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Zero placement and the 'squaring down' problem: a polynomial matrix approach

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Certain results in the theory of polynomial matrices, free $\mathbb{R}[s]$-modules and minimal bases of rational vector spaces are used in order to investigate the 'squaring down' and zero placement problem.

1. Introduction

Standard linear multivariable feedback control schemes consider feedback loops between a *selected* set of measured output variables and an *equal* number of independent control inputs.

In the case of a multivariable system whose number of measured output variables is greater than the number of control inputs, the problem of combining all outputs together into a new set of outputs, whose number is equal to the number of control inputs, has been called a 'squaring down' (SD) problem (Kouvaritakis and MacFarlane 1976).

One extreme case of the above problem has been studied by Rosenbrock and Rowe (1970, 1970 Theorem 4.1) and briefly is the following: Let the equation $\dot{x} = Ax + Bu$ be given with (A, B) completely controllable, and let the matrix C in: $y = Cx$ be subject to choice. Then the problem is: can the 'measurement' map C be chosen so that (i) the pair (A, C) is observable and (ii) the transfer function matrix $C(sI-A)^{-1}B$ is *square* and its Smith–MacMillan form $M(s)$ has a desired set of numerator polynomials $\epsilon_i(s)$?

Rosenbrock's result is that a C satisfying the above requirements can always be chosen provided that the desired polynomials $\epsilon_i(s)$ in $M(s)$ and their degrees satisfy certain necessary conditions.

It is evident that the solution of the general SD problem has significant consequences on the zero structure of the corresponding loop transmission transfer function matrix and therefore that it vitally affects the final design procedure.

The general SD problem has been studied by Kouvaritakis and MacFarlane (1976), and it consists in finding an $m \times r$ $(m < r)$ constant post-compensator K such that $KG(s)$ (where $G(s)$ is a given $r \times m$ transfer function matrix) is a square $(m \times m)$ rational matrix which, apart from the zeros (if any) inherited from the non-square $(r \times m)$ $G(s)$, has also an additional number of desired zeros. The existing approach does not provide much insight into the limitations imposed by the given $G(s)$ on the possible numerator polynomials $\epsilon'_i(s)$ (and their degrees) of the Smith–MacMillan forms of $KG(s)$ that can be obtained by varying K.

The primary purpose of this paper is to provide a more detailed study of the SD problem as well as to give answers to the questions regarding the

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possible Smith–MacMillan forms of $KG(s)$. Specifically it is shown that the limitations, imposed by $G(s)$, on the possible zero structures of $KG(s)$, can be associated with what Forney (1975) calls the '*invariant dynamical indices*' of a certain rational vector space obtained from the given $G(s)$.

In § 2, a review of some known but scattered results concerning polynomial matrices, free $\mathbb{R}[s]$-modules and their connection with Forney's (1975) results on the algebraic structure of minimal bases of rational vector spaces is given. In § 3, we commence the investigation of the SD problem by considering an irreducible (right) matrix fraction description (MFD) of $G(s)$: $G(s)=C(s)D(s)^{-1}$. Using the results of § 2, it is then shown that the 'numerator' matrix $C(s)$ can always be factored as $C(s)=P(s)C_R(s)$, where $P(s)$ is a *minimal basis* for the rational vector space spanned by the columns of $C(s)$, and $C_R(s)$ is a square polynomial matrix which 'contains' all the zeros of the non-square transfer function matrix $G(s)$, (if any). Based on this result, the solution to the SD problem is shown to be equivalent to the existence of a real $(m\times r)$ matrix K such that $KP(s)=C_1(s)$ has a desired Smith-form structure. The fact that $P(s)$ is a minimal basis gives rise finally to Theorem 1, which can be seen as a generalization of Rosenbrock's theorem.

2. Mathematical background

In this section we collect some basic known results regarding the algebraic structure of rational vector spaces and their polynomial bases.

In the following, $\mathbb{R}$ denotes the field of real numbers, $\mathbb{R}(s)$ the field of rational functions in s with coefficients in $\mathbb{R}$, and $\mathbb{R}[s]$ the ring of polynomials in s with coefficients in $\mathbb{R}$. $\text{Rank}_{\mathbb{F}}(\cdot)$ denotes the rank of $(\cdot)$ over the field $\mathbb{F}$ and $\dim_{\mathbb{F}}$ denotes the dimension of a vector space defined over $\mathbb{F}$. Finally the symbol $\mathbf{m}$ denotes the set of integers $\{1, 2, \ldots, m\}$.

Let $V(s)$ be an $r\times m$ $(r>m)$ polynomial matrix with $\text{rank}_{\mathbb{R}(s)} V(s)=m$ and write it in terms of its m column polynomial vectors $\mathbf{v}_j(s)$ as

$$V(s)=[\mathbf{v}_1(s), \mathbf{v}_2(s), \ldots, \mathbf{v}_m(s)], \quad \text{where } \mathbf{v}_j(s)=[v_{1j}(s), v_{2j}(s), \ldots, v_{rj}(s)]^T, \ j\in\mathbf{m}.$$

Definition 1

The *degree* of $\mathbf{v}_j(s)$ is the highest degree occurring among the degrees of its polynomial elements $v_{ij}(s)$, $i\in\mathbf{r}$, i.e.

$$\deg \mathbf{v}_j(s)=\max_{i\in\mathbf{r}} \{\deg v_{ij}(s)\} \quad j\in\mathbf{m}$$

Definition 2 (Rosenbrock 1974)

The *complexity c of* $V(s)$ is the sum of the degrees of its column polynomial vectors, i.e. $c=\sum_{j=1}^{m} \deg \mathbf{v}_j(s)$.

Definition 3 (Rosenbrock 1974)

The *degree d of* $V(s)$ is the highest degree occurring among the degrees of all its m-order minors.

Since an m-order minor of $V(s)$ is a sum of products of polynomials one from each column, the maximum degree occurring among all the m-order minors of

$V(s)$, i.e. its degree d, cannot exceed its complexity c, i.e. we have $c \geq d$ (Rosenbrock 1974). Now let $g_j = \deg \mathbf{v}_j(s)$, $j \in \mathbf{m}$, and write $\mathbf{v}_j(s) = \sum_{k=0}^{g_j} \mathbf{v}_j^k s^k$, $j \in \mathbf{m}$. Then $V(s)$ can be written as

$$V(s) = [\mathbf{v}_1^{g_1}, \mathbf{v}_2^{g_1}, \ldots, \mathbf{v}_m^{g_m}] \begin{bmatrix} s^{g_1} & & \bigcirc \\ & s^{g_2} & \\ & & \ddots \\ \bigcirc & & s^{g_m} \end{bmatrix} + V_b Z(s) \tag{1}$$

where V_b is a real $r \times c$ $\left(c = \sum_{j=1}^{m} g_j\right)$ matrix and

$$Z(s) = \begin{bmatrix} 1 & & \bigcirc \\ s & & \\ \vdots & & \\ s^{g_1 - 1} & & \\ & \ddots & \\ & & 1 \\ & & s \\ & & \vdots \\ \bigcirc & & s^{g_m - 1} \end{bmatrix} \tag{2}$$

a $c \times m$ polynomial matrix. The (real) matrix $[\mathbf{v}_1^{g_1}, \mathbf{v}_2^{g_1}, \ldots, \mathbf{v}_m^{g_m}] = V_a$ is called the ***highest (column) degree coefficient matrix*** of $V(s)$.

Definition 4

$V(s)$ is said to be *column proper* (or equivalently is said to have least complexity) if the matrix V_a has full rank (m) (Wolovich 1974).

If by $M(s)^{i_1, i_2, \ldots, i_m}$, $M_a^{i_1, i_2, \ldots, i_m}$, we denote the $\binom{r}{m}$, m-order minors of $V(s)$ and V_a respectively, which are formed from their rows $i_1, i_2, \ldots, i_m$, then from (1) we have that

$$M(s)^{i_1, i_2, \ldots, i_m} = s^c M_a^{i_1, i_2, \ldots, i_m} + \text{lower degree terms} \tag{3}$$

Now if $V(s)$ is column proper, then $\text{rank}_{\mathbb{R}}\, V_a = m$, i.e. for at least one set of indices (say) $i_1, i_2, \ldots, i_m$ we have: $M_a^{i_1, i_2, \ldots, i_m} \neq 0$, and from definition 3 and (3) we have

$$\deg V(s) = d \triangleq \max_{\substack{\text{all possible sets} \\ \text{of indices } i_1, \ldots, i_m}} \{\deg M(s)^{i_1, \ldots, i_m}\} = c$$

The above gives rise to the following.

Proposition 1

A polynomial matrix $V(s)$ is column proper if its complexity c is equal to its degree d.

The following result implicit in Forney's (1975) reduction algorithm (stem no. 3) is a generalization of a result concerning the reduction of a square polynomial matrix to column proper form which was first published by Wolovich (1973).

Proposition 2

Let $V(s)$ be an $r \times m$ $(r > m)$ polynomial matrix which is not column proper. Then there always exists an $m \times m$ unimodular matrix $U(s)$ such that the polynomial matrix $V'(s) = V(s)U(s)$ is column proper.

Proof

Let $V(s)$ be not column proper ; i.e. from (1) write

$$V(s) = V_a \begin{bmatrix} s^{g_1} & & \bigcirc \\ & s^{g_2} & \\ & & \ddots \\ \bigcirc & & s^{g_m} \end{bmatrix} + V_b(s) \tag{4}$$

(where $V_b(s) = V_b Z(s)$ is an $r \times m$ polynomial matrix with the degree of its jth column : $\mathbf{v}_{bj}(s)$ being less than g_j, $j \in \mathbf{m}$), and let $\operatorname{rank}_{\mathbb{R}} V_a < m$. Then there exists a real $m \times 1$ vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m)^{\mathrm{T}} \neq 0$ such that $V_a \boldsymbol{\alpha} = 0$, or $\sum_{j=1}^{m} \alpha_j \mathbf{v}_j^{g_j} = 0$. Let $g_0 = \max_{j \in \mathbf{m}} \{g_j\}$ and assume that this degree occurs in the jth column of $V(s)$: then define

$$\boldsymbol{\alpha}(s) \triangleq [\alpha_1 s^{g_0 - g_1}, \alpha_2 s^{g_0 - g_2}, \ldots, \alpha_{j0}, \ldots, \alpha_m s^{g_0 - g_m}]^{\mathrm{T}} \tag{5}$$

Obviously $\deg \boldsymbol{\alpha}(s) < g_0$ and if we define the polynomial vector

$$\begin{aligned} \mathbf{v}'_{j_0}(s) &= V(s)\boldsymbol{\alpha}(s) = [V_a \operatorname{diag}(s^{g_1}, s^{g_2}, \ldots, s^{g_m}) + V_b(s)]\boldsymbol{\alpha}(s) \\ &= V_a \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix} s^{g_0} + V_b(s)\boldsymbol{\alpha}(s) = s^{g_0} \sum_{j=1}^{m} \alpha_j \mathbf{v}_j^{g_j} + V_b(s)\boldsymbol{\alpha}(s) \\ &= V_b(s)\boldsymbol{\alpha}(s) \end{aligned}$$

i.e. $\deg v_{j_0}(s) = g'_0 < g_0$. Now post-multiplying $V(s)$ by $\boldsymbol{\alpha}(s)$ to get $\mathbf{v}'_{j_0}(s)$ is equivalent to post-multiplying $V(s)$ by the $(m \times m)$ unimodular matrix

$$U_{j_0} = \left[\begin{array}{cccc:ccc} 1 & 0 & \ldots & \alpha_1 s^{g_0 - g_1} & 0 & \ldots & 0 \\ 0 & 1 & \ldots & \alpha_2 s^{g_0 - g_1} & 0 & \ldots & 0 \\ & & \ddots & & & & \\ 0 & 0 & \ldots & \alpha_{j_0} & 0 & \ldots & 0 \\ \hdashline 0 & 0 & \ldots & & 1 & \ldots & 0 \\ \vdots & & & \vdots & & \ddots & \\ 0 & 0 & \ldots & \alpha_m s^{g_0 - g_m} & 0 & \ldots & 1 \end{array}\right] \begin{array}{l} \\ \\ \\ \to \text{row } j_0 \\ \\ \\ \\ \end{array}$$

$\uparrow$ Column j_0

and the resulting matrix $V'_{j_0}(s)=V(s)U_{j_0}(s)$ will have all its columns (except its j_0th column) equal to the columns of $V(s)$. In particular, the degree of the j_0th column $\mathbf{v}'_{j_0}(s)$ of $V'_{j_0}(s)$ is strictly less than g_0, the degree of the j_0th column of $V(s)$. This then implies that: complexity of $V'_{j_0}(s)<$ complexity of $V(s)$, while: $\deg V'_{j_0}(s)=\deg V(s)$ (since $U_{j_0}(s)$ is unimodular). Now as the complexity c of a polynomial matrix is in general greater than its degree d, a number of elementary column operations (represented by unimodular matrices, which are formed as $U_{j_0}(s)$ above) can be determined so that the resulting matrix $V'(s)$ will have complexity equal to its degree, i.e. it will be column proper. Q.E.D.

Definition 5 (Emre and Silverman, 1975)

An $m\times m$ polynomial matrix $Q(s)$ with $\deg\det Q(s)\geqslant 1$ (i.e. not unimodular) is said to be a *right divisor* (RD) of an $r\times m$ polynomial matrix $\bar{V}(s)$ if there exists an $r\times m$ polynomial matrix $V(s)$, such that $\bar{V}(s)=V(s)\ Q(s)$. Let $Q_G(s)$ be RD of $\bar{V}(s)$. Then $Q_G(s)$ is said to be a *greatest right divisor* (GRD) of $\bar{V}(s)$ iff $\deg\det Q_G(s)\geqslant\deg\det Q(s)$ for every RD $Q(s)$ of $\bar{V}(s)$.

Definition 6

Given r $1\times m$ (row) polynomial vectors $\mathbf{v}_i^{\mathrm{T}}(s)$, $i\in\mathbf{r}$, they are said to be *relatively right prime* (RRP) iff one of the following equivalent conditions is satisfied.

(i) The polynomial matrix

$$V(s)=\begin{bmatrix}\mathbf{v}_1^{\mathrm{T}}(s)\\ \mathbf{v}_2^{\mathrm{T}}(s)\\ \vdots\\ \mathbf{v}_r^{\mathrm{T}}(s)\end{bmatrix}$$

has no RD.

(ii) The Smith form of $V(s)$ is $\begin{bmatrix}I_m\\ 0\end{bmatrix}$

(iii) The greatest common devisor (GCD) of all m-order minors of $V(s)$ is 1.

Definition 7

A polynomial matrix satisfying the equivalent conditions of Definition 6 is said to have *least degree.*†

Definition 8

Let the r $1\times m$ polynomial (row) vectors $\bar{\mathbf{v}}_i^{\mathrm{T}}(s)$, $i\in\mathbf{r}$ be not RRP. Then an $m\times m$ polynomial matrix $Q(s)$ is said to be a *common right divisor* (CRD) of the r $1\times m$ vectors $\bar{\mathbf{v}}_i^{\mathrm{T}}(s)$ iff it is a RD of

$$\bar{V}(s)=\begin{bmatrix}\bar{\mathbf{v}}_1^{\mathrm{T}}(s)\\ \bar{\mathbf{v}}_2^{\mathrm{T}}(s)\\ \vdots\\ \bar{\mathbf{v}}_r^{\mathrm{T}}(s)\end{bmatrix}$$

† Woolwich *et al.* (1977) call such a polynomial matrix ' prime basis '.

An $m \times m$ polynomial matrix $Q_G(s)$ is said to be a *greatest common right divisor* (GCRD) of the r $1 \times m$ vectors $\bar{\mathbf{v}}_i^T(s)$ iff it is a GRD of $\bar{V}(s)$.

It is well known (Wolovich 1974, MacDuffee 1956) that any GRD $Q_G(s)$ of $\bar{V}(s)$ can be obtained via a number of elementary row operations ; i.e. there exists always an $r \times r$ unimodular matrix $U_L(s)$ such that

$$U_L(s)\bar{V}(s) = \begin{bmatrix} Q_G(s) \\ \cdots\cdots \\ 0 \end{bmatrix}$$

Now let $\bar{V}(s)$ be a given $r \times m$ polynomial matrix which is *not least degree*, and consider the set of all *polynomial* vectors $\bar{\mathbf{v}}(s) \in \mathbb{R}^m[s]$ which can be written as linear combinations of the columns $\bar{\mathbf{v}}_j(s)$, $j \in \mathbf{m}$ of $\bar{V}(s)$ (with coefficients of course in the ring $\mathbb{R}[s]$). This set is a free $\mathbb{R}[s]$-module† which we denote by $\mathscr{M}_{\bar{V}}$. The polynomial matrix $\bar{V}(s)$ is a basis for $\mathscr{M}_{\bar{V}}$ and the rank of $\mathscr{M}_{\bar{V}}$ is equal to the number of independent elements in any basis for $\mathscr{M}_{\bar{V}}$. Obviously $\mathscr{M}_{\bar{V}}$ has more than one basis, and if $U(s)$ is any $m \times m$ *unimodular* matrix, then $\hat{V}(s) = \bar{V}(s)\ U(s)$ is another basis for $\mathscr{M}_{\bar{V}}$. From the fact that the m-order minors of $\bar{V}(s)$ multiplied by det $U(s)$ (= constant), we have the following proposition.,

Proposition 3

If $\bar{V}(s)$ and $\hat{V}(s)$ are both bases of $\mathscr{M}_{\bar{V}}$ then

$$\bar{d} = \deg\ \bar{V}(s) \equiv \deg\ \hat{V}(s)$$

If now $Q(s)$ is a RD of $\bar{V}(s)$ (i.e. $Q(s)$ is not unimodular), and $\bar{V}(s) = V_1(s)\ Q(s)$ for some $r \times m$ polynomial matrix $V_1(s)$, and we consider the module $\mathscr{M}_{V_1}$ generated by (the columns $\mathbf{v}_j(s)$, $j \in \mathbf{m}$ of) $V_1(s)$, then $\bar{V}(s)$ is not a basis for $\mathscr{M}_{V_1}$ and the module $\mathscr{M}_{\bar{V}}$ generated by (the columns $\bar{\mathbf{v}}_j(s)$, $j \in \mathbf{m}$ of) $\bar{V}(s)$ is in this case a submodile of $\mathscr{M}_{V_1}$ i.e. $\mathscr{M}_{\bar{V}} \subset \mathscr{M}_{V_1}$.

Obviously since $Q(s)$ is a RD of $\bar{V}(s)$, we have that

$$\bar{d} = \deg\ \bar{V}(s) > \deg\ V_1(s) = d_1$$

In general, if $Q_i(s)$, $i = 1, 2, \ldots$, are RDs of $\bar{V}(s)$, i.e. $\bar{V}(s) = V_i(s)\,Q_i(s)$, $i = 1, 2, \ldots$, and the deg det $Q_i(s) = q_i$ are such that $q_1 \leqslant q_2 \leqslant q_3 \leqslant \ldots$,then we will have that

$$\mathscr{M}_{\bar{V}} \subsetneq \mathscr{M}_{V_1} \subseteq \mathscr{M}_{V_2} \subseteq \mathscr{M}_{V_3} \subseteq \ldots \tag{7}$$

and

$$\deg\ \bar{V}(s) > \deg\ V_1(s) \geqslant \deg\ V_2(s) \geqslant \deg\ V_3(s) \geqslant \ldots \tag{8}$$

Moreover if $Q_G(s)$ is a GRD of $\bar{V}(s)$ so that $\bar{V}(s) = V(s)\ Q_G(s)$, then

$$\deg\ V_i(s) \geqslant \deg\ V(s) \quad \text{and } \mathscr{M}_{V_i} \subseteq \mathscr{M}_V \forall i = 1, 2, \ldots,$$

The importance of the module $\mathscr{M}_V$ and its (least degree) basis $V(s)$ both defined above through the matrix $\bar{V}(s)$ can now become clear. To see this, consider the set of all linear combinations (over the field $\mathbf{R}(s)$) of the columns $\bar{\mathbf{v}}_j(s)$, $j \in \mathbf{m}$ of $\bar{V}(s)$. This set represents a *vector space* $\mathscr{V}(s)$ (over $\mathbf{R}(s)$) whose

† See Sain (1975, 1976) for a brief introduction to module theory.

dimension (over $\mathbb{R}(s)$) is equal to $\mathrm{rank}_{\mathbb{R}(s)}\bar{V}(s)$ and which is called a *rational vector space* (Forney 1975). The polynomial matrix $\bar{V}(s)$ is a basis for $\mathscr{V}(s)$ and any other basis (polynomial or not) of $\mathscr{V}(s)$ is given by $\bar{V}(s)Q(s)$, where $Q(s)$ is any non-singular (over $\mathbb{R}(s)$) $m \times m$ matrix with elements in $\mathbb{R}(s)$. If we consider the set of all *polynomial* vectors in $\mathscr{V}(s)$, then this set *coincides* with the module $\mathscr{M}_V$ defined above.

Considering again the rational vector space $\mathscr{V}(s)$ defined as above via a given polynomial matrix $\bar{V}(s)$, we now have the following definition.

Definition 9

A polynomial matrix $V(s)$ is said to be a *minimal basis* of $\mathscr{V}$ iff $\bar{V}(s) = V(s)Q_G(s)$ for some $m \times m$ polynomial matrix $Q_G(s)$ and $V(s)$ has the following properties : (i) $V(s)$ is least-degree and (ii) $V(s)$ is column-proper (Forney 1975).

Given in general an $r \times m$ rational matrix $\bar{V}(s)$, then Forney (1975) describes a way of computing a minimal basis for the rational vector space spanned by its columns. He then shows that the column degrees $g_j = \deg \mathbf{v}_j(s)$, $j \in \mathbf{m}$, of a minimal basis : $V(s) = [\mathbf{v}_1(s), \mathbf{v}_2(s), \ldots, \mathbf{v}_m(s)]$ of $\mathscr{V}(s)$ are the same† (i.e. invariant) for every minimal basis of $\mathscr{V}(s)$, i.e. the g_j, $j \in \mathbf{m}$, characterize $\mathscr{V}(s)$. Forney calls these degrees the '*invariant dynamical indices*' of $\mathscr{V}(s)$, and their sum : $\sum_{j=1}^{m} g_j$(i.e. the complexity c of $V(s)$) he calls the '*invariant dynamical order*' of $\mathscr{V}(s)$.

3. Squaring down and zero placement

Let Σ be a linear, time invariant, multivariable system giving rise to an $r \times m$ $(r > m)$, strictly proper transfer function matrix $G(s)$ which has full rank (m) over $\mathbb{R}(s)$. The squaring down problem is the following.

Problem

Determine under what conditions an $m \times r$ constant post-compensator K exists such that the $m \times m$ rational matrix $KG(s)$ has a desired zero structure.

Let

$$G(s) = C(s)D(s)^{-1} \tag{9}$$

be an irreducible (right) matrix fraction description (MFD) of $G(s)$, i.e. let $C(s)$, $D(s)$ be, respectively, $r \times m$ and $m \times m$ relatively (right) prime (rrp) (Rosenbrock 1970) polynomial matrices both of which have full rank (over $\mathbb{R}(s)$). Now from the results of § 2 we have the following proposition.

Proposition 4

Let $C(s)$ be the $r \times m$ polynomial numerator matrix of the irreducible (right) MFD of $G(s)$ in (9). Then $C(s)$ can always factored (in a non-unique way) as

$$C(s) = P(s)C_R(s) \tag{10}$$

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† Within reordering.

where $C_R(s)$ is a GRD of $C(s)$ and $P(s)$ is an $r \times m$ polynomial matrix which is a minimal basis for the rational vector space $\mathscr{C}(s)$ spanned by the columns of $C(s)$.

An algorithmic procedure (implicit in Forney's (1975) reduction algorithm) to obtain a factorization of $C(s)$ with the above properties is given in the Appendix. Without loss of generality we can also assume that in the factorization (10), $P(s)$ is a 'degree ordered' minimal basis (Wolovich *et al* 1977): i.e. that if $P(s)=[\mathbf{p}_1(s), \mathbf{p}_2(s), \ldots, \mathbf{p}_m(s)]$ and $\delta_j = \deg \mathbf{p}_j(s)$; $j \in \mathbf{m}$, are the 'invariant dynamical indices' of $\mathscr{C}(s)$, then $\delta_1 \leqslant \delta_2 \leqslant \ldots \leqslant \delta_m$.

From Proposition 4 it simply follows that the finite zeros of $G(s)$ are equal to the zeros of $\det C_R(s)$. Now define the $(m+c) \times m$ ($c = \sum_{j=1}^{m} \delta_j$, i.e. the complexity or equally the degree of $P(s)$) polynomial matrix

$$Z(s) = \begin{bmatrix} 1 & & \\ s & & \\ \vdots & & \bigcirc \\ s^{\delta_1} & & \\ & \ddots & \\ & & 1 \\ & & s \\ \bigcirc & & \vdots \\ & & s^{\delta_m} \end{bmatrix} \tag{11}$$

and let P be the unique $r \times (m+c)$ (real) matrix satisfying

$$P(s) = PZ(s) \tag{12}$$

Then the SD problem is that of determining an $m \times r$ matrix K such that $KPZ(s) = C_1(s)$ has a desired Smith form.

Let $KP = C_1$ and $\Psi(s) = \operatorname{diag}[\psi_m(s), \psi_{m-1}(s), \ldots, \psi_1(s)]$ be the Smith form of $D(s)$ in (9) (see Rosenbrock 1970). Then we can state the following.

Theorem 1

An $m \times (m+c)$ real matrix C_1 can always be chosen so that the $m \times m$ polynomial matrix $C_1(s) = C_1 Z(s)$ is such that the polynomial matrices: $C_{sq}(s) = C_1(s)\, C_R(s)$ and $D(s)$ are relatively (right) prime and $C_1(s)$ has Smith form $E_{C1}(s) = \operatorname{diag}[\bar{\epsilon}_1(s), \bar{\epsilon}_2(s), \ldots, \bar{\epsilon}_2(s), \ldots, \bar{\epsilon}_m(s)]$, where $\bar{\epsilon}_j(s)$, $j \in \mathbf{m}$, may be any prescribed monic polynomials subject to the following necessary conditions:

(i) $\bar{\epsilon}_j(s)$ devides $\bar{\epsilon}_{j+1}(s)$, $j \in \mathbf{m}-1$.

(ii) The degrees $\deg \bar{\epsilon}_j(s)$ satisfy

$$\sum_{j=1}^{k} \deg \bar{\epsilon}_j(s) \leqslant \sum_{j=1}^{k} \delta_j, \quad k \in \mathbf{m}$$

where δ_j, $j \in \mathbf{m}$, are the 'invariant dynamical indices' of the rational vector

† Notice that the numerators of all irreducible (right) MFDs of $G(s)$ are unimodular (column) equivalent.

space $\mathscr{C}(s)$ spanned by the columns of any 'numerator' polynomial matrix $C(s)$ of $G(s)$†, arranged so that $\delta_1 \leqslant \delta_2 \leqslant \ldots \leqslant \delta_m$.

(iii) $\mathrm{GCD}\{\bar{\epsilon}_m(s), \psi_1(s)\} = 1$.

Proof

The proof of Theorem 1 follows directly from Rosenbrock's theorem (Rosenbrock 1970, Theorem 4.1). To see this, let

$$A_1 = \text{block diag}\,(A_{11}, A_{22}, \ldots, A_{mm}) \tag{13}$$

$$B_1 = \begin{bmatrix} B_{11} \\ B_{12} \\ \vdots \\ B_{1m} \end{bmatrix} \tag{14}$$

where

$$A_{1j} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & & \ldots & \\ 0 & 0 & 0 & \ldots & 1 \\ x & x & x & \ldots & x \end{bmatrix} \begin{matrix} j \in \mathbf{m} \\ (\delta_j + 1) \times (\delta_j + 1) \end{matrix} \tag{15}$$

and

$$B_{1j} = \begin{bmatrix} & & & 0 & & \\ 0 & 0 & \ldots & 1 & \ldots & 0 \end{bmatrix} \begin{matrix} j \in \mathbf{m} \\ (\delta_j + 1) \times m \end{matrix} \tag{16}$$

$\uparrow$ jth column

and choose the xs in A_1 so that the minimal polynomial of A_1 is $\psi_1(s)$. Then (A_1, B_1) is controllable with controllability indices: $\lambda_i = \delta_i + 1$, $i \in \mathbf{m}$ $(\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_m)$ and according to Rosenbrock's theorem a $m \times (m+c)$ matrix C_1 can be chosen so that the Smith form of $C_1(s) = C_1 Z(s)$ is E_{C_1} and the $\bar{\epsilon}_j(s)$, $j \in \mathbf{m}$, satisfy the conditions (i), (ii) and (iii) of Theorem 1. Condition (iii) also guarantees that $C_{sq}(s) = C_1(s)\, C_R(s)$ and $D(s)$ are rrp.

Corollary

The maximum possible number of additional zeros that can be placed by squaring down is equal to the *invariant dynamical order* of $\mathscr{C}(s)$: $c = \sum_{j=1}^{m} \delta_j$.

Now if C_1 is chosen so that $C_1(s)$ has invariant polynomials $\bar{\epsilon}_j(s)$, the existence of a 'squaring down' matrix K that will assign as additional zeros of $KG(s)$ the zeros of the $\bar{\epsilon}_j(s)$ merely depends on whether the matrix equation

$$KP = C_1 \tag{17}$$

has solution with respect to K, i.e. whether C_1 belongs to the row space of P.

From the fact that $P(s)$ is column-proper, it follows that the p_jth, $j \in \mathbf{m}$, columns $\left(p_j = \sum_{i=1}^{j} (\delta_i + 1)\right)$ of P are linearly independent ; hence all we know about the $r \times (m+c)$ matrix P is that

$$m \leqslant \operatorname{rank}_{\mathbb{R}} P \leqslant \min(r, m+c) \tag{18}$$

As a consequence we have the following.

Proposition 5

If $r = m + c$ and $\operatorname{rank}_{\mathbb{R}} P = r$, then a solution to the squaring down problem exists always. If C_1 is chosen as in Theorem 1, then the squaring down matrix K is given by $K = C_1 P^{-1}$.

Example

Let

$$G(s) = \begin{bmatrix} \dfrac{s+1}{s(s+2)^2} & \dfrac{s+1}{s(s+2)^2(s+3)} \\[2ex] \dfrac{1}{s(s+2)^2} & \dfrac{(s+1)^2}{s(s+2)^2(s+3)} \\[2ex] 0 & \dfrac{s+1}{(s+2)(s+3)} \end{bmatrix}$$

which has Smith–MacMillan form

$$M(s) = \begin{bmatrix} \dfrac{1}{s(s+2)^2(s+3)} & 0 \\[2ex] 0 & \dfrac{s+1}{s+2} \\[2ex] 0 & 0 \end{bmatrix}$$

i.e. the poles of $G(s)$ are $0, -2, -2, -2, -3$, and it also has a zero at -1. An irreducible MFD of $G(s)$ is

$$G(s) = C(s)D(s)^{-1} = \begin{bmatrix} s+1 & 0 \\ 1 & 1 \\ 0 & s+1 \end{bmatrix} \begin{bmatrix} s(s+2)^2 & -(s+2) \\ 0 & (s+3)(s+2) \end{bmatrix}^{-1}$$

Following the procedure described in the Appendix, the 'numerator' matrix $C(s)$ can be factored as

$$C(s) = \begin{bmatrix} s+1 & 0 \\ 1 & 1 \\ 0 & s+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & s+1 \end{bmatrix} \begin{bmatrix} s+1 & 0 \\ 1 & 1 \end{bmatrix} = P(s)C_R(s)$$

where $P(s)=[\mathbf{p}_1(s),\ \mathbf{p}_2(s)]$ is a minimal basis of $\mathscr{C}(s)$, and $\delta_1=\deg \mathbf{p}_1(s)=0$, $\delta_2=\deg \mathbf{p}_2(s)=1$ are the invariant dynamical indices of $\mathscr{C}(s)$. Now write

$$P(s)=\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & s+1 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & s \end{bmatrix}=PZ(s)$$

According to Theorem 1, a 2×3 matrix C_1 can be chosen so that the Smith form of $C_1Z(s)=C_1(s)$ is $E_{C_1}(s)=\operatorname{diag}[\bar{\epsilon}_1(s),\ \bar{\epsilon}_2(s)]$, where $\bar{\epsilon}_1(s)|\bar{\epsilon}_2(s)$ and $\deg \bar{\epsilon}_1(s)\leqslant 0$, $\deg \bar{\epsilon}_1(s)+\deg \bar{\epsilon}_2(s)\leqslant 1$. So the only possibilities arising are (i) $\deg \bar{\epsilon}_1(s)=0$, $\deg \bar{\epsilon}_2(s)=0$, i.e. no additional zeros, and (ii) $\deg \bar{\epsilon}_1(s)=0$, $\deg \bar{\epsilon}_2(s)=1$, i.e. one additional zero. Let

$$C_1=\begin{bmatrix} 1 & 0 & 0 \\ \beta & \alpha & 1 \end{bmatrix}$$

with $\alpha,\ \beta\in\mathbb{R}$, $\alpha>0$

$$C_1(s)=\begin{bmatrix} 1 & 0 \\ \beta & s+\alpha \end{bmatrix}\quad E_{C_1}(s)=\begin{bmatrix} 1 & 0 \\ 0 & s+\alpha \end{bmatrix}$$

In this case $r=m+c$, and P is invertible ; hence a squaring down 2×3 matrix K that will assign an additional zero of $KG(s)$ at $-\alpha$ is

$$K=C_1P^{-1}=\begin{bmatrix} 1 & 0 & 0 \\ \beta & \alpha & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ \beta+1 & \alpha-1 & 1 \end{bmatrix}$$

For $\beta=-1$ and $\alpha=1$, i.e. one additional zero at -1,

$$KG(s)=\begin{bmatrix} \dfrac{s+1}{s(s+2)^2} & \dfrac{s+1}{s(s+2)^2(s+3)} \\ 0 & \dfrac{s+1}{(s+2)(s+3)} \end{bmatrix},\quad M'(s)=\begin{bmatrix} \dfrac{s+1}{s(s+2)^2(s+3)} & 0 \\ 0 & \dfrac{s+1}{s+2} \end{bmatrix}.$$

where $M'(s)$ is the Smith–MacMillan form of $KG(s)$.

4. Conclusions

In this paper the general zero placement and squaring down problem has been examined in the light of Forney's theory of minimal bases of rational vector spares. It has been shown that the resticitions on the possible zero structures of the $m\times m$ rational matrix $KG(s)$ which are imposed by the given $r\times m$ $(r>m)$ transfer function matrix $G(s)$ can be associated with the invariant dynamical indices of the rational vector space $\mathscr{C}(s)$ which is defined by any $r\times m$ $(r>m)$ numerator polynomial matrix appearing in any *irreducible* (right) matrix fraction description of (Gs). It has also been shown that the squaring down and *arbitrary* zero placement problem has in general no solution. From eqn. (17) it can easily be seen that this problem is analogous to the still unresolved problem of pole assignment by constant output feedback.

Appendix

We summarize the steps involved in the decomposition (10). $C(s)$ is an $r \times m$ $(r > m)$ polynomial matrix.

(1) By unimodular row operations (represented by an $r \times r$ unimodular matrix $U_L(s)$), reduce $C(s)$ to upper (right) triangular form (see Wolovich 1974, Theorems 2.5.11 and 2.5.16) : i.e. determine $U_L(s)$ such that

$$U_L(s)C(s) = \begin{bmatrix} \bar{C}_R(s) \\ 0 \end{bmatrix}$$

Then $\bar{C}_R(s)$ is a GRD of $C(s)$.

(2) Compute $U_L^{-1}(s)$ and let

$$U_L^{-1}(s) = \begin{bmatrix} L_1(s) & L_2(s) \\ L_3(s) & L_4(s) \end{bmatrix}$$

Then

$$C(s) = U_L^{-1}(s) \begin{bmatrix} \bar{C}_R(s) \\ 0 \end{bmatrix} = \begin{bmatrix} L_1(s) \\ L_3(s) \end{bmatrix} \bar{C}_R(s)$$

and $\begin{bmatrix} L_1(s) \\ L_3(s) \end{bmatrix} = \bar{P}(s)$ has least degree.

(3) If $\bar{P}(s)$ is column-proper, then $P(s) = \bar{P}(s)$, $C_R(s) = \bar{C}_R(s)$. If $\bar{P}(s)$ is not column-proper, then determine (see proof of Proposition 2) an $m \times m$ unimodular matrix $U_R(s)$ such that $\bar{P}(s)U_R(s) = P(s)$ is column-proper. Then $C_R(s) = U_R(s)^{-1}\bar{C}_R(s)$.

References

Emre, E., and Silvermann, L. M., 1975, *I.E.E.E. Conference on Decision and Control*, WP5-3, 191.

Forney, G. D., 1975, *J. SIAM Control*, **13**, 493.

Kouvaritakis, B., and MacFarlane, A. G. J., 1976, *Int. J. Control*, **23**, 167.

MacDuffee, C. C., 1956, *Theory of Matrices* (New York : Chelsea).

Rosenbrock, H. H., and Rowe, B. A., 1970, *Proc. Instn. elect. Engrs.*, **117**, 1079.

Rosenbrock, H. H., 1970, *State Space and Multivariable Theory*, (London : Nelson).

Rosenbrock, H. H., 1974, *Int. J. Control*, **19**, 323.

Sain, M. K., 1975, Preprints of 6th IFAC World Congress, Paper 9.1, Parl IB, 1 ; 1976, *Proc. Instn. elect. electron. Engrs.*, **63**, 96.

Wolovich, W. A., 1973, *Automatica*, **9**, 97 ; 1974, *Linear Multivariable Systems* (New York : Springer-Verlag).

Wolovich, W. A., Antsaklis, P., and Elliot, H., 1977, *I.E.E.E. Trans. autom. Control.*, **22**, 88.