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(A, B)-invariant subspaces and polynomial matrix algebra— towards a more integrated approach. Part I: Square systems

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(*A*, *B*)-invariant subspaces and polynomial matrix algebra—towards a more integrated approach.
Part I: Square systems†

P. N. R. STOYLE‡ and A. I. G. VARDULAKIS§

The algebra of (A, B)-invariant subspaces $\mathscr{V}$ having null intersection with range $B = \mathscr{B}$ is examined from a viewpoint of the algebra of controllability subspaces in their polynomial vector parametrization and decoupling theory. Thus a characterization of such subspaces is arrived at in terms of an equivalence class of numerator matrices associated with the Wolovich–Falb canonical matrix fraction decomposition of a linear reachable time-invariant multivariable system. A simple and useful formula is obtained for reading off the maximal (A, B)-invariant subspace $\mathscr{V}^{\max} \subseteq \ker C$ in terms of any related numerator matrix. It is also shown how to obtain the important class of feedback $\mathbf{F}(\mathscr{V})$ of interest to typical geometric theory calculations by an inspection method akin to that used in the solution of the restricted decoupling problem. The set of all (A, B)-invariant subspaces in some fixed subspace, ker C say, is classified in terms of suitably defined divisor classes of the equivalence class of numerator matrices belonging to $\mathscr{V}^{\max} \subseteq \ker C$.

1. Introduction

The purpose of this paper is to extend the work of Warren and Eckberg (1975), Stoyle and Vardulakis (1979 a, b) on controllability subspaces (c.s.) in their minimal polynomial vector parametrization (the so-called ‘decoupling vector’) to the other main type of (A, B)-invariant subspace, that having null intersection with the input space. Once again, direct connections with the Wolovich–Falb (1969) theorem are exposed and this is the key to the reconciliation of two major state feedback theories: the Wonham and Morse (1970) geometric theory of (A, B)-invariant subspaces and the Wolovich (1974 a) theory based on the Luenberger form and associated matrix fraction decompositions of the transfer function. The approach here is strongly c.s. biased which may be surprising since c.s. are of course just a special case of (A, B)-invariant subspaces, but investigations in Stoyle and Vardulakis (1979 a, b) indicate that c.s. algebra is appealingly simple, so it becomes advantageous where appropriate to turn other problems of the geometric theory around into the form of, say, a decoupling problem.

2. Preliminaries

Basic properties of (A, B)-invariant subspaces are summarized in Morse and Wonham (1971) while a particularly good summary of the notation for the Wolovich–Falb feedback theorem is given in Wolovich (1972), see also

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Wolovich (1974 a, b). The best reference on general algebra of polynomial matrices is Forney (1975).

An (A, B)-invariant subspace $\mathscr{V}$ is ' proper ' if $\mathscr{V} \cap \mathscr{B} = \varnothing$ (Karcanias and Kouvaritakis 1977). From first principles, it is easy to show that any arbitrary (A, B)-invariant subspace can be decomposed as

$$\mathscr{V} = \mathscr{V}^{\text{prop}} \oplus \mathscr{R}^{\max}$$

where $\mathscr{R}^{\max}$ is the maximal controllability subspace contained in $\mathscr{V}$ and $\mathscr{V}^{\text{prop}}$ is proper. To put the difference in perspective, the eigenvalues of $A + BF$ are fixed and independent of the choice of $F \in \mathbf{F}(\mathscr{V})$ and these eigenvalues are the ' invariant zeros ' (Kouvaritakis and Shaked 1976) of the system. On $\mathscr{R}^{\max}$ by contrast the eigenvalues are freely assignable by choice of $F \in \mathbf{F}(\mathscr{V})$. These properties are implicit in Wonham and Morse's work and are developed and interpreted in Karcanias and Kouvaritakis (1977) and MacFarlane and Karcanias (1978).

3. Notation

3.1. *System terms*

$\mathscr{X} = \mathbb{R}^n$, state space (script letters used for subspaces).

$\Sigma = (A, B, C)$, usual description of a linear, time-invariant, finite-dimensional multivariable system, via state transition matrix $A \in \mathbb{R}^{n \times n}$, input matrix $B \in^{n \times m}$, and output matrix $C \in \mathbb{R}^{p \times n}$. We assume this triple is in Luenberger (1967) *canonical form.*

$v_1, v_2, \ldots, v_m$ Controllability (or Kronecker) indices of Σ (see e.g. Luenberger 1967). $\mathscr{X}$ can always be decomposed as $\mathscr{X} = \mathscr{R}_1 \oplus R_2 \oplus \ldots \oplus \mathscr{R}_m$ (Wonham 1974) and then $v_i = \dim \mathscr{R}_i$. $\{\mathscr{R}_i, i \in \mathbf{m}\}$ are ' elementary c.s.' of Σ.

$A_F = A + BF$, $F \in \mathbb{R}^{m \times n}$ is linear constant state feedback.

$S(s) = \text{block diag } \{s_1, s_2, \ldots, s_m\} : \ s_i = (1, s, \ldots, s^{v_{m-i+1}})$.

$[s^v] = \operatorname{diag}_{i \in \mathbf{m}} (s^{v_{m-i+1}})$.

$[s^{v_m - v}] = \operatorname{diag}_{i \in \mathbf{m}} (s^{v_m - v_{m-i+1}})$.

$T_{F,G}(s) = C(sI - A_F)^{-1}BG = C(s)\delta_F^{-1}(s)B_m G$: closed-loop transfer function of s. On the right is the Wolovich matrix fraction decomposition of the c.l. system ; $\delta_F(s) = [s^v] - (A_m + F)S(s)$ is the Wolovich denominator matrix (A_m is read off from the Luenberger A matrix) and $C(s) = CS(s)$ is the (Wolovich) numerator matrix. The canonical input transformation $B_m \in \mathbb{R}^{m \times m}$, an upper triangular matrix with 1's along the diagonal will, for the sake of mathematical tidiness, be removed from the Wolovich formulae by defining input $u'(t) = B_m u(t)$ as the new input to the system, yielding the so-called Luenberger input-transformed coordinates (Stoyle and Vardulakis 1979 a, b).

3.2. *Polynomial matrix terms* (Forney 1975)

$\mathbb{R}^{p\times m}[s]$ The class of $p\times m$ matrices with polynomial elements over $\mathbb{R}$, the field of real numbers.

$\mathbb{R}^{p\times m}(s)=$ The class of $p\times m$ matrices with rational (i.e. quotient of polynomials) elements.

$\partial p(s)$ The degree of a polynomial $p(s)\in\mathbb{R}[s]$.

$\partial P(s)=\partial \det P(s),\ P(s)\in\mathbb{R}^{m\times m}[s]$. The ' zeros ' of $P(s)$ are the roots of the equation $\partial P(s)=0$. The zeros of the Wolovich numerator matrix (Σ assumed reachable) are just the invariant zeros of § 2.

$\partial_{ri}Q(s)\ (\partial_{ci}Q(s))$ The degree of the ith row (column) of $Q(s)\in\mathbb{R}^{p\times m}[s]$.

$[Q(s)]_r^h$ The highest row coefficient of $Q(s)$ (due to Wolovich 1973). When $[Q(s)]_r^h$ is of full rank, $Q(s)$ is ' row-proper '.

$P(s)\ Q(s)$ $P(s)$ right-divides $Q(s)$, i.e. $Q(s)=K(s)P(s)$ for some polynomial matrix $K(s)$.

g.c.d. Greatest common divisor (of a set of polynomials).

m.p. $\{P(s)\}=\det P(s)/$g.c.d. $(m-1)$th order minors of $P(s)\in\mathbb{R}^{m\times m}[s]$.

adj $P(s)=$ the classical (determinantal) adjoint of $P(s)$.

3.3. *Vector space terms* (Wonham 1974, Gantmacher 1959)

ker C Kernel of the (matrix) map $C:\ \mathbb{R}^n\to\mathbb{R}^p$.

$\mathscr{V}_C^{\max}\ (\mathscr{R}_C^{\max})$ The maximal $(A,\ B)$-invariant subspace (resp. c.s.) contained in ker C.

$\mathscr{V}_C^{\max,\,\mathrm{cyc}}$ The maximal cyclic subspace in $\mathscr{V}_C^{\max}$.

$\mathscr{V}^{\mathrm{stab}}$ (when $\mathscr{V}$ is proper), see Remark 5.19.

$\mathbf{F}(\mathscr{V})=\{F:\ (A+BF)\mathscr{V}=\mathscr{V}\}$.

$\mathscr{Q}^\perp$ The orthogonal complement of subspace $\mathscr{Q}$.

$\mathscr{Q}/\mathscr{V}$ ($\mathscr{Q}$ mod $\mathscr{V}$) Factor space modulo a vector subspace $\mathscr{V}$. The elements or cosets of this factor space are written $a+\mathscr{V}$, or a (mod $\mathscr{V}$).

$A|_{\mathscr{V}}$ Map $A:\ \mathscr{X}\to\mathscr{X}$ restricted to a subspace $\mathscr{V}$ such that $A\mathscr{V}=\mathscr{V}$.

$\bar{A}=A/\mathscr{V}$ The map induced by A on the canonical projection $\mathscr{X}\to\mathscr{X}/\mathscr{V}$, $\mathscr{V}$ an invariant subspace (i.e. $A\mathscr{V}=\mathscr{V}$).

m.p. $\{A\}$ The minimum polynomial of a map $A\in\mathbb{R}^{n\times n}$.

$$\mathrm{span}\ \{v_1\ \dots\ v_l\}=\left\{\sum_{l\in\mathbf{t}} a_l v_l:\ a_l\in\mathbb{R}\quad\text{for } l\in\mathbf{t}\right\},\ v_l,\ l\in\mathbf{t}\ \text{are vectors.}$$

$\{A\,|\mathscr{B}\}=\{\mathscr{B},\ A\mathscr{B},\ \dots,\ A^{n-m}\mathscr{B}\}$.

3.4. *Abbreviations and definitions in the text*

s.i.s.o.	Single input single output system.
l.h.h.p. (r.h.h.p.)	Left (right) hand half plane (of the complex plane).
t.f.	Transfer function.
n.m.c.d.c.	See Definition 5.1.
t.r.d.	See Algorithm 5.3.
decoupling vector $\beta(s)$, cyclic listing R, and germ (R) of a c.s. $\mathscr{R}$	Defined in Stoyle and Vardulakis (1979 a).
degree deficiency or decoupling indices $\{f_i : i \in \mathbf{m}\}$	See eqn. 5.4.
(valid) numerator matrix	See Definition 5.1.
proper numerator matrix	See Definition 5.12.
stem of a c.s.	See Definitions 4.4, 5.7.
region of inter-twining of a c.s.	See Definition 5.8.
$\mathscr{K}^{\mathscr{V}}(s)$, $\mathscr{L}^{\mathscr{V}}(s)$	See Definition 5.12.
$\mathscr{C}^{\mathscr{V}}(s)$, $\mathscr{C}_0^{\mathscr{V}}(s)$	See text prior to Definition 5.14.
C^c, $C^c(s)$, $Z(s)$	See Theorem 5.5.

3.5. *Other symbols*

$\sum$	Sum.
$\prod$	Product.
$\sim$	Geometric/polynomial matrix equivalence (for decoupling vectors and c.s., see Stoyle and Vardulakis 1979 a; for numerator matrices and proper (A, B)-invariant subspaces, see Definitions 4.6, 5.14.
$\subseteq$	Subspace containment.
$\oplus$, $\sum^{\oplus}$	Direct sum of vector subspaces.
$\varnothing$	Null space.
$\Rightarrow$	'Implies'.
T	Matrix transpose.
$\mathbf{m}$	The set $\{1, 2, \ldots, m\}$.

4. Zeros and $(A,\ B)$-invariant subspaces for s.i.s.o. systems

Let a s.i.s.o. system triple $\Sigma=(A,\ b,\ c)$ be characterized by a strictly proper transfer function $t_0(s)=c(s)/\delta_0(s)$ where $\partial c(s)=m<n=\partial\delta_0(s)$. Our object is to calculate the maximal $(A,\ B)$-invariant subspace $\mathscr{V}^{\max}$ contained in the kernel of the o.p. map c ; and also determine the class of state feedbacks $\mathbf{F}(\mathscr{V}^{\max})$ which 'achieve' $\mathscr{V}^{\max}$.

First let us prove the case when all the roots of $c(s)$ are distinct. The proof for the non-generic case of repeated roots follows and is only slightly more involved, but the reader may wish to skip this section.

Theorem 4.1

Let $a(s)$ be an arbitrary monic polynomial of degree $n-m$, and let $\bar{f}$ be that unique feedback solving the equation

$$k\delta_{\bar{f}}(s)=a(s)c(s) \tag{4.1}$$

where k is the highest coefficient of $c(s)$. Form the polynomial matrix

$$C^{\mathrm{e}}(s)=\begin{bmatrix} c(s) \\ sc(s) \\ \vdots \\ s^{n-m-1}c(s)\end{bmatrix}$$

and let the $(n-m)\times m$ 'stripe' matrix C^{e} be uniquely defined from the identity

$$C^{\mathrm{e}}(s)=C^{\mathrm{e}}S(s)$$

Then the maximal (A,b)-invariant subspace contained in $\ker c$, or equally the maximal $(A+b\bar{f})$-invariant subspace in $\ker c$, is given by

$$\mathscr{V}^{\max}=\ker C^{\mathrm{e}} \quad \text{and} \quad \bar{f}\in\mathbf{F}(\mathscr{V}^{\max})$$

$\mathbf{F}(\mathscr{V}^{\max})$ is obtained as $a(s)$ is allowed to vary freely. The minimal polynomial of $A+b\bar{f}|\mathscr{V}^{\max}$ is $a(s)$ (independent of the $\bar{f}$ solving eqn. (4.1)) and the transfer function of the closed loop system with feedback $\bar{f}$ and input transformation $\bar{g}=k^{-1}$ applied is given by $t_{\bar{f},\bar{g}}(s)=a(s)^{-1}$.

Proof (Case (i)—distinct roots of $c(s)$)

Let

$$c(s)=\sum_{i\in\mathbf{m}}(s-s_i),\quad s_i\neq s_j \quad \text{for } i\neq j$$

For full state variable feedback f, we have the Wolovich–Falb identity (refer to § 3 on notation) :

$$(sI-A_f,\ -b)\begin{bmatrix} S(s) \\ \cdots\cdots \\ \delta_f(s)\end{bmatrix}=0 \tag{4.2}$$

Now let $\bar{f}$ be determined directly from eqn. (4.1) by equating coefficients of powers of s. From eqn. (4.2)

$$g(sI-A_{\bar{f}})S(s)=ba(s)c(s) \tag{4.3}$$

Provided $c(s)$ is not a constant (i.e. considering case $m>0$), let us put $s=s_i$, $i\in\mathbf{m}$ in eqn. (4.3) obtaining :

$$(s_iI-A_{\bar{f}})S(s_i)=0$$

Thus s_i is an eigenvalue of $A_{\bar{f}}$ and $S(s_i)$ is the corresponding eigenvector. (Note if s_i complex, instead of the complex eigenvectors $S(s_i)$, $S(s_i{}^*)$ we should consider the related real pair : $\frac{1}{2}\{(S(s_i)+S(s_i{}^*)\}$, $\frac{1}{2}j\{(S(s_i)-S(s_i{}^*)\}$.)

Next we define the subspace $\mathscr{V}=\text{span}\ \{S(s_i),\ i\in\mathbf{m}\}$, then $\mathscr{V}$ is a subspace of the generalized eigenspace of $A+b\bar{f}$, plainly $(A+b\bar{f})$-invariant. Moreover

$$cS(s_i)=c(s_i),\quad i\in\mathbf{m}$$

Define the matrix C^c as in the statement of Theorem 4.1, then

$$S(s_i)\in\ker C^c\quad \text{iff}\ C^c(s_i)=0$$

It is immediately verified that the latter condition holds iff $c(s_i)=0$. That is to say that : $\mathscr{V}\subseteq\ker C^c$ iff $\mathscr{V}\subseteq\ker c$. From the form of the matrix C^c it is clear that $\dim\mathscr{C}^c=n-m$ where $\mathscr{C}^c$ is the subspace of $\mathscr{X}$ spanned by the rows of the matrix C^c, and also $\dim\mathscr{V}=m$, so $\mathscr{V}\oplus\mathscr{C}^c=\mathscr{X}$. Also, as is well known, $\mathscr{V}$ is an (A,b)-invariant subspace iff for some feedback $\bar{f}$ it is $(A+b\bar{f})$-invariant. Putting the last three observations together we infer that $\mathscr{V}^{\max}=\mathscr{V}=\text{span}\ \{S(s_i),\ i\in\mathbf{m}\}=(\mathscr{C}^c)^\perp$. We should lastly consider the case when $c(s)$ is a non-zero constant, but then $\mathscr{C}^c$ is the whole space $\mathscr{X}$, whence we verify that $\mathscr{V}^{\max}=\varnothing$. Next, $\mathscr{V}^{\max}$ is cyclic under $A+b\bar{f}$ as it is a subspace of $\mathscr{X}$ which is itself cyclic with generator b. Thus the minimal polynomial of $\mathscr{V}^{\max}$ coincides with the characteristic polynomial, and the latter has been seen to be $c(s)$ as the roots of $c(s)$ are eigenvalues with multiplicity (see also Case (ii) below) of $A+b\bar{f}$ on $\mathscr{V}^{\max}$, irrespective of $\bar{f}\in\mathbf{F}(\mathscr{V}^{\max})$. The statement that $t_{\bar{f},\bar{g}}(s)=a(s)^{-1}$ is immediate from $t_{\bar{f},\bar{g}}(s)=c(s)/(k\delta_{\bar{f}}(s))$ and eqn. (4.1). $a(s)$ is arbitrary i.e. the poles of the c.l. system can be arbitrarily assigned by choice of f.

Conversely if $\bar{f}\in\mathbf{F}(\mathscr{V}^{\max})$ then $A+b\bar{f}$ is well defined on the non-trivial factor space $\mathscr{R}=\mathscr{X}/\mathscr{V}^{\max}$ and has an induced map $\overline{A+b\bar{f}}$ there. It follows that

$$a(s)\triangleq\text{m.p.}\ \overline{(A+b\bar{f})}=\frac{\text{m.p.}\ (A+b\bar{f})}{\text{m.p.}\ (A+b\bar{f}\,|\,\mathscr{V}^{\max})}$$

This then shows that the class of f's defined by (4.1) (as $a(s)$ is allowed to vary freely) is the most general possible, i.e. coincides with $\mathbf{F}(\mathscr{V}^{\max})$.

Proof (Case (ii)-repeated roots of $c(s)$)

Let $c(s)=\sum\limits_{i\in\mathbf{m}'}(s-s_i)^{p_i}$, $m'<m$, $\sum\limits_{i\in\mathbf{m}'}p_i=m$, suppose at least one positive integer $p_i>1$. Differentiate (4.3) j times

$$(sI-A_{\bar{f}})\frac{1}{j-1}\frac{d^jS(s)}{ds^j}+\frac{d^{j-1}S(s)}{ds^{j-1}}=b\frac{d^j}{ds^j}(a(s)c(s))\qquad(4.4)$$

where $d^{-1}S(s)/ds^{-1}\triangleq 0$. Setting $s=s_i$ in this equation we see that case $j=0$ is just the eigenvector identity for that eigenvector belonging to s_i while (4.4) for $1 \leqslant j < p_i$ may be recognized as the standard generalized eigenvector relation from which the useful expression

$$\frac{1}{(j-1)} \frac{d^j S(s)}{ds^j}\bigg|_{s=s_i}$$

for the jth generalized eigenvector may be read off.

Let $\mathscr{V}_i$ be defined as the subspace of the state space spanned by the set of generalized eigenvectors above belonging to s_i and let $\mathscr{V} = \sum^{\oplus} \mathscr{V}_i$. Once again $\mathscr{V}$ is trivially $(A + b\bar{f})$-invariant. Now

$$\frac{d^j c(s)}{ds^j}\bigg|_{s=s_i} = 0 = c\,\frac{d^j S(s)}{ds^j}\bigg|_{s=s_i}, \quad j = 0, 1, \ldots, p_i - 1$$

showing that $\mathscr{V}_i \subseteq \ker c$, $i \in \mathbf{m}$, or $\mathscr{V} \subseteq \ker c$. Also, as before, it is simple to check from the definition of $C^c(s)$ from $c(s)$ that:

$$\frac{d^j C^c(s)}{ds^j}\bigg|_{s=s_i} = 0 \quad \text{iff} \quad \frac{d^j c(s)}{ds^j}\bigg|_{s=s_i} = 0$$

so indeed $\mathscr{V} \subseteq \ker C^c$. Then following the dimensionality argument of case (i) we may again conclude that $\mathscr{V} = \mathscr{V}^{\max} = \ker C^c \subseteq \ker c$. This concludes the proof.

Remark 4.2

If a non-zero system initial state is set up $x_0 \in \mathscr{V}^{\max}$ then in the c.l. system we note the well-known property following from the $(A + b\bar{f})$-invariance of $\mathscr{V}^{\max}$ that the subsequent disturbance remains confined to $\mathscr{V}^{\max}$ so that no signal may be seen at the sysem o.p. This implies:

$\mathscr{V}^{\max} \subseteq$ maximal unobservable subspace of the feedback system

On the other hand it is easily checked that:

$$[c, c(A + b\bar{f}), \ldots, c(A + b\bar{f})^{n-m-1}] = C^c \tag{4.5}$$

Whence it follows that

$\mathscr{C}^c \subseteq$ maximal observable subspace of the feedback system

Since $\mathscr{C}^c \oplus \mathscr{V}^{\max} = \mathscr{X}$, the above subspace inequality signs may be replaced with equality. The unobservability of $\mathscr{V}^{\max}$ causes the order of the system to drop down by $\dim \mathscr{V}^{\max} = m$ when feedback $\bar{f} \in \mathbf{F}(\mathscr{V}^{\max})$ is applied, another very well known state-space phenomenon corresponding to the cancellation of the zeros of $c(s)$ by the action of feedback $\bar{f}$.

Remark 4.3

Equation (4.3) puts $a(s)$ on a par with $c(s)$ and so we may determine the subspace $\mathscr{A}$ of the state space on which the zeros of $a(s)$ (the poles of the c.l.

system) are placed. This is done by forming $A(s)$

$$A(s)=\begin{bmatrix} a(s) \\ sa(s) \\ \vdots \\ s^{m-1}a(s) \end{bmatrix}$$

from which we proceed to define A by the now familiar type of identity : $A(s)=AS(s)$. Then by analogy with the foregoing :

$$\mathscr{A}=\ker A$$

It is also to be observed that $\mathscr{A}$ is the $(A+b\bar{f})$-invariant ' complement ' of $\mathscr{V}^{\max}$ for that f solving eqn. (4.1), i.e. $A+b\bar{f}$ is ' reduced ' by the pair of subspaces $(\mathscr{V}^{\max}, \mathscr{A})$ (see Halmos 1958, § 40).

Definition 4.4

The subspace $\mathscr{A}$ just defined is called the *stem* of the system (A, b, c) with respect to $(\mathscr{V}^{\max}, \bar{f}\in\mathbf{F}(\mathscr{V}^{\max}))$. The subspace $\mathscr{A}^0$ corresponding to $a(s)=s^{n-m}$ is called just the *stem* of $\mathscr{V}^{\max}$.

Remark 4.5

We have seen that every non-zero polynomial of degree $m<n$ is the minimum polynomial for a unique associated ' proper ' (A, b)-invariant subspace (of dimension m). By virtue of b being a generator of $\mathscr{X}$ for any A_f, it is easily seen that any non-zero (A, b)-invariant subspace $\mathscr{V}$ such that $b\in\mathscr{V}$ must be the whole space itself i.e. $\mathscr{V}=\mathscr{X}$ in this case and moreover m.p. $\{A+b\bar{f}\,|\,\mathscr{V}^{\max}\}=\delta_f(s)$. Thus we can deduce that the $\mathscr{V}^{\max}$ associated with a system numerator matrix $c(s)$ is necessarily proper. Conversely from the fact (stated in § 3) that on any proper (A, b)-invariant subspace $\mathscr{V}$, which by properness is necessarily of dimension $m<n$, the eigenvalue/eigenvector structure is fixed for all $f\in\mathbf{F}(\mathscr{V})$, this means that the m.p. $\{A+bf\,|\,\mathscr{V}\}=c(s)$ is fixed of degree m and then the last theorem details the simple relations between $c(s)$, the $\mathscr{V}^{\max}$ kernel problem and the eigenvector decomposition of $\mathscr{V}=\mathscr{V}^{\max}$. These observations lead to the following definition.

Definition 4.6†

Every non-zero polynomial $c(s)$ of degree $m<n$ is the minimum polynomial for a unique associated proper (A, b)-invariant subspace $\mathscr{V}$. Conversely any proper (A, b)-invariant subspace $\mathscr{V}$ has a unique associated minimum polynomial $c(s)$. $\mathscr{V}$ is just $\mathscr{V}^{\max}$ for the ker c problem, where $c(s)=cS(s)$. We may accordingly write this correspondence :

$$c(s)\sim\mathscr{V}$$

† The last two results were originally announced for the s.i.s.o. case by Kalman (informal communication January 1977). Kalman reported obtaining them as a corollary of Lemma 10*b* in Kalman *et al.* (1969). Very recently they have been written up and presented by Fessas (1978).

We end this section by giving a proof of a nice property of this equivalence which we shall need later on for the multivariable case.

Corollary 4.7

Let $\mathscr{V}_0 \subset \mathscr{V}^{\max}$ be an (A, b)-invariant subspace. Let $\mathscr{V}^{\max}$ be proper (A, b)-invariant and let $\mathscr{V}_0 \sim c_0(s)$, $\mathscr{V}^{\max} \sim c(s)$. Then $c_0(s) \mid c(s)$.

Proof

We shall verify the corollary beginning from Wonham's definition of an (A, b)-invariant subspace. For

$$\mathscr{V}_0, \mathscr{V}^{\max}: \ A\mathscr{V}_0 \subseteq \mathscr{V}_0 + \{b\}, \quad A\mathscr{V}^{\max} \subseteq \mathscr{V}^{\max} + \{b\}$$

As $\mathscr{V}_0 \subset \mathscr{V}^{\max}$, take a basis $\{u_1, u_2, \ldots, u_r\}$ for $\mathscr{V}_0$ and extend it to a basis $\{u_1, u_2, \ldots, u_m\}$ for $\mathscr{V}^{\max}$. There exist scalars v_i such that:

$$Au_i = bv_i + w_i, \quad w_i \in \mathscr{V}_0, \quad i \in \mathbf{r}$$

and

$$Au_i = bv_i + w_i, \quad w_i \in \mathscr{V}^{\max}, \quad i = r+1, r+2, \ldots, m$$

As $\{u_1, u_2, \ldots, u_m\}$ is a basis for $\mathscr{V}^{\max}$, there exists $\bar{f} \in \mathbb{R}^n$ such that $\bar{f}u_i = -v_i$, $i \in \mathbf{m}$, whence $(A + b\bar{f})u_i = w_i$, $i \in \mathbf{m}$. For this $\bar{f} \in \mathbf{F}(\mathscr{V}^{\max})$ (a little more work would show that for all $f \in \mathbf{F}(\mathscr{V}^{\max})$)

$$(A + b\bar{f})\mathscr{V}^{\max} \subseteq \mathscr{V}^{\max}, \quad (A + b\bar{f})\mathscr{V}_0 \subseteq \mathscr{V}_0$$

Thus $A + b\bar{f}$ is well defined on the non-trivial factor space $\mathscr{V}^{\max}/\mathscr{V}_0$ and therefore will have a minimum polynomial $u(s) = c(s)/c_0(s)$ of degree $\geqslant 1$. This proves the assertion.

This corollary provides a simple characterization of all (A, b)-invariant subspaces contained in a given (A, b)-invariant subspace. Finally we may ask whether given an (A, b)-invariant subspace $\mathscr{V}$ there is any way we can bypass determination of the eigenvalues of $A + b\bar{f}$ for some $\bar{f} \in \mathbf{F}(\mathscr{V})$ or equivalently the zeros of some suitable polynomial matrix and go directly by some *rational* procedure to $c(s)$. This indeed is the case, but the process brings in and interacts with the theory of generalized resultants.

5. Multivariable extension of the *(A, B)*-invariant subspace results

Problem

Again we simply wish to determine all (A, B)-invariant subspaces $\mathscr{V} \subseteq \ker C$, and the class $\mathbf{F}(\mathscr{V})$. In particular we shall be interested in the maximal such $\mathscr{V}$, $\mathscr{V}^{\max}$. First of all we shall need the following definition, lemma and algorithm.

Definition 5.1

A numerator matrix $C(s)$ satisfies the numerator matrix column degree condition (*n.m.c.d.c.*) if

$$\partial_{ci}\{C(s)\} \leqslant v_{m-i+1} - 1, \quad i \in \mathbf{m}$$

or equivalently

$$\partial_{ci}\{C(s)[s^{v_m - v}]_r\} \leqslant v_m - 1$$

Then $C(s)$ is a *valid numerator matrix* if it satisfies this condition.

Lemma 5.2

Let a numerator matrix $C(s)\in\mathbb{R}^{p\times m}[s]$ be given and let $\mathcal{V}$ be an (A, B)-invariant subspace contained in $\ker C$. If $K(s)$ is an arbitrary polynomial matrix dimensioned such that the product $C_1(s)=K(s)C(s)$ exists and is furthermore a valid numerator matrix, then $\mathcal{V}\subseteq\ker C\Rightarrow\mathcal{V}\subseteq\ker C_1$ where C_1 is defined through the identity $C_1(s)=C_1S(s)$.

Proof

The result is obvious when C and C_1 are related by a constant matrix K such that $C_1=KC$ for then $\ker C\subseteq\ker C_1$. More generally, $K(s)$ will be expressible as $K_1S_1(s)$ where $S_1(s)$ is some polynomial matrix of form akin to $S(s)$ but with some set of indices $\{e_i,\ i\in\mathbf{p}\}$ instead of $\{v_{m-i+1},\ i\in\mathbf{m}\}$:

$$S_1(s)=\begin{pmatrix} 1 & s & s & \ldots & s^{e_1} & 0 & \ldots & & & & 0\\ 0 & 0 & 0 & \ldots & 0 & 1 & s & s & \ldots & s^{e_2} \ \ldots & 0\\ 0 & \ldots & & & & & & & & & 0\\ \ldots & & & & & & & & & & \\ 0 & \ldots & & & & & & 1 & s & \ldots & s^{e_p}\end{pmatrix}^{\mathrm{T}}$$

so that

$$C_1(s)=K(s)C(s)=KS_1(s)C(s)$$

$$=K\begin{pmatrix} c_1(s)\\ sc_1(s)\\ \cdots\\ s^{e_1}c_1(s)\\ c_2(s)\\ \cdots\\ \cdots\\ s^{e_p}c_p(s)\end{pmatrix}$$

To complete the proof, it is enough to show that if $\mathcal{V}\subseteq\ker c$ for some row vector c of C, then $\mathcal{V}\subseteq\ker c'$ whenever $c'(s)=sc(s)$ satisfies n.m.c.d.c. Now for any $F\in\mathbf{F}(\mathcal{V})$ the system $\Sigma_F{}^c=(A+BF,\ B,\ c)$ having t.f. $c(s)\delta_F{}^{-1}(s)$ may be thought of in discrete time (and this of course entails no loss of generality) as being just the system $\Sigma_F{}^{c'}=(A+BF,\ B,\ c')$ with its whole o.p. sequence shifted forward by one unit of time (for any given initial state and input sequence). Now any state x_0 set up in $\mathcal{V}$ as an initial condition will have its resultant transient nulled by the output matrix c, clearly also by c', which is to say $\mathcal{V}\subseteq\ker c\Rightarrow\mathcal{V}\subseteq\ker c'$. The gist of this idea will be familiar from the Kalman *et al.* (1969) module theory.

Next, we shall focus attention on full-rank square $(p=m)$ numerator matrices which are the exact analogues of the s.i.s.o. numerators examined in the previous section. As we shall be confirming in this multivariable set-up, they are characterized as governing only *proper* underlying (A, B)-invariant subspaces.

Consider then a general $m\times m$ polynomial matrix $G(s)$ of full rank. In the event of $[G(s)]_r^h$ being singular, we shall need to apply the following algorithm due to Sain (1975), originally inspired by the work of Wolovich (1973), Wang and Davison (1973) and Forney (1975) and restated here in a form adapted to our purposes. First note if $\partial_{ri}G(s)=g_i$, $G(s)$ can always be expressed uniquely in the following manner :

$$G(s)=\operatorname*{diag}_{i\in\mathbf{m}}(s^{g_i})[G(s)]_r^h+\text{similar terms of lower total row degree} \qquad (5.1)$$

where the '*total row degree*' (*t.r.d.*) g of the first term of the r.h.s. is defined as $g=\sum_{i\in\mathbf{m}} g_i$.

Algorithm 5.3

Suppose that $G(s)\in\mathbb{R}^{m\times m}[s]$ is not row-proper i.e. that $[G(s)]_r^h$ is singular, then $G(s)$ can always be systematically reduced to row-proper form by left multiplications with elementary unimodular matrices (Gantmacher 1959), as follows : Define $G^{(0)}(s)=G(s)$ and enter the following sequence of operations : Let $k=0$.

Step 0 : Define $g_i^{(k)}=\partial_{ri}G^{(k)}(s)$.

Step 1 : Determine row-vector $\alpha\in\mathbb{R}^m$ such that

$$\alpha[G^{(k)}(s)]_r^h=0 \qquad (5.2)$$

Let $g_{\max}^{(k)}=\max_{\substack{i\in\mathbf{m}\\ i:\ \alpha_i\neq 0}} g_i^{(k)}$ occurring at row $i_0^{(k)}$ say.

Step 2 : Define polynomial row-vector $\alpha(s)=(\alpha_1(s),\ \dots,\ \alpha_m(s))$ by

$$\alpha_i(s)=\alpha_i s^{g_{\max}^{(k)}-g_i^{(k)}}$$

and let

$$g_0^{(k)}(s)=\alpha(s)G^{(k)}(s)$$

Then $\partial g_0^{(k)}(s)<g_{\max}^{(k)}$. Replace row $i_0^{(k)}$ of $G^{(k)}(s)$ with $g_0^{(k)}(s)$, set $k=k+1$ and call the resulting matrix $G^{(k)}(s)$.

Step 3 : Test whether $[G^{(k)}(s)]_r^h$ is singular. If YES, go to Step 0 ; if NO, put $H(s)=G^{(k)}(s)$ and STOP.

The above row replacement process (Step 2) is clearly equivalent to left multiplication by a unimodular 'elementary operation' :

$$K^{(k)}(s)=\begin{bmatrix} 1 & 0\dots 0 & & 0 & 0\\ 0 & 1 & & & \vdots\\ \vdots & & \ddots & & \vdots\\ \vdots & & 1 & & \vdots\\ \alpha_1(s) & \dots & \alpha_i(s) & \dots & \alpha_m(s)\\ 0 & 0 & 0 & 1 & \\ \vdots & & & & \ddots\\ 0 & 0 & \dots\dots & 0 & 1\end{bmatrix}$$

From the construction $\alpha_i(s)$ is just a constant which may be taken equal to 1 without loss of generality and also $H(s)=K(s)G(s)$ where

$$K(s)=K^{(k-1)}(s) \ldots K^{(1)}(s)K^{(0)}(s)$$

and k takes the value it has on exit from the algorithm.

Note how, on each four-step cycle of the algorithm, the t.r.d. of $G^{(k)}(s)$ is strictly less than the last value so that the algorithm will certainly terminate after a finite number of steps, and the reduction will be complete when we finally have $g=\partial\,\{\det G^{(k)}(s)\}\geqslant 0$ satisfied. This follows from the observation that

$$\det G(s)=\det\,[G(s)]_r^h s^g + \text{terms of lower degree} \tag{5.3}$$

In the theorems that follow, $G(s)$ will be taken to be $C(s)[s^{v_m-v}]$, a polynomial matrix whose system-theoretic significance in the context of decoupling was made clear in Stoyle and Vardulakis (1978 b). Note that generically in the parameter space of those $C(s)$ which are valid numerator matrices $[C(s)[s^{v_m-v}]]_r^h$ will be non-singular and there is consequently no need to apply the reduction procedure. Lastly in this connection, observe the very important fact that the left unimodular operations of the algorithm leave our kernel problem invariant. To be more precise :

Lemma 5.4

Let two valid numerator matrices $C_1(s)$, $C_2(s)$ be related by a unimodular matrix $P(s)$ thus : $C_2(s)=P(s)C_1(s)$. Then any (A, B)-invariant subspace $\mathscr{V}$ is such that $\mathscr{V}\subseteq\ker C_1$ iff $\mathscr{V}\subseteq\ker C_2$ (so in particular $\mathscr{V}_{C_1}{}^{\max}=\mathscr{V}_{C_2}{}^{\max}$).

Proof

Immediate from Lemma 5.2.

Theorem 5.5

Reduce the $m\times m$ polynomial matrix $C(s)[s^{v_m-v}]$ to $\Gamma(s)[s^{v_m-v}]$ by Algorithm 5.3 (if no reduction is necessary, let us still rename $C(s)$ to $\Gamma(s)$).

Let

$$f_i \triangleq \min_{j\in\mathbf{m}}\,\{v_{m+1-i}-1-\partial\Gamma_{ij}(s)\},\quad i\in\mathbf{m} \tag{5.4}$$

where

$$\partial k\triangleq 0,\quad k\in\mathbb{R}$$

f_i is the ' degree deficiency ' of the ith row of $\Gamma(s)$. Construct a polynomial matrix $C^e(s)$ and a matrix C^e in the following straightforward manner analogous to the corresponding construction of Theorem 4.1.

Define

$$Z(s)=\text{block diag}\,[\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_m],\quad \hat{s}_i=(1, s, \ldots, s^{f_i}),\quad i\in\mathbf{m} \tag{5.5}$$

and

$$C^e(s)=Z(s)\Gamma(s) \tag{5.6}$$

Obtain C^e from

$$C^e S(s)=C^e(s) \tag{5.7}$$

Then $\mathscr{V}^{\max}\subseteq\ker C$ is given by

$$\mathscr{V}^{\max}=\ker C^e \tag{5.8}$$

where the latter is of dimension $z=\partial\,\{\det C(s)\}$.

Proof

On account of the non-singularity of $\Gamma(s)[s^{v_m-v}]$, there exists a feedback F solving the decoupling type of equation (Stoyle and Vardulakis 1978 b)

$$[\Gamma(s)[s^{v_m-v}]]_r^h\, \delta_F(s) = \text{diag}\,(s^{f_i+1})\Gamma(s) \tag{5.9}$$

Taking determinants and then degrees on both sides of this equation :

$$n = \sum_{i\in\mathbf{m}} (f_i+1) + \partial\,\{\det C(s)\} \tag{5.10}$$

since $\partial\,\{\det C(s)\} = \partial\,\{\det \Gamma(s)\}$. Also from (5.4) and (5.9)

$$\partial_{ci}\{[s^{f_i}]\Gamma(s)\} = v_{m+1-i} - 1$$

so $C^e(s) = Z(s)\Gamma(s)$ is an $f\times m$ valid numerator matrix, where

$$f = \sum_{i\in\mathbf{m}} (f_i+1)$$

By Lemma 5.4 $\mathscr{V}^{\max} \subseteq \ker C$ is just $\mathscr{V}^{\max} \subseteq \ker \Gamma$ and by Lemma 5.2 $\mathscr{V}^{\max} \subseteq \ker \Gamma$ implies that $\mathscr{V}^{\max} \subseteq \ker C^e$. To show that the 'stripe' matrix C^e has full rank, let us assume the contrary and suppose there is a linear relation holding between its rows c_i^e, $i\in\mathbf{f}$, i.e.

$$\sum_{i\in\mathbf{f}} a_i c_i^e = 0, \quad a_i \in \mathbb{R}$$

Then

$$\left(\sum_{i\in\mathbf{f}} a_i c_i^e\right) S(s) = \sum_{i\in\mathbf{m}} a_i(s) c_i(s) = 0$$

where $a_i(s)$ are polynomials of degree $\leqslant f_i$, and $c_i(s)$, $i\in\mathbf{m}$, are the rows of $C(s)$. This implies that $C(s)$ is singular over $\mathbb{R}(s)$, and this contradiction shows that :

$$\text{rank } C^e = f = n - \partial\{\det C(s)\} \tag{5.11}$$

from (5.10). Hence dim $\{\mathscr{C}^e\}^\perp = n$-rank $C^e = \{\det C(s)\}$. Thus we have shown that

$$\dim \mathscr{V}^{\max} \leqslant \partial\{\det C(s)\} \tag{5.12}$$

At this point we can conclude from a dimensionality argument that $\mathscr{V} = \ker C^e$, for it is well known that the number of invariant zeros, for the reachable system under consideration, is equal to the dimension of the c.l. eigenspace for that class of feedbacks which make the maximal set of c.l. eigenvectors lie in the kernel of C, and this maximal c.l. unobservable subspace is just $\mathscr{V}^{\max}$ (Bengtsson 1973, MacFarlane 1975). However it is certainly instructive to exhibit this eigenspace in a manner consistent with the approach advanced here and generalizing the s.i.s.o. analysis. This is done in the next theorem which poses our problem as one of diagonal decoupling (Stoyle and Vardulakis 1978 b) and goes some of the way towards integrating the decoupling vector concept (Stoyle and Vardulakis 1978 a) into the present theory.

Theorem 5.6

Let f_i be defined from (5.4) and let $\delta_i(s)$, $i\in\mathbf{m}$, be arbitrary monic polynomials of degree f_i+1. Define $\Delta(s) = \text{diag}\,(\delta_i(s))$ and determine feedback $\bar{F}$

uniquely from the 'decoupling equation' (refer to Stoyle and Vardulakis 1978 b)

$$K\delta_F(s) = \Delta(s)\Gamma(s) \tag{5.13}$$

$\Gamma(s)$ is as in Theorem 5.5 and $\bar{G}^{-1} = K = [C(s)[s^{v_m - v}]]_r^h$. Then $\bar{F} \in \mathbf{F}(\mathscr{V}^{\max})$ and with this feedback closing the loop, and input transformation $\bar{G}$, the transfer function becomes $T_{F,\bar{G}}(s) = L^{-1}(s)\Delta^{-1}(s)$ where $\Gamma(s) = L(s)C(s)$, $L(s)$: unimodular.

Proof (Case (i))

In this case $C(s)$ has Smith form

$$\left[\begin{array}{c|c} I_{m-1} & 0 \\ \hline 0 & \det C(s) \end{array}\right]$$

$\Gamma(s)$ therefore has the same form.

As in the diagonal decoupling (see Wolovich and Falb 1969, Stoyle and Vardulakis 1978 b) we first gather up all non-trivial polynomial factors of rows of $\Gamma(s)$ into a diagonal left factor matrix $\Gamma_D(s)$; thus $\Gamma(s)$ factorizes as

$$\Gamma(s) = \Gamma_D(s)\Gamma_R(s) \tag{5.14}$$

where $\Gamma_D(s) = \text{diag}\,[\gamma_i(s)]$, $\gamma_i(s) = \text{g.c.d.}\,\{\Gamma_{ij}(s),\ j \in \mathbf{m}\}$, $i \in \mathbf{m}$, monic.

Now in our current coordinates (refer to the introductory section on notation) the feedback version of the Wolovich–Falb structure theorem for linear m.v. systems reads :

$$[sI - A_F - B]\begin{bmatrix} S(s) \\ \delta_F(s) \end{bmatrix} = 0 \tag{5.15}$$

By the choice of K, the columns on either side of (5.13) may be equated allowing F to be determined uniquely as $\bar{F}$ (Stoyle and Vardulakis 1978 b). Substitution of (5.13) in (5.15) and post-multiplication by the diagonal decoupling matrix $B(s)$ introduced in the afore-mentioned reference yields :

$$[sI - A_F]S(s)B(s) = BK^{-1}\Delta(s)\Gamma_D(s)\det \Gamma_R(s) M(s)^{-1} \tag{5.16}$$

analogously to eqn. (4.3)

$$M(s) = \text{diag}\,[m_i(s)], \quad \text{where } m_i(s) = \text{g.c.d.}\,\{i\text{th col of adj } \Gamma_R(s)\}$$

(Let us for the moment assume that $m_i(s) = 1$, $i \in \mathbf{m}$; generically this is so.) Also $\bar{G} = K^{-1} = (h_1, h_2, \ldots, h_m)$ where h_i is the so-called '*germ*' of the ith decoupling controllability subspace of the system triple (A, B, Γ) (again see Stoyle and Vardulakis 1978 a, b for definitions and interpretations of these terms).

Let us next concentrate attention on the interpretation of the ith column of this last polynomial matrix equation, namely :

$$[sI - A_F]S(s)\beta_i(s) = h_i^c\, \delta_i(s)\gamma_i(s) \det \Gamma_R(s) \tag{5.17}$$

where $h_i^c = Bh_i$ is the generator of the ith decoupling c.s. of (A, B).

Suppose :

$$\det \Gamma_R(s) = \prod_{i \in \mathbf{l}} (s - s_k)^{z_k} \quad \text{with} \sum_{k \in \mathbf{l}} z_k = z_R \triangleq \partial\{\det \Gamma_R(s)\} \tag{5.18 a}$$

$$\gamma_i(s) \det \Gamma_R(s) = \prod_{k \in \mathbf{l}_i} (s - s_k)^{z_k^i} \quad \text{with } l_i \geqslant l, \quad \sum_{k \in \mathbf{l}_i} z_k{}^i \triangleq z_R{}^i \geqslant z_R \quad \text{for } k \in \mathbf{l} \tag{5.18 b}$$

We may now restrict ourselves to the c.s. $\mathscr{R}_i$ having cyclic listing $R_i \sim \beta_i(s)$ and generating function $x_i(s) = S(s)\beta_i(s)$ and we may consider the s.i.s.o. subsystem defined on this reachable subspace $\mathscr{R}_i$ by the triple :

$$\Sigma^i \triangleq \{A + B\bar{F} \mid \mathscr{R}_i, h_i{}^c, \gamma_i\} \tag{5.19}$$

the 'ith decoupling subsystem'; γ_i is the ith row of $\Gamma(s)$.

Comparing the form of (5.17) with (4.3) of the s.i.s.o. result of the previous section, we see that the largest $(A + B\bar{F}, h_i{}^c)$-invariant subspace (indeed, the largest (A, B)-invariant subspace, as $\mathscr{R}_i \cap \mathscr{B} = \{h_i{}^c\}$) contained in $\mathscr{R}_i \cap \ker \gamma_i$ is $\mathscr{V}_{\gamma i}{}^{\max} \sim \gamma_i(s) \det \Gamma_R(s)$. As

$$\mathscr{R}_i \subseteq \bigcap_{\substack{j \in \mathbf{m} \\ j \neq i}} \ker \gamma_j$$

in its capacity as the ith decoupling c.s. we see that $\mathscr{V}_{\gamma i}{}^{\max}$ is a proper (A, B)-invariant subspace contained in $\ker \Gamma = \bigcap_{j \in \mathbf{m}} \ker \gamma_j$. Using Corollary 4.7, let $\mathscr{V}_{\Gamma_R} \sim \det \Gamma_R(s)$ in Σ^i, then we assert that $\mathscr{V}_{\Gamma_R}$ is the largest (A, B)-invariant subspace contained in $\ker \Gamma_R$ i.e. that $\mathscr{V}_{\Gamma_R} = \mathscr{V}_{\Gamma_R}{}^{\max}$.

To appreciate this last statement, recall from the proof of Theorem 4.1 that

$$\left. \begin{aligned} \mathscr{V}_{\Gamma_R} &= \sum_{k \in \mathbf{l}}^{\oplus} \mathscr{V}_{\Gamma_R}{}^k \\ \mathscr{V}_{\Gamma_R}{}^k &= \text{span} \left\{ \left. \frac{d^j x_i(s)}{ds^j} \right|_{s = s_k} ; \quad j = 0, 1, \ldots, z_k - 1 \right\} \end{aligned} \right\} \tag{5.20}$$

for any $i \in \mathbf{m}$. (In the last formula there is of course assumed the usual device of pairing generalized eigenvectors for complex s_k.) Thus we have exhibited z_R independent vectors spanning an (A, B)-invariant subspace contained in $\ker \Gamma_R$, whence by Theorem 5.5, $\mathscr{V}_{\Gamma_R} = \mathscr{V}_{\Gamma_R}{}^{\max}$ ($= \dim \ker \Gamma_R{}^c$). Thus we have constructively demonstrated the above assertion.

For any $i, j \in \mathbf{m}$, note that we also have the expression :

$$\mathscr{V}_{\Gamma_R}{}^{\max} = \mathscr{R}_i \cap \mathscr{R}_j \tag{5.21}$$

as clearly by construction $\mathscr{V}_{\Gamma_R}{}^{\max} \subseteq \mathscr{R}_i$ for all $i \in \mathbf{m}$, and $\mathscr{R}_i \subseteq \ker \bar{\Gamma}_{R,i}$, $\mathscr{R}_i \subseteq \ker \bar{\Gamma}_{R,j}$ implies that $\mathscr{R}_i \cap \mathscr{R}_j \subseteq \ker \Gamma_R$. (Recall that $\bar{\Gamma}_{R,i}$ stands for Γ_R with its ith row omitted.)

Let us finally enumerate the remaining '(A, B)-generalized eigenvectors' (Karcanias and Kouvaritakis 1977) which are in $\ker C$ but which are not elements of $\mathscr{V}_{\Gamma_R}{}^{\max}$. From (5.17), (5.18) and the s.i.s.o. analysis, these occur associated with the real root s_k of det $C(s)$ in the decoupling c.s. $\mathscr{R}_i$ if $z_k^i > z_k$ for $k \in \mathbf{l}$ or if $k \notin \mathbf{l}$, and are given by :

$$\frac{1}{(j-1)} \frac{d^j x_i(s)}{ds^j}\bigg|_{s=s_k}, \quad j = z_k, z_{k+1}, \ldots, z_{k-1}{}^i \quad \text{for } k \in \mathbf{l} \tag{5.22 a}$$

$$\frac{1}{(j-1)} \frac{d^j x_i(s)}{ds^j}\bigg|_{s=s_k}, \quad j = 0, 1, \ldots, z_k - 1 \quad \text{for } k \notin \mathbf{l} \tag{5.22 b}$$

with the customary trivial modification for the case of non-real s_k. Let $\mathscr{V}_i$ be defined as the span of these generalized eigenvectors. We need only check that they are independent for $i \neq j$. To this end consider the class $\mathbf{F}'$ solving the decoupling equation closely related to (5.13)

$$K\delta_{F'}(s) = \Delta'(s)\Gamma_R(s) \tag{5.23}$$

where $\Delta'(s) = \text{diag}\,[\delta'_i(s)]$, $\partial\{\delta'_i(s)\} = f_i + d_i + 1 \triangleq f'_i + 1$, $d_i \triangleq \partial\{\gamma_i(s)\}$. Let us 'factor out' m.p. $(\Gamma_R(s))$ which is just det $\Gamma_R(s)$ itself and let us consider the factor space $\mathscr{R}_i/\mathscr{V}_0$ on which $A + B\bar{F}'$ is well-defined, where we are renaming $\mathscr{V}_{\Gamma_R}{}^{\max}$ to $\mathscr{V}_0$. As $\mathscr{R}_i$, $i \in \mathbf{m}$, are decoupling c.s. for the decouplable system (A, B, Γ) (or equally for (A, B, Γ_R)) they cover the whole space $\mathscr{X}$, whence $\mathscr{R}_i/\mathscr{V}_0$, $i \in \mathbf{m}$, cover $\mathscr{X}/\mathscr{V}_0$ (i.e. $\sum_{i \in \mathbf{m}} \mathscr{R}_i/\mathscr{V}_0 = \mathscr{X}/\mathscr{V}_0$). From the way we have defined the relevant factor spaces,

$$(\Sigma^i)'/\mathscr{V}_0 = \{\overline{A + B\bar{F}'}, h_i{}^c + \mathscr{V}_0, \gamma_i\} \tag{5.24}$$

(where $\overline{A + B\bar{F}'}$ is the map induced by $A + B\bar{F}'$ on the canonical projection $\mathscr{R}_i \xrightarrow{P'} \mathscr{R}_i/\mathscr{V}_0$) is a s.i.s.o. subsystem having t.f. $\gamma_i(s)/\delta'_i(s)$. Since $\delta'_i(s) \in \mathbb{R}[s]$ may be assigned arbitrarily and independently for each $i \in \mathbf{m}$, this is saying that arbitrary pole-placement is possible on $\mathscr{R}_i/\mathscr{V}_0$, $i \in \mathbf{m}$, and it is intuitively clear that this can only be so if all these latter subspaces are independent as vector subspaces of $\mathscr{X}/\mathscr{V}_0$. For $\partial\delta'_i(s) = f'_i + 1 = \dim \mathscr{R}_i/\mathscr{V}_0$, but if $\dim \sum_{i \in \mathbf{m}} \mathscr{R}_i/\mathscr{V}_0 \triangleq r < \sum_{i \in \mathbf{m}} f'_i$ as would be the case if the $\mathscr{R}_i/\mathscr{V}_0$ were not independent, then we would be placing by suitable choice of $F' \in \mathbf{F}'$ $\sum_{i \in \mathbf{m}} (f'_i + 1)$ poles on an r-dimensional $(\overline{A + B\bar{F}'})$-invariant subspace, which is clearly a contradiction by the eigenvalue interpretation of poles that we already have from the above methods.

The subspaces $\mathscr{R}_i/\mathscr{V}_0$ are isomorphic to the 'stem' subspaces $\mathscr{A}'_i \subset \mathscr{X}$ of the s.i.s.o. subsystems $(\Sigma_R{}^i)' \triangleq \{A + B\bar{F}', h_i{}^c, \gamma_{Ri}\}$ in the sense that the diagram :

$$\begin{array}{ccc} \mathscr{A}'_i & \xrightarrow{A + BF'} & \mathscr{A}'_i \\ P' \downarrow & & \downarrow \\ \mathscr{R}_i/\mathscr{V}_0 & \xrightarrow{\overline{A + BF'}} & \mathscr{R}_i/\mathscr{V}_0 \end{array} \qquad \text{where } P' \text{ is 1–1 on } \mathscr{A}'_i \tag{5.25}$$

commutes.

As the eigenvectors associated with the $\gamma_i(s)$ and given by (5.22) are clearly localized on $\mathscr{A}'_i$ and since these latter subspaces are all independent from pulling back along $(P')^{-1}$ all the eigenvectors associated with the zeros of $C(s)$ have therefore been accounted for, either as s.i.s.o. type 'stem eigenspaces' $\mathscr{V}_i$ or as belonging to an eigenspace created by the 'region of intertwining' $\mathscr{V}_0$ of the c.s. $\mathscr{R}_i$, $i \in \mathbf{m}$. Having thus produced a z-dimensional $(A+B\bar{F})$-eigenspace :

$$\mathscr{V} = \mathscr{V}_0 \oplus \sum_{i \in \mathbf{m}}^{\oplus} \mathscr{V}_i \tag{5.26}$$

contained in ker C, we may conclude from Theorem 5.5 that this is just $\mathscr{V}^{\max}$, and so we have demonstrated that $\bar{F} \in \mathbf{F}(\mathscr{V}^{\max})$.

This ends the proof of case (i) of the theorem, except to remark that the non-generic subcases when not all $m_i(s)$ are unity (refer to (5.16)) goes through essentially the same as the case treated ; the proof is slightly longer and more involved, for now

$$\mathscr{V}_{\Gamma_R}{}^{\max} = \mathscr{V}_0 = \sum_{i<j} \mathscr{R}_i \cap \mathscr{R}_j. \tag{5.27}$$

Proof (Case (ii))

In this case $C(s)$ has Smith form diag $[\epsilon_1(s), \epsilon_2(s), \ldots, \epsilon_m(s)]$ and m.p. $(C(s)) = \epsilon_m(s) = \det C(s))$. For a sketch proof, see Stoyle (1980).

Definition 5.7

With reference to Definition 4.4, by analogy we can define the stem $\mathscr{S}_i$ of the ith decoupling c.s. $\mathscr{R}_i$ of the system $\Sigma = (A, B, \Gamma)$ with respect to $(\mathscr{V}^{\max}, \bar{F} \in \bigcap_{i \in \mathbf{m}} (\mathbf{F}(\mathscr{R}_i)) \cap \mathbf{F}(\mathscr{V}^{\max}))$ to be the stem $\mathscr{A}_i \subset \mathscr{X}$ with respect to $(\mathscr{V}^{\max} \cap \mathscr{R}_i, \bar{F})$ of the s.i.s.o. subsystem $\Sigma^i = (A + B\bar{F}, h_i{}^c, \gamma_i)$. Equally, via the canonical projection $P : \mathscr{X} \to \mathscr{X}/\mathscr{V}^{\max}$, we can identify the stem $\mathscr{S}_i$ with the factor space $\mathscr{S}_i/\mathscr{V}^{\max}$ which is itself isomorphic to $\mathscr{R}_i/\mathscr{V}^{\max}$ equipped with the appropriate induced map $A + B\bar{F}$. This latter can serve as the definition of $\mathscr{S}_i$ in the non-cyclic case too.

Definition 5.8

With reference to the theorem above, define the *region of intertwining* of the c.s. $\mathscr{R}_i$, $i \in \mathbf{m}$, as the (A, B)-invariant subspace

$$\mathscr{V}_0 = \sum_{i<j} \mathscr{R}_i \cap \mathscr{R}_j \quad (= \mathscr{V}_{\Gamma_R}{}^{\max, \text{cyc}}) \tag{5.28}$$

Remark 5.9

We record an additional interpretation of the full set of m stems : $\{\mathscr{S}_i{}^0, i \in \mathbf{m}\}$ as the set of elementary c.s. of the factored input-to-state system :

$$\Sigma^{i/s}/\mathscr{V}^{\max} \triangleq (A, B + \mathscr{V}^{\max}) \bmod \mathscr{V}^{\max} \tag{5.29}$$

This then implies that the list of decoupling invariants $(f_i + 1)$ (which are well-known (F, G)-invariants of the system) are unique, when ordered, under that class of (left) unimodular transformations leaving $C(s)[s^{v_m - v}]$ in row-proper form, since these indices are now seen as just the Kronecker indices of the system $\Sigma^{i/s}/\mathscr{V}^{\max}$ (alternatively this uniqueness can be seen as a formal consequence of Forney's (1975) main theorem).

Proposition 5.10

Recall from (5.1) that $\Gamma(s)[s^{v_m-v}]$ may be uniquely expressed as :

$$\Gamma(s)[s^{v_m-v}] = \text{diag}\,(s^{g_i})K + \text{terms of lower t.r.d.} \tag{5.30}$$

Then we have the relation :

$$K = [\Gamma(s)[s^{v_m-v}]]_r^h \tag{5.31}$$

Proof

Multiplying (5.13) on the right by $[s^{v_m-v}]$ we get :

$$K\{[s^v]-(A_m+F)S(s)\}[s^{v_m-v}]$$
$$= \Delta(s)\,\text{diag}\,(s^{g_i})[C(s)[s^{v_m-v}]]_r^h + \text{terms of lower t.r.d.} \tag{5.32}$$

i.e. Ks^{v_m} + terms of lower t.r.d. = diag $[\delta_i(s)s^{g_i}]K + \dots$. The result now follows from consideration of the highest degree terms.

Note incidentally that the above device can simultaneously furnish a quick proof of the fact that $K = [C(s)[s^{v_m-v}]]_r^h$ first proved as Proposition 1 and sequel of Stoyle and Vardulakis (1978 b).

At this point there remain three obvious questions to be answered in order to complete the task of carrying over the s.i.s.o. analogy to the m.i.m.o. case.

Question 1

Is the $\bar{F}$ determined by (5.13) the most general element of $\mathbf{F}(\mathscr{V}^{\max})$ as $\Delta(s)$ is allowed to vary freely within the degree constraint : $\deg \delta_i(s) = f_i + 1$ for the monic polynomial $\delta_i(s)$?

Question 2

Does any (A, B)-invariant subspace $\mathscr{V}$ which is proper have an associated $m \times m$ full rank numerator matrix $C(s)$?

Question 3

Given that a $\mathscr{V}^{\max} \subseteq \ker C$ has a whole class of associated numerator matrices, which so far have been seen to be unique only up to left unimodular transformations preserving the row-properness of $\Gamma(s)[s^{v_m-v}]$, is the latter class of admissible left transformations large enough for our purposes, in a sense to be defined ? And can we then go on to define a suitable equivalence class of numerator matrices and a well-defined concept of division for such equivalence classes so as to generalize Corollary 4.7 ?

In answer to Question 1 let us first prove the partial result :

Proposition 5.11

The feedback $\bar{F}$ determined by (5.13) is the most general achieving $\mathscr{V}^{\max}$ through the decoupling c.s. $\mathscr{R}_i$ of the system (A, B, Γ). More precisely stated, if $\mathscr{F} \triangleq \{\bar{F} :\ \bar{F}$ solves (5.13) for some $\Delta(s)\}$ then $\mathscr{F} = \mathbf{F}(\mathscr{V}^{\max}) \cap \left\{ \bigcap_{i \in \mathbf{m}} \mathbf{F}(\mathscr{R}_i) \right\}$.

Proof

On each of the compatible decoupling c.s. $\mathscr{R}_i$ we have seen from the proof of Theorem 5.6 that : $\mathscr{V}_0 \oplus \mathscr{V}_i = \mathscr{V}_{\gamma_i}{}^{\max} \subset \mathscr{R}_i$ where $\mathscr{V}_0 = \mathscr{V}_{\Gamma_R}{}^{\max,\text{cyc}}$ and

$\mathscr{V}_i \sim \gamma_i(s)$ on the associated subsystem Σ^i (see (5.19)) also $\mathscr{V}_{\gamma_i}^{\max, \text{cyc}} \sim \gamma_i(s) \det \Gamma_R(s) m_i^{-1}(s)$. Now combining the corresponding s.i.s.o. result (see proof of Theorem 4.1) with Theorem 4 of Stoyle and Vardulakis (1978 a) we know that : $\bar{F} \in \bigcap_{i \in \mathbf{m}} \mathbf{F}(\mathscr{R}_i) \cap \mathbf{F}(\mathscr{V}^{\max})$ implies that there exist polynomials $\delta_i(s)$ such that $\bar{F}$ solves :

$$\delta_{F}(s)\beta_i(s) = h_i^{c}\, \delta_i(s)\gamma_i(s) \det \Gamma_R(s) m_i^{-1}(s) \quad \text{for all } i \in \mathbf{m}$$

From this relation, by arguments presented in Stoyle and Vardulakis (1978 b) Proposition 1 and Theorem 1, it may be concluded that $\bar{F}$ solves the equation :

$$K\delta_F(s) = \Delta(s) \operatorname{diag} \gamma_i(s) \Gamma_R(s) = \Delta(s)\Gamma(s)$$

The reverse inclusion $\mathscr{F} \subseteq \bigcap_{i \in \mathbf{m}} \mathbf{F}(\mathscr{R}_i) \cap \mathbf{F}(\mathscr{V}^{\max})$ is inherent in the proof of Theorem 5.6 and this then ends the proof.

Theorem 5.12

Let $\mathscr{V}$ be a given proper (A, B)-invariant subspace, then there exists an $m \times m$ output matrix C for which $\mathscr{V} = \mathscr{V}_C^{\max}$.

Proof

Let F be a feedback for which $\mathscr{V}$ is $(A+BF)$-invariant. By the reachability (assumed throughout this paper) of the pair $(A, B) = \Sigma^{i/s}$, $\mathscr{X} = \{A \mid \mathscr{B}\} = \{A_F \mid \mathscr{B}\}$. By the $(A+BF)$-invariance of $\mathscr{V}$, $\mathscr{X}/\mathscr{V} = \{\bar{A}_F \mid \bar{\mathscr{B}}\}$ where $\bar{A}_F$ is the map induced by A_F on $\mathscr{X}/\mathscr{V}$ and $(\bar{\mathscr{B}} = (\mathscr{B} + \mathscr{V})/\mathscr{V}$. The system $(\bar{A}_F, \bar{B}) \triangleq \Sigma^{i/s}/\mathscr{V}$ (sometimes more loosely, but unambiguously written as the pair $(A_F, B+\mathscr{V})$ mod $\mathscr{V}$) is thus well-defined and reachable, therefore there will be a Brunovski canonical form and therefore a set of elementary† c.s. $\mathscr{R}_i^0/\mathscr{V}$ associated with it for some feedback F_0 say. Suppose $\dim \mathscr{R}_i^0/\mathscr{V} = f_i + 1$ and that $\mathscr{R}_i^0/\mathscr{V} \cap \bar{\mathscr{B}} = \{b_i + \mathscr{V}\}$. It is clear from the definitions that

$$\mathscr{R}_i = \text{span}\,\{b_i, A_{F_0} b_i, \ldots, A_{F_0}^{\,n-m+1}\, b_i\}$$

is a c.s. and satisfies (i) $\mathscr{R}_i \cap \mathscr{B} = b_i$ and (ii) $\mathscr{R}_i$, $i \in \mathbf{m}$ are compatible since, for example, $F_0 \in \bigcap_{i \in \mathbf{m}} \mathbf{F}(\mathscr{R}_i)$. In other words the c.s. $\mathscr{R}_i$, $i \in \mathbf{m}$ are cyclically independent. Now by results on the cyclic independence in Stoyle and Vardulakis (1978 b), Theorem 2, F_0 solves an equation of the form :

$$K\delta_{F_0}(s) = \Delta(s)\Gamma(s)$$

for some valid numerator matrix $\Gamma(s)$, a diagonal matrix $\Delta(s)$ with monic polynomial entries and where $K = [\Gamma(s)[s^{v_m - v}]]_r^h$. Furthermore from the interpretations already available from the preceding theory we see that $\mathscr{V} = \mathscr{V}_\Gamma^{\max}$ and $\deg \delta_i(s) = f_i + 1$ when these degrees and these indices are ordered. By the choice of F as the regulating feedback F_0, in fact $\delta_i(s) = s^{f_i + 1}$.

† For a system $\Sigma^{i/s} = (A, B)$ which has been rendered into input-transformed Luenberger canonical form, the set of m ' elementary ' c.s. are understood to be those having decoupling vectors simply e_i, $i \in \mathbf{m}$ where $e_i \in \mathbb{R}^m$ is the unit vector having a 1 in the ith position and zeros elsewhere.

Definition 5.12

A valid numerator matrix $\Gamma(s)$ such that $[\Gamma(s)[s^{v_m-v}]]_r^h$ is row-proper will be said to be a *proper numerator matrix*. *Define* $\mathscr{K}^{\mathscr{V}}(s)$ to be the class of all unimodular transformations acting on the left of $\Gamma(s)$, a proper numerator matrix such that $\mathscr{V}=\mathscr{V}_\Gamma^{\max}$, and preserving the properness property (from Proposition 5.10, equivalent to preserving the row degree indices g_i of $[\Gamma(s)[s^{v_m-v}]$, or the degree deficiency indices f_i). Define $\mathscr{L}^{\mathscr{V}}(s)$ to be the class of all unimodular matrices acting on the left of $\Gamma(s)$ and preserving valid numerator matrix form. Occasionally we need $\mathscr{U}(s)$, the class of all unimodular transformations.

For $\mathscr{K}^{\mathscr{V}}(s)$, $\mathscr{L}^{\mathscr{V}}(s)$ to be well-defined with the superscript $\mathscr{V}$ rather than $\mathscr{V}_\Gamma^{\max}$ we need the following :

Lemma 5.13

The class $\mathscr{L}^{\mathscr{V}}(s)$ of admissible left transformations leaving our kernel problem invariant forms a group (under normal matrix multiplication of $m\times m$ matrices) which is *complete* in the sense that if $\mathscr{V}_{C_1}^{\max}=\mathscr{V}_{C_2}^{\max}$ then there exists $L(s)\in\mathscr{L}^{\mathscr{V}}(s)$ such that :

$$C_2(s)=L(s)C_1(s) \tag{5.33}$$

The class $\mathscr{K}^{\mathscr{V}}(s)$ likewise is a group and is complete in an entirely analogous sense.

Proof

The non-trivial thing to check out is the completeness property. From Theorem 5.5, $\mathscr{V}_{C_2}^{\max}=\mathscr{V}_{C_2}^{\max}$ implies that

$$\mathscr{C}_1^{e}=\mathscr{C}_2^{e} \tag{5.34}$$

or in matrix terms

$$C_1^{e}=K^{e}C_2^{e} \tag{5.35}$$

where C_1^e, C_2^e are written down as in (5.6) from some proper $\Gamma_1(s)$, $\Gamma_2(s)$ respectively obtained from equations

$$\Gamma_j(s)=L_j(s)C_j(s), \quad j=1,2\,; \quad L_j(s)\in\mathscr{L}^{\mathscr{V}}(s) \tag{5.36}$$

Furthermore

$$K^{e}=\begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_{f^1} \end{bmatrix}, \quad k_i \text{ are row vectors in } R^{f^2}$$

$\{f_i^j, i\in\mathbf{m}\}$ are the decoupling indices for $j=1,2$ of $\Gamma_1(s)$, $\Gamma_2(s)$ respectively, and $f^j=\sum_{i\in\mathbf{m}}(f_i^j+1)$. Selecting the rows indexed :

$$1,\ f_1^1+2,\ f_1^1+f_2^1+3,\ \ldots,\ \sum_{i\in\mathbf{m}-1} f_i^1+m$$

of eqn. (5.35) in that order and writing them as the rows of a matrix : $\Gamma_1=KC_2^e$; K, $m\times f^2$. Multiplying this last relation on the right by $S(s)$ we derive

$$\Gamma_1(s)=KC_2^{e}(s)=KZ_2(s)\Gamma_2(s)=K(s)\Gamma_2(s) \tag{5.37}$$

where $Z_i(s)$, $i=1, 2$ is defined after the manner of $Z(s)$ (viz. eqn. (5.5)). Since $\det \Gamma_1(s) = \det \Gamma_2(s)$ $(=\det C_1(s) = \det C_2(s))$ we deduce that the polynomial matrix $K(s)$ is unimodular and clearly then $K(s) \in \mathscr{K}^{\mathscr{V}}(s)$. This in effect proves that $\mathscr{K}^{\mathscr{V}}(s)$ is complete, and the extension to $\mathscr{L}^{\mathscr{V}}(s)$ is trivial via (5.36).

We have now verified that every proper (A, B)-invariant subspace $\mathscr{V}$ can be associated with a unique equivalence class $\mathscr{C}^{\mathscr{V}}(s)$ of which $C(s)$ is a member if $\mathscr{V} = \mathscr{V}_C{}^{\max}$, and the equivalence relation defined is: $C_1(s) \sim C_2(s)$ if there exists $L(s) \in \mathscr{L}^{\mathscr{V}}(s)$ such that: $C_1(s) = L(s) C_2(s)$. Let us also define $C_0{}^{\mathscr{V}}(s)$ to be the subclass of $\mathscr{C}^{\mathscr{V}}(s)$ comprising the proper members of $\mathscr{C}^{\mathscr{V}}(s)$ with the induced equivalence relation.

$\Gamma_1(s) \sim \Gamma_2(s)$ if there exists $K(s) \in \mathscr{K}^{\mathscr{V}}(s)$ such that: $\Gamma_1(s) = K(s)\Gamma_2(s)$.

Definition 5.14

With reference to the preceding remarks, there is a 1–1 correspondence between proper (A, B)-invariant subspaces and numerator classes. Taking the cue from Definition 4.6, write this as:

$$\mathscr{C}^{\mathscr{V}}(s) \sim \mathscr{V} (= \mathscr{V}_C{}^{\max})$$

where the subscript C and superscript $\mathscr{V}$ are optional when the context is unambiguous. We can also very easily define a right division of equivalence classes as follows. Let $C_1(s) \in \mathscr{C}^1(s)$, $C_2(s) \in \mathscr{C}^2(s)$, then say $\mathscr{C}^1(s) | \mathscr{C}^2(s)$ if $C_1(s) | C_2(s)$. It is trivial to check that this division is well-defined, and now we can state

Theorem 5.15 (Multivariable extension of Corollary 4.7)

Let $\mathscr{V}_1$, $\mathscr{V}_2 \subseteq \mathscr{V}_1$ be (A, B)-invariant subspaces and let $\mathscr{V}_j \sim \mathscr{C}^j(s)$, $j=1, 2$. Then $\mathscr{C}^2(s) | \mathscr{C}^1(s)$ in the sense of the preceding definition. Conversely if $\mathscr{C}^2(s) | \mathscr{C}^1(s)$, and $\mathscr{C}^j(s) \sim \mathscr{V}_j$ for $j=1, 2$ then $\mathscr{V}_2 \subseteq \mathscr{V}_1$.

Proof

$(\Rightarrow)$ Suppose that $\mathscr{V}_2 = \mathscr{V}_{C_2}{}^{\max} \subseteq \mathscr{V}_{C_1}{}^{\max} = \mathscr{V}_1$, or equivalently that

$$\mathscr{V}_{\Gamma_2}{}^{\max} \subseteq \mathscr{V}_{\Gamma_1}{}^{\max} \quad \text{where} \quad \Gamma_j(s) \in \mathscr{C}_0{}^j(s), \quad j=1, 2 \tag{5.38}$$

Taking orthogonal complements by Theorem 5.5, this is equivalent to

$$\mathscr{C}_1{}^{\mathrm{e}} \subseteq \mathscr{C}_2{}^{\mathrm{e}} \tag{5.39}$$

or in matrix terms as in (5.35)

$$C_1{}^{\mathrm{e}} = K^{\mathrm{e}} C_2{}^{\mathrm{e}} \tag{5.40}$$

for some suitably dimensioned K^{e}, from which we derive (5.37) as before, call it now

$$\Gamma_1(s) = K(s)\Gamma_2(s) \tag{5.41}$$

i.e. $\Gamma_2(s) | \Gamma_1(s)$ which implies that $\mathscr{C}^2(s) | \mathscr{C}^1(s)$.

$(\Leftarrow)$ If $\mathscr{C}^2(s) | \mathscr{C}^1(s)$, there exist $\Gamma_j(s) \in \mathscr{C}_0{}^j(s)$ such that $\Gamma_2(s) | \Gamma_1(s)$ i.e. there exists a polynomial matrix $K(s)$ such that $\Gamma_1(s) = K(s)\Gamma_2(s)$

$$\Gamma_1(s)[s^{v_m - v}] = K(s)\Gamma_2(s)[s^{v_m - v}] \tag{5.42}$$

By virtue of the row-properness of $\Gamma_2(s)[s^{v_m-v}]$ and the fact that $\Gamma_1(s)$ is valid numerator matrix it may be seen that {rows of $K(s)$} $\subseteq$ span {rows of $Z_2(s)$} (over $\mathbb{R}$) i.e. $K(s)=KZ_2(s)$ for some real constant matrix K. In fact by similar reasoning

$$Z_1(s)\Gamma_1(s)=C_1{}^{e}(s)=K^{e}(s)\Gamma_2(s) \tag{5.43}$$

for some extension $K^{e}(s)$ of $K(s)$ such that $K^{e}(s)=K^{e}Z_2(s)$. Now we get $\mathscr{V}_{C_2}{}^{\max}\subseteq\mathscr{V}_{C_1}{}^{\max}$ by a simple reversal of the first part of the argument above.

Remark 5.16

Note that the second part of the proof does not actually require that $C_2(s)$ is a valid numerator matrix when we find it is a factor of $C_1(s)$ as long as it can be reduced to one by left unimodular operations. It is easy to check that this latter is ensured as long as $C_2(s)$ can be reduced to a valid numerator matrix by left unimodular operations.

Corollary 5.17

Let two matrices $P_1(s)$, $P_2(s)$ be given, both reducible by elements of $\mathscr{U}(s)$ to valid numerator matrices $\Gamma_1(s)$, $\Gamma_2(s)$, then $P_1(s)\,|\,P_2(s)$ if and only if $\mathscr{V}_2\subseteq\mathscr{V}_1$ where $\mathscr{V}_2\sim\Gamma_2(s)$, $\mathscr{V}_1\sim\Gamma_1(s)$.

Proof

Immediate from Theorem 5.15 and the remark following it. Clearly the corollary is also a statement in pure mathematics and can straightforwardly be lifted to this general context.

Theorem 5.18

The most general state feedback F achieving $\mathscr{V}^{\max}\subseteq\ker C$ is obtained by solving for $F=\bar{F}$:

$$K\delta_F(s)=\delta_{F^0}{}^{0}(s)\Gamma(s) \tag{5.44}$$

as F^0 is allowed to vary freely, K is as usual $[\Gamma(s)[s^{v_m-v}]]_{\mathrm{r}}^{\mathrm{h}}$

$$\delta_{F^0}{}^{0}(s)=\operatorname*{diag}_{i\in\mathbf{m}}(s^{f_i})-F^0Z(s) \tag{5.45}$$

where f_i, $i\in\mathbf{m}$ are the decoupling indices of the system (A, B, Γ) and $Z(s)$ is as in Theorem 5.5.

Proof

As a direct consequence of Theorem 5.12 the system $(A, B+\mathscr{V})$ mod $\mathscr{V}$ is well-defined reachable system having Kronecker indices : f_{m+1-i} w.l.o.g. with the f_i assumed ordered $f_1\leqslant f_2\leqslant\ \dots\ \leqslant f_m$. We have seen from the special case of Theorem 5.6 that pole-placement is being achieved on each of the independent elementary c.s. (see footnote to Theorem 5.12) of this system, and that for the restricted class $\mathscr{F}$ of feedbacks solving (5.13) the closed loop system denominator matrix is $\Delta(s)=\operatorname{diag}\{\delta_i(s)\}$. By direct analogy with the Wolovich–Falb structure theorem for the original system with Kronecker indices : v_{m+1-i}, $i\in\mathbf{m}$, we see that the most general form of closed loop denominator matrix is that given by (5.45). This is of course more general than that initially suggested by way of answer to Question 1 above. By consideration of highest degree terms (viz. Proposition 5.10 and eqn. 5.32) it may be seen that K is as before. Furthermore degree considerations permit F^0 to be restricted to one having covariant form (Dickinson 1976) without any loss of generality, for the factored system. This may be done if the latter is desired in canonical form after cancellation of the numerator matrix.

Lastly to check that $\bar{F}$ as determined by (5.44) is indeed an element of $\mathbf{F}(\mathscr{V}^{\max})$, we can rederive (5.17) in the modified form :

$$[sI - A_F]S(s)\beta_i(s) = i\text{th column } \{\bar{G}\delta_{F^{o0}}(s)\}\gamma_i(s) \det \Gamma_R(s) \tag{5.46}$$

where again $\bar{G} = K^{-1}$. Therefore all the analysis of Theorem 5.6 exhibiting $\mathscr{V}^{\max}$ as the region of intertwining of decoupling c.s. together with (in the non-generic case of $\gamma_i(s) \neq 1$) certain stem subspaces, with the non-cyclic $C(s)$ as a limiting case, goes through as before, showing finally that $\bar{F} \in \mathbf{F}(\mathscr{V}^{\max})$. Finally observe that for each distinct F^0 we obtain a distinct solution F.

Remark 5.19 (Cancellation of a subset of invariant zeros)

Let $\mathscr{V}_1$, $\mathscr{V}_2$, $\mathscr{V}_2 \subseteq \mathscr{V}_1$ be as in Theorem 5.15, with $\mathscr{C}^2(s) | \mathscr{C}^1(s)$. Let

$$\Gamma_j(s) \in \mathscr{C}_0{}^j(s), \quad j = 1, 2$$

then from (5.41)

$$\Gamma_1(s) = K(s)\Gamma_2(s) \quad \text{where } K(s) = KZ_2(s)$$

The form of the latter expression implies that

$$\partial_{ci}\{K(s)\} \leqslant f_{m-i+1}$$

From Theorem 5.17 if $\bar{F} \in \mathbf{F}(\mathscr{V}_2)$ then F solves an equation like (5.44) :

$$K_2\delta_F(s) = \delta_{F^{o0}}(s)\Gamma_2(s), \quad K_2 \triangleq [\Gamma(s)[s^{v_m - v}]]_r^h \tag{5.47}$$

where

$$\partial_{ci}\{\delta_{F^{o0}}(s)\} = f_{m-i+1}$$

whence

$$\Gamma_1(s)\delta_F{}^{-1}(s)K_2{}^{-1} = K(s)(\delta_{F^{o0}}(s))^{-1} \tag{5.48}$$

Thus a Popov–Wolovich type of matrix fraction decomposition with lower controllability indices and order than the original system is indeed reproduced on feedback-induced cancellation of the zeros of $\Gamma_2(s)$ corresponding to the subspace $\mathscr{V}_2$ of the state-space being rendered unobservable by state feedback (this is Wolovich's (1974 b) result in rather sharper form).

Let us observe that splitting of a numerator matrix $C_1(s)$ rendered in proper form $\Gamma_1(s)$ becomes necessary for example when $C_1(s)$ has zeros in the r.h.h.p. (or more generally ' bad zeros '—Wonham 1974). Suppose $\Gamma_1(s)$ factorizes as

$$\Gamma_1(s) = K^{\text{bad}}(s)\Gamma_2{}^{\text{good}}(s) \tag{5.49}$$

where $\Gamma_2{}^{\text{good}}(s)$ is a proper numerator matrix having as zeros only those zeros of $C_1(s)$ designated as ' good ' (i.e. lying in some chosen region of the complex plane, for instance the l.h.h.p.—in which case $\Gamma_2{}^{\text{good}}(s)$ is denoted $\Gamma_2{}^{\text{stab}}(s) \sim \mathscr{V}^{\text{stab}}$). It is then permissible to cancel $\Gamma_2{}^{\text{good}}(s) \sim \mathscr{V}^{\text{good}}$ say via an (F, G) pair, $F \in \mathbf{F}(\mathscr{V}^{\text{good}})$. Any further cancellation involving any zeros of $K^{\text{bad}}(s)$ leads, as is well-known, to an ill-conditioned closed-loop system.

Finally, $\delta_{F^{o0}}(s)$ in (5.47) or (5.44) will itself have to be selected to have suitable 'good' poles in the l.h.h.p. For a method of doing this, see Stoyle (1980) where pole-placement techniques are fully treated.

Remark 5.20 (Canonical form for $\Gamma(s)\in\mathscr{C}_0^{\mathscr{V}}(s)$)

There is a certain degree of freedom stemming from the arbitrariness in the choice of representative of $\mathscr{C}_0^{\mathscr{V}}(s)$. Instead of solving (5.44) with $\Gamma(s)\in\mathscr{C}_0(s)$ we could solve (5.44′) for (F', G') with $G'=(K')^{-1}$

$$K'\delta_{F'}(s)=\delta_{F^{o0}}(s)\Gamma'(s) \tag{5.44′}$$

where $\Gamma'(s)\in\mathscr{C}_0(s)$ is distinct from $\Gamma(s)$ but related by

$$\Gamma'(s)=K(s)\Gamma(s), \quad K(s)\in\mathscr{K}^{\mathscr{V}}(s)$$

It is natural to enquire whether there is an easily obtainable canonical member of each class $\mathscr{C}_0^{\mathscr{V}}(s)$. Now Forney's (1975) construction of 'echelon form', with Forney operations limited to those of Step 3 of his reduction algorithm, can serve to produce such a canonical member of $\mathscr{C}_0(s)$ i.e. there is a unique member $\hat{\Gamma}(s)$ of $\mathscr{C}_0(s)$ such that

(i) $g_1\leqslant g_2\leqslant \dots \leqslant g_m$
 where $g_i\triangleq\partial_{ri}\{\hat{\Gamma}(s)[s^{v_m-v}]\}$, $\Gamma(s)\in\mathscr{C}^0(s)$.

(ii) The polynomials $\gamma_{ii}(s)$ are monic of degree g_i
 where $\gamma_{ij}(s)\triangleq\{i, j\}$th element of $\hat{\Gamma}(s)$.

(iii) For any i, i' such that $i>i'$

$$\partial\gamma_{i,i'}(s)<g_i$$

Now if we solve an equation such as (5.44) for $\hat{\Gamma}(s)$ in place of $\Gamma(s)$ the reader may easily check that conditions (i)–(iii) imply that $\hat{K}^{-1}$ and hence $\hat{G}$ have upper triangular form with 1's along the diagonal.

Let us remark that this canonical form together with a similar canonical form for the denominator matrix $\delta_F(s)$ derived in Stoyle (1980), are linked to the work of Dickinson (1976) and Wang and Davison (1976) on canonical forms. They can be used to show how an equation of the form of (5.44) may be solved to obtain directly an (F, G) pair such tht $\Sigma_{F,G}\triangleq(A+BF, BG, C)$ is in Luenberger canonical form.

6. Examples

We now give an example which illustrates some of the main points of the foregoing theory, with the notable exception of Algorithm 5.3, which is illustrated in the succeeding example.

Example 6.1

Check that the following system satisfies Wonham's (1974) disturbance decoupling condition with stability (DDPS). The system is specified in

Luenberger canonical form as

$$n=7, \quad v_1=2, \quad v_2=2, \quad v_3=3, \quad p=3=m$$

$$A_m=\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -3 & 0 & 1 \end{pmatrix}; \quad B_m=I_3$$

$$C=\begin{pmatrix} 2 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Synthesize a feedback $\bar{F}$ and i.p.-transformation $\bar{G}$ which decouple a disturbance input $e(t)$ which enters the state equation as :

$$\dot{x}=Ax+Bu+Ee, \quad E=\begin{bmatrix} 1 & 0 & -2 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Let the c.l. system be furthermore constrained to be statically decoupled (Wolovich 1974) and have c.l. poles at $s=-1, -1, -2, -2, -3$.

We briefly remind the reader that Wonham (1974) shows by a compact geometrical argument that the system $\Sigma=(A, B, C)$ can be isolated from the effect of the disturbance $e(t)$ iff

$$\mathscr{V}_C^{\max,\text{prop},\text{stab}} \oplus \mathscr{R}_C^{\max} \supseteq \text{column span } (E) \triangleq \mathscr{E}$$

and a suitable f.b. is chosen from

$$\mathbf{F}(\mathscr{V}^{\max,\text{prop},\text{stab}} \oplus \mathscr{R}_C^{\max})$$

where

$$\mathscr{V}_C^{\max,\text{prop},\text{stab}} = (\mathscr{V}_C^{\max,\text{prop}})^{\text{stab}}$$

However, computing $C(s)$ as

$$C(s)=CS(s)=C\begin{pmatrix} 1 & s & s^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & s & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & s \end{pmatrix}^{\mathrm{T}}$$

$$=\begin{pmatrix} s^2+2s+2 & 1 & 0 \\ 0 & s-2 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

we see that $\mathrm{rank}_{\mathbf{R}(s)}\, C(s)=3$, hence the case treated in § 5 obtains, and $\mathscr{R}_C={}^{\max}\varnothing$ and $\mathscr{V}_C^{\max,\mathrm{prop}}=\mathscr{V}_C^{\max}$. The zeros of $C(s)$ are located at $s=2$, $s=1\pm i$, so in line with Remark 5.19 let us factorize $C(s)$ as $K^{\mathrm{unstab}}(s)C^{\mathrm{stab}}(s)$ in an obvious notation.

$$C(s)=\begin{pmatrix} s^2+2s+2 & 1 & 0 \\ 0 & s-2 & 2 \\ 0 & 0 & 1 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & s-2 & 2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} s^2+2s+2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $C^{\mathrm{unstab}}(s)$ has a r.h.h.p. zero at $s=2$ and $C^{\mathrm{stab}}(s)\sim\mathscr{V}_C^{\max,\mathrm{stab}}$ has l.h.h.p. zeros at $s=1+i$. Now $C^{\mathrm{stab}}(s)$ is already in proper numerator matrix form since

$$[C^{\mathrm{stab}}(s)[s^{v_m-v}]]_{\mathrm{r}}^{\mathrm{h}}=\left[\begin{pmatrix} s^2+2s+2 & s & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix}\right]_{\mathrm{r}}^{\mathrm{h}}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is non-singular, therefore there is no cause to apply Algorithm 5.3 to reduce $C(s)$ to a proper numerator matrix, and $C^{\mathrm{stab}}(s)=\Gamma^{\mathrm{stab}}(s)$. Next eqn. (5.4) determines the degree deficiency indices of $\Gamma^{\mathrm{stab}}(s)$ as : $f_1=0$, $f_2=f_3=1$ which then permit the C^{c} matrix for $C^{\mathrm{stab}}(s)$ to be written down directly from $\Gamma^{\mathrm{stab}}(s)$ (see (5.6), (5.7)) :

$$(C^{\mathrm{stab}})^{\mathrm{c}}=\left[\begin{array}{ccc|cc|cc} 2 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

It is immediately verified that $(C^{\mathrm{stab}})^{\mathrm{c}}E=0$ i.e. $\ker\,(C^{\mathrm{stab}})^{\mathrm{c}}=\mathscr{V}^{\max,\mathrm{stab}}\supseteq\mathscr{E}$, which verifies Wonham's condition above, so that DDPS is indeed solvable here. Let us choose $\bar{F}\in\mathbf{F}(\mathscr{V}^{\max,\mathrm{stab}})$ to place c.l. poles as desired by solving (5.44) :

$$K\delta_F(s)=\delta_{F^{\mathrm{o}0}}(s)\Gamma^{\mathrm{stab}}(s)$$

for F, with $\delta_{F^{\mathrm{o}0}}(s)$ chosen as :

$$\delta_{F^{\mathrm{o}0}}(s)=\begin{pmatrix} s+3 & 0 & 0 \\ 0 & (s+1)^2 & \alpha \\ 0 & 0 & (s+2)^2 \end{pmatrix}$$

Obtain K as $K=[\Gamma^{\mathrm{stab}}(s)[s^{v_m-v}]]_{\mathrm{r}}^{\mathrm{h}}=I_3$ from above.

α is included in order to satisfy the static decoupling condition (where $\bar{G}=K^{-1}=I_3$)

$$\lim_{t\to\infty}\{\text{c.l. step response}\}=\lim_{s\to 0} sT_{F,\bar{G}}(s)(1/s)$$

$$=C(0)\delta_F^{-1}(0)\bar{G}=K^{\text{unstab}}(0)(\delta_{F^{\circ}}{}^{0}(0))^{-1}\bar{G}$$

$$=\begin{pmatrix}1 & 0 & 0\\ 0 & -2 & 2\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}3 & 0 & 0\\ 0 & 1 & \alpha\\ 0 & 0 & 4\end{pmatrix}^{-1}$$

$=$ a diagonal matrix when α is given the value -1.

$$\delta_F(s)=\begin{pmatrix}s^3 & 0 & 0\\ 0 & s^2 & 0\\ 0 & 0 & s^2\end{pmatrix}-(A_m+\bar{F})S(s)=K^{-1}\delta_{F^{\circ}}{}^{0}(s)\Gamma^{\text{stab}}(s)$$

$$=\begin{pmatrix}s^3+5s^2+8s+6 & s+3 & 0\\ 0 & s^2+2s-2 & -1\\ 0 & 0 & s^2+4s+4\end{pmatrix}$$

hence

$$\bar{F}S(s)=\begin{pmatrix}-5s^2-8s-7 & -s-2 & 0\\ -2 & -3s-1 & 1\\ -s & 3s-1 & -5s-4\end{pmatrix}$$

therefore

$$\bar{F}=\left(\begin{array}{ccc:cc:cc}-7 & -8 & -5 & -2 & -1 & 0 & 0\\ -2 & 0 & 0 & -1 & -3 & 1 & 0\\ 0 & -1 & 0 & -1 & 3 & -4 & -5\end{array}\right)$$

Thus an (F, G)-pair satisfying the problem synthesis conditions has been specified, although it is still far from being unique since the problem is somewhat under-specified ; there remains leeway to accommodate other constraint(s) as well.

Example 6.2

Reduce numerator matrix

$$C(s)=\begin{pmatrix}s+1 & 0 & 1\\ s^2+2 & -1 & s\\ -2s & -2 & 0\end{pmatrix}$$

to proper form, where the system controllability indices are given as $v_1=3$, $v_2=v_1=2$.

Referring to Algorithm 5.3, determine

$$[s^{v_m-v}]=\begin{pmatrix}1&0&0\\0&s&0\\0&0&s\end{pmatrix}$$

then

$$G(s)=C(s)[s^{v_m-v}]=\begin{pmatrix}s+1&0&s\\s^2+2&-s&s^2\\-2s&-2s&0\end{pmatrix}$$

Cycle 1

$$k=0,\ G^{(0)}(s)=G(s),\ [G^{(0)}(s)]_r^h=\begin{pmatrix}1&0&1\\1&0&1\\-2&-2&0\end{pmatrix}\text{ singular.}$$

Step 0. $g_1^{(0)}=1,\ g_2^{(0)}=2,\ g_3^{(0)}=1.$

Step 1. $(1\ \ -1\ \ 0)[G^{(0)}(s)]_r^h=0$, so $\alpha=(1\ \ -1\ \ 0)$, $g_{\max}^{(0)}=2$, $i_0^{(0)}=2$.

Step 2. $\alpha_1(s)=1\times s^1=s,\ \alpha_2(s)=-1\times s^0=-1,\ \alpha_3(s)=0\times s^0=0.$
$g_0^{(0)}(s)=(s\ \ -1\ \ 0)\ G^{(0)}(s)=(s-2\ \ s\ \ 0)$—replaces row $i_0^{(0)}=2$ of $G^{(0)}(s)$.

$$\text{Set } k=1,\ G^{(1)}(s)=\begin{pmatrix}s+1&0&s\\s-2&s&0\\-2s&-2s&0\end{pmatrix}$$

Step 3. $[G^{(1)}(s)]_r^h=\begin{pmatrix}1&0&1\\1&1&0\\-2&-2&0\end{pmatrix}$ which is still singular so enter :

Cycle 2

Step 0. $g_1^{(1)}=1,\ g_2^{(1)}=1,\ g_3^{(1)}=1.$

Step 1. $(0\ \ 2\ \ 1)[G^{(1)}(s)]_r^h=0$ so $\alpha=(0\ \ 2\ \ 1)$, $g_{\max}^{(1)}=1$, $i_0^{(1)}=3$ (could equally well be 2).

Step 2. $\alpha_1(s)=0\times s^0=0,\ \alpha_2(s)=2\times s^0=2,\ \alpha_3(s)=1\times s^0=1.$
$g_0^{(1)}(s)=(0\ \ 2\ \ 1)G^{(1)}(s)=(-4\ \ 0\ \ 0)$—replaces row $i_0^{(1)}=3$ of $G^{(1)}(s)$.

$$\text{Set } k=2,\ G^{(2)}(s)=\begin{pmatrix}s+1&0&s\\s-2&s&0\\-4&0&0\end{pmatrix}.$$

Siep 3. $[G^{(2)}(s)]_r^h = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -4 & 0 & 0 \end{pmatrix}$ which is at last non-singular so set

$$H(s) = G^{(2)}(s) = \Gamma(s)[s^{v_m - v}] \text{ and STOP.}$$

Thus

$$\Gamma(s) = \begin{pmatrix} s+1 & 0 & 1 \\ s-2 & 1 & 0 \\ -4 & 0 & 0 \end{pmatrix}$$

the desired proper numerator matrix.

We may write down the two Sain operations involved as

$$K^{(0)}(s) = \begin{pmatrix} 1 & 0 & 0 \\ s & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad K^{(1)}(s) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}.$$

$$K(s) = K^{(1)}(s)K^{(0)}(s) = \begin{pmatrix} 1 & 0 & 0 \\ s & -1 & 0 \\ 2s & -2 & 1 \end{pmatrix}, \quad \Gamma(s) = K(s)C(s)$$

7. Conclusion

The paper has examined in some detail the intimate relation existing between the so-called proper (A, B)-invariant subspaces $\mathscr{V} \subseteq \ker C$ and system matrix fraction decompositions bound up with the feedback version of the Wolovich–Falb (1969) structure theorem for linear time-invariant multivariable systems. A conjecture of Bhattacharyya (1975) regarding the extension of his algorithm for the rapid calculation of the maximal (A, B)-invariant subspace $\mathscr{V}^{\max} \subseteq \ker C$ to non-decouplable systems has been solved in complete generality. For decouplable systems Bhattacharya's result in effect states that

$$\mathscr{V}^{\max} = \ker \begin{pmatrix} c_1 \\ c_1 A \\ \vdots \\ c_m A^{f_m} \end{pmatrix} = \ker C^c$$

by straight computation in our current coordinates. In this way Remark 4.2 is easily generalized to the multivariable case.

For the square full rank systems considered here the disturbance decoupling problem with stability (DDPS) (Wonham 1970, 1974) can now be solved by the choice of a stable $\delta_{\mu'^0}{}^0(s)$ (viz. eqn. (5.44) in essentially the same way as the decoupling vector solution of the restricted diagonal decoupling problem (RDP)

Stoyle and Vardulakis (1978 b), for the two problems are seen to be very closely related.

The methods will be seen to be rich in direct system-theoretic interpretations and in most cases lead to a more compact solution of a given geometrically-posed problem than a purely geometric approach. More examples are to be found in Stoyle (1980) where the theory is fully extended using similar c.s.-theoretic analysis to non-square systems, with application to minimal-order observers and to general problems of non-interacting control.

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