros one of degree $f_1+1=2$ and the other of degree $f_2+1=1$.

Also from (14) $K_R(s)=\begin{bmatrix}2&0\\0&s\end{bmatrix}$ and from (19)

$$\begin{bmatrix}s^2&0\\0&s\end{bmatrix}\begin{bmatrix}2&0\\0&s\end{bmatrix}=M\begin{bmatrix}1&0\\s&0\\s^2&0\\0&s\\0&s^2\end{bmatrix},\quad \text{i.e. } M=\left[\begin{array}{ccc|cc}0&0&2&0&0\\0&0&0&0&1\end{array}\right]$$

so that from (18)

$$\tilde{F}^*{}_1=-K^{-1}M-\tilde{A}_m=\begin{bmatrix}0&0&-2&1&-2\\0&0&2&9&5\end{bmatrix}\quad\text{and}\quad \tilde{G}^*{}_1=K^{-1}=\begin{bmatrix}1&0\\-1&1\end{bmatrix}$$

Finally

$$T_{\tilde{F}^*_1,\,\tilde{G}^*_1}(s)=\begin{bmatrix}1/s^2&0\\0&1/s\end{bmatrix}$$

4. Infinite zero structure of non-decouplable systems

Suppose now that $\Sigma=(A,\,B,\,C)$ is not decouplable, i.e. let $K(s)=\tilde{C}(s)[s^{v_m-v}]$ be not row proper. Then it is well known (Wolovich 1973 b), that there exists a $m\times m$ unimodular matrix $Q(s)$ such that

$$K'(s)=Q(s)K(s)=Q(s)\tilde{C}(s)[s^{v_m-v}]=\Gamma(s)[s^{v_m-v}] \tag{36}$$

is (i) row proper and (ii) deg {jth column of $\Gamma(s)\} \leqslant v_{m+1-j}-1$, $j \in \mathbf{m}$, where

$$\Gamma(s)=Q(s)\tilde{C}(s)=[\gamma_{ij}(s)], \quad i \in \mathbf{m}, \quad j \in \mathbf{m} \tag{37}$$

Now write

$$\Gamma(s)=\Gamma S(s) \tag{38}$$

and consider the system $(\tilde{A}, \tilde{B}, \Gamma)=\Sigma_\Gamma$. It can be easily proved that Σ_Γ is observable and by (36) also decouplable. By analogy to (10) define the indices

$$f_i^\Gamma = \min_{j \in \mathbf{m}} \{v_{m+1-j}-1-\deg \gamma_{ij}(s)\}, \quad i \in \mathbf{m} \tag{39}$$

and let

$$K'=[K'(s)]^h{}_r=[\Gamma(s)[s^{v_m-v}]]^h{}_r \tag{40}$$

(Where $[\cdot]^h{}_r$ denotes the matrix with elements in $\mathbb{R}$ consisting of the coefficients of the highest degree s terms in each row of the expression inside the brackets.) Obviously rank $K'=m$ and by Proposition 4 there exists a state feedback matrix (say) $\tilde{F}^*{}_2$ such that

$$K'\delta_{\tilde{F}^*_2}(s)=\text{diag }(s^{f_1^\Gamma+1}, \ldots, s^{f_m^\Gamma+1})\Gamma(s) \tag{41}$$

Now the number of finite zeros of Σ is $q=\deg \det \tilde{C}(s)=\deg \det Q(s)^{-1} \det \Gamma(s)=\deg \det \Gamma(s)$, since $Q(s)^{-1}=R(s)$ is unimodular, and from (41) by taking determinants and then degrees of both sides it simply follows that

$$q=n-m-\sum_{i=1}^{m} f_i^\Gamma \tag{42}$$

Equation (41) can be written as

$$\Gamma(s)\delta_{\tilde{F}^*_2}(s)^{-1}=[\text{diag }(s^{f_1^\Gamma+1}, \ldots, s^{f_m^\Gamma+1})]^{-1}K' \tag{43}$$

and if

$$\tilde{G}^*=(K')^{-1} \tag{44}$$

then

$$T_{\tilde{F}^*_2, \tilde{G}^*}(s)=\tilde{C}(s)\delta_{\tilde{F}^*_2}{}^{-1}(s)\tilde{G}^*=R(s) \text{ diag }(1/s^{f_1^\Gamma+1}, \ldots, 1/s^{f_m^\Gamma+1}) \tag{45}$$

Making the substitution $s=1/w$ we obtain

$$T_{\tilde{F}^*_2, \tilde{G}^*}\left(\frac{1}{w}\right)=R\left(\frac{1}{w}\right) \text{diag }(w^{f_1^\Gamma+1}, \ldots, w^{f_m^\Gamma+1}) \tag{46}$$

As $R(s)$ is unimodular (by Corollary 3 in Pugh and Ratcliffe (1979)) it has no finite zeros. With respect to the infinite poles and infinite zeros of $R(s)$ we can state the following.

Lemma

For every unimodular matrix $R(s)$ we have :

number of infinite poles = number of infinite zeros = $\delta(R(s))$

where : $\delta(R(s))$ denotes the McMillan degree of $R(s)$ and we know that (Rosenbrock 1970, p. 137)

$$\delta(R(s))=\nu(R(1/w)) \tag{47}$$

where $\nu(\cdot)$ denotes the *least degree* (Rosenbrock 1970) of the indicated matrix.

Proof

It is known that a polynomial (in this case also unimodular) matrix may have infinite poles and infinite zeros (Pugh and Ratcliffe 1979). From Corollary 8 in Pugh and Ratcliffe (1979) the total number of poles of $R(s)$ is equal to the total number of zeros and since $R(s)$ has no finite poles and no finite zeros its total number of infinite poles equals its total number of infinite zeros. But the total number of infinite poles of $R(s)$, is by Corollary 6 in Pugh and Ratcliffe (1979) equal to $\delta(R(s))$, hence the lemma. □

From (46) and the above lemma we can state the following.

Proposition 8

If $\Sigma=(A, B, C)$ is not decouplable then Σ (or $T(s)$) has m infinite zeros of degrees η_i, $i\in\mathbf{m}$. The degrees η_i are given by the degrees of the elementary divisors w^{η_i} of the polynomial matrix

$$R\left(\frac{1}{w}\right) \operatorname{diag}\,(w^{f_1{}^\Gamma+1}, \ldots, w^{f_m{}^\Gamma+1}) \tag{48}$$

Example

Consider the system Σ with $n=4$, $m=2$, $v_2=2$, $v_1=2$ and

$$\hat{A}=\tilde{A}=\left[\begin{array}{cc:cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ \hdashline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right], \quad \hat{B}=\tilde{B}=\left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \\ \hdashline 0 & 0 \\ 0 & 1 \end{array}\right], \quad \hat{C}=\tilde{C}=\left[\begin{array}{cc:cc} 0 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right]$$

(The above is the Luenberger controllable canonical form of the system in Owens (1978).)

$$\tilde{C}(s)=\tilde{C}S(s)=\begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ s & 0 \\ 0 & 1 \\ 0 & s \end{bmatrix}=\begin{bmatrix} s & -1 \\ 1-s & 1 \end{bmatrix}$$

$$K(s)=\tilde{C}(s)\operatorname{diag}\,(1, s^{v_2-v_1})=\tilde{C}(s), \quad K=[K(s)]^h{}_r=\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \quad \text{rank } K=1$$

i.e. Σ is not decouplable. The unimodular matrix

$$Q(s)=\begin{bmatrix} 1 & 1 \\ s & s+1 \end{bmatrix}$$

is such that

$$K'(s)=Q(s)K(s)=\begin{bmatrix} 1 & 1 \\ s & s+1 \end{bmatrix}\begin{bmatrix} s & -1 \\ 1-s & 1 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

is row proper. In this case $K'(s)=\Gamma(s)$ and from

$$\Gamma(s)=\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}=\Gamma S(s)\rightarrow\Gamma=\left[\begin{array}{cc:cc} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right]$$

and $\Sigma_\Gamma = (\tilde{A}, \tilde{B}, \Gamma)$ is decouplable. From (39)

$$f^\Gamma{}_1 = \min \{v_2 - 1 - \deg (1), v_1 - 1 - \deg (0)\} = \min \{2 - 1 - 0, 2 - 1 - (-\infty)\} = 1$$

$$f^\Gamma{}_2 = \min \{v_2 - 1 - \deg (1), v_1 - 1 - \deg (1)\} = \min \{2 - 1 - 0, 2 - 1 - 0\} = 1$$

so that Σ (and also Σ_Γ) has no finite zeros : $q = n - m - (f^\Gamma{}_1 + f^\Gamma{}_2) = 4 - 2 - 2 = 0$. Also

$$R(s) = Q(s)^{-1} = \begin{bmatrix} s+1 & -1 \\ -s & 1 \end{bmatrix}$$

and from (45) and (46)

$$T_{F^*_2, G^*}(s) = R(s) \begin{bmatrix} 1/s^2 & 0 \\ 0 & 1/s^2 \end{bmatrix} = \begin{bmatrix} s+1 & -1/s^2 \\ -1/s & 1/s \end{bmatrix}$$

$$T_{F^*_2, G^*}\left(\frac{1}{w}\right) = R\left(\frac{1}{w}\right) \begin{bmatrix} w^2 & 0 \\ 0 & w^2 \end{bmatrix} = \begin{bmatrix} (1+w)/w & -1 \\ -1/w & 1 \end{bmatrix} \begin{bmatrix} w^2 & 0 \\ 0 & w^2 \end{bmatrix}$$

$$= \begin{bmatrix} w(1+w) & -w \\ -w & w^2 \end{bmatrix}$$

which has Smith form

$$S(T_{F^*_2, G^*})(w) = \begin{bmatrix} w & 0 \\ 0 & w^3 \end{bmatrix}$$

i.e. Σ has $m = 2$ infinite zeros one of of degree 1 and the other of degree 3.

5. Asymptotic behaviour of closed loop eigenvectors under high gain output feedback

Let us consider now a decouplable system Σ under the output feedback control law

$$u = K_y y + Gv \tag{49}$$

where $K_y \in \mathbb{R}^{m \times m}$, $G \in \mathbb{R}^{m \times m}$, det $G \neq 0$ and let

$$K_y = \hat{B}_m{}^{-1} H, \quad H \in \mathbb{R}^{m \times m}, \quad G = \hat{B}_m{}^{-1}$$

Then the transfer function matrix of the closed loop system $\Sigma_{K_y, G} = (A + BK_y C, BG, C)$ is

$$T_{K_y, G}(s) = C(sI - A - BK_y C)^{-1} BG = \tilde{C}(sI - \tilde{A} - \tilde{B}H\tilde{C})^{-1}\tilde{B} \tag{50}$$

Let $\tilde{A}_k = \tilde{A} + \tilde{B}H\tilde{C}$ and $\tilde{A}_{km}$ represent the $m \times n$ matrix composed of the p_i, $i \in \mathbf{m}$ ordered rows of $\tilde{A}_k$. Then $\tilde{A}_{km} = \tilde{A}_m + H\tilde{C}$ and from the Wolovich–Falb structure theorem we will have that $T_{K_y, G}(s)$ can be written also as

$$T_{K_y, G}(s) = \tilde{C}(sI - \tilde{A} - \tilde{B}H\tilde{C})^{-1}\tilde{B} = \tilde{C}(s)\delta_H{}^{-1}(s) \tag{51}$$

where det $\delta_H(s) = \Delta_H(s)$ is the closed loop characteristic polynomial and

$$\delta_H(s) = [s^v] - \tilde{A}_{km} S(s) = [s^v] - (\tilde{A}_m + H\tilde{C})S(s) = \delta_0(s) - H\tilde{C}(s) \tag{52}$$

From (51) and for $H = kI_m$ we have the identity

$$(sI - \tilde{A} - k\tilde{B}\tilde{C})S(s) = \tilde{B}\delta_k(s) \quad (53)$$

Assume now that $k \neq 0$ is fixed and let $\Lambda = \{s_i = s_i(k),\ i \in \mathbf{n}\} \subset \mathbb{C}$ be the (assumed to be distinct) zeros of $\Delta_k(s)$, i.e. the eigenvalues of $\tilde{A} + k\tilde{B}\tilde{C}$. Then rank $\delta_k(s_i) = m - 1$ for every s_i, $i \in \mathbf{n}$ and therefore for every $s_i \in \Lambda$ there will be some vector $\beta_i = \beta_i(k) \neq 0$ such that

$$\delta_k(s_i)\beta_i = 0, \quad i \in \mathbf{n} \quad (54)$$

and from (53) for $s = s_i \in \Lambda$

$$[s_i(k)I - \tilde{A} - k\tilde{B}\tilde{C}]S(s_i(k))\beta_i = 0, \quad i \in \mathbf{n} \quad (55)$$

If we now put

$$\tilde{x}_i = S(s_i(k))\beta_i, \quad i \in \mathbf{n} \quad (56)$$

then (55) says that $\tilde{x}_i$ is a (closed loop) eigenvector of $\tilde{A} + k\tilde{B}\tilde{C}$ corresponding to its eigenvalue $s_i(k)$. It is well known that as $k \to \infty$, $q = n - m - \sum_{i=1}^{m} f_i$ of the eigenvalues $s_i(k)$ of $\tilde{A} + k\tilde{B}\tilde{C}$ move towards the q finite zeros along the root loci and the remaining $n - q = \sum_{i=1}^{m} (f_i + 1)$ eigenvalues become arbitrarily large in magnitude and in ' bunches ' (Butterworth patterns) of $f_i + 1$ eigenvalues move towards the m infinite zeros (each of which, if Σ is decouplable has degree $f_i + 1$). The q eigenvectors $\tilde{x}_i$ of $\tilde{A} + k\tilde{B}\tilde{C}$ (which correspond to those eigenvalues that as $k \to \infty$ move towards the q finite zeros) in the limit span $\mathscr{V}^{\max}$ the maximal $(\tilde{A}, \tilde{B})$-invariant subspace in ker $\tilde{C}$. If the q finite zeros of Σ are distinct and we denote them by $\Lambda_z = \{z_1, z_2, \ldots, z_q\}$ then it can easily be proved that as $k \to \infty$ the limiting eigenvectors of $\tilde{A} + kB\tilde{C}$ that correspond to the limiting eigenvalues $z_i \in \Lambda_z$ (and span $\mathscr{V}^{\max}$) are given by

$$\tilde{x}_i{}^* = S(z_i)\beta_i{}^*, \quad i \in \mathbf{q} \quad (57)$$

where $\beta_i{}^* = \ker \tilde{C}(z_i)$, $z_i \in \Lambda_z$, $i \in \mathbf{q}$.

With respect to the limiting values of the rest $n - q$ of the eigenvectors of $\tilde{A} + k\tilde{B}\tilde{C}$ as $k \to \infty$ we can state the following.

Proposition 9

As $k \to \infty$ the $n - q$ eigenvectors $\tilde{x}_i(s_i(k))$ of $\tilde{A} + k\tilde{B}\tilde{C}$ (that correspond to the $n - q$ eigenvalues $s_i(k)$ that tend to infinity, i.e. to the m infinite zeros of Σ) tend into $\mathscr{B} = \operatorname{Im} \tilde{B}$, i.e.

$$\lim_{\substack{k \to \infty \\ s_i \to \infty}} \tilde{x}_i(s_i(k)) \subset \mathscr{B}, \quad i \in \mathbf{t}, \quad t = n - q \quad (58)$$

Proof

Let $k \neq 0$ be fixed, $s_i(k) \neq 0$, $i \in \mathbf{n}$ be the finite eigenvalues (assumed to be distinct) of $\tilde{A} + k\tilde{B}\tilde{C}$, and

$$\beta_i = \beta(s_i(k)) = [\beta_{i1}(s_i), \ldots, \beta_{im}(s_i)] = \ker \delta_k(s_i(k))$$

If we denote by $\|\tilde{x}_i\|$ the norm of the eigenvector $\tilde{x}_i$ and we assume that $\|\tilde{x}_i\| = 1$, then from (56)

$$\|\tilde{x}_i\|^2 = |\beta_{i1}(s_i)|^2(1 + |s_i|^2 + \dots + |s_i|^{2v_m-2}) + \dots + |\beta_{im}(s_i)|^2(1 + |s_i|^2 + \dots + |s_i|^{2v_1-2}) = 1$$

and dividing by $|s_i|^2$

$$\frac{\|\tilde{x}_i\|^2}{|s_i|^2} = |\beta_{i1}(s_i)|^2\left(\frac{1}{|s_i|^2} + 1 + \dots + |s_i|^{2v_m-4}\right) + \dots + |\beta_{im}(s_i)|^2\left(\frac{1}{|s_i|^2} + 1 + \dots + |s_i|^{2v_1-4}\right) = \frac{1}{|s_i|^2} \tag{59}$$

From (59) if $k \to \infty$ and $s_i(k) \to \infty$ then

$$\lim_{\substack{k\to\infty\\ s_i\to\infty}} \frac{\|\tilde{x}_i\|^2}{|s_i|^2} = 0 \tag{60}$$

Let $\tilde{z}_i = \tilde{z}_i(s_i)$ be the projection of $\tilde{x}_i$ on the orthogonal complement $\mathscr{B}^\perp$ of $\mathscr{B}$, then

$$\tilde{z}_i = \tilde{B}^\perp \tilde{x}_i \tag{61}$$

where by the canonical structure of $\tilde{B}$ we have that

$$\tilde{B}^\perp = \text{block diag}\,[\tilde{B}_1{}^\perp, \tilde{B}_2{}^\perp, \dots, \tilde{B}_m{}^\perp] \tag{62}$$

$$\tilde{B}_i{}^\perp = \underbrace{\begin{bmatrix} 1 & 0 & 0 \dots\dots\dots 0 \\ 0 & 1 & 0 \dots\dots\dots 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 \quad 1 & 0 \end{bmatrix}}_{v_{m+1-i}} \; v_{m+1-i} - 1, \quad i \in \mathbf{m}$$

Then from (61) and (56)

$$\tilde{z}_i(s_i) = \tilde{B}^\perp S(s_i)\beta_i \begin{bmatrix} 1 & & \\ s_i & & \bigcirc \\ \vdots & & \\ s_i{}^{v_m-2} & & \\ & \ddots & \\ & & 1 \\ \bigcirc & & s_i \\ & & \vdots \\ & & s_i{}^{v_1-2} \end{bmatrix} \begin{bmatrix} \beta_{i1}(s_i) \\ \beta_{i2}(s_i) \\ \vdots \\ \beta_{im}(s_i) \end{bmatrix}$$

and therefore

$$\|\tilde{z}_i\|^2 = |\beta_{i1}(s_i)|^2(1 + |s_i|^2 + \dots + |s_i|^{2v_m-4}) + \dots + |\beta_{im}(s_i)|^2(1 + |s_i|^2 + \dots + |s_i|^{2v_1-4}) < \frac{\|\tilde{x}_i\|^2}{|s_i|^2} = \frac{1}{|s_i|^2}$$

i.e.

$$\lim_{\substack{k\to\infty \\ s_1\to\infty}} \|\tilde{z}_i\|^2 = 0$$

which implies (58). □

Conclusions

In this paper the infinite zero structure of controllable and observable linear multivariable systems which give rise to square, strictly proper transfer function matrices has been investigated.

By adopting the definition of infinite zeros and their degrees given by Pugh and Ratcliffe (1979) and using certain results from the polynomial matrix approach to the solution of the decoupling problem by state feedback, the relation between the degrees of infinite zeros of decouplable systems and the set of feedback invariants known as ' decoupling invariants ' has been established. Using a similar approach the infinite zero structure of non-decouplable systems has also been elucidated. As the definition of infinite zeros and their degrees in Pugh and Ratcliffe does not involve the concept of high gain, an interpretation of the above results with respect to the root-locus infinite zeros needs further investigation.

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