[15] J. C. Doyle, "Robustness of multiloop linear feedback systems," in *Proc. 1978 IEEE Conf. on Decision and Contr.*, San Diego, CA, Jan. 10–12, 1979.
[16] G. Stein and J. C. Doyle, "Singular values and feedback: Design examples," in *Proc. Allerton Conf. on Commun. Contr., and Comput.*, Monticello, IL, Oct. 4–6, 1978.
[17] N. R. Sandell, "Robust stability of multivariable feedback systems," in *Proc. Allerton Conf. on Commun. Contr., and Comput.*, Monticello, IL, Oct. 4–6, 1978.
[18] A. J. Laub, "Computational aspects of singular value decomposition and some applications," in *Proc. Allerton Conf. on Commun., Contr., and Comput.*, Monticello, IL, Oct. 4–6, 1978.
[19] M. G. Safonov and M. Athans, "On stability theory," in *Proc. IEEE Conf. on Decision and Contr.*, San Diego, CA, Jan. 10–12, 1979.
[20] M. G. Safonov, *Stability and Robustness of Multivariable Feedback Systems.* Cambridge, MA: MIT Press, to be published.
[21] ——"Tight bounds on the response of multivariable systems with component uncertainty," in *Proc. Allerton Conf. on Commun. Contr., and Comput.*, Monticello, IL, Oct. 4–6, 1978.
[22] G. Zames, "On the input-output stability of time-varying nonlinear feedback systems—Part II: Conditions involving circles in the frequency plane and sector nonlinearities," *IEEE Trans. Automat. Contr.*, vol. AC-11, pp. 465–476, 1966.
[23] D. W. Nuzman and N. R. Sandell, Jr., "An inequality arising in robustness analysis of multivariable systems," *IEEE Trans. Automat. Contr.*, vol. AC-24, pp. 492–493, 1979.
[24] M. G. Safonov and M. Athans, "Gain and phase margin for multiloop LQG regulators," *IEEE Trans. Automat. Contr.*, AC-22, pp. 173–178, 1977.
[25] A. G. J. MacFarlane and D. F. A. Scott-Jones, "Vector gain," *Int. J. Contr.*, vol. 29, pp. 65–91, 1979.
[26] L. Zadeh and C. A. Desoer, *Linear System Theory.* New York: McGraw-Hill, 1963.
[27] G. Zames, "Nonlinear operators for system analysis," Res. Lab. for Electron., Mass. Inst. Technol., Cambridge, MA, Tech. Rep. 370, Sept. 1960.
[28] ——, "Functional analysis applied to nonlinear feedback systems," *IEEE Trans. Circuit Theory*, vol. CT-10, pp. 392–404, 1963.
[29] ——, "On the stability of nonlinear, time-varying feedback systems," *Proc. NEC*, vol. 20, pp. 725–730, Oct. 1964.

On the Structure of Maximal (A, B)-Invariant Subspaces: A Polynomial Matrix Approach

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***Abstract*—Given a controllable and observable triple (A, B, C) describing a linear time invariant multivariable system Σ, which gives rise to a full rank transfer function matrix $T_o(s)$, the structure of the maximal (A, B)-invariant subspace contained in $\ker C$ is investigated using a polynomial matrix approach. Thus, certain connections between the geometric and the polynomial matrix approaches to linear system theory are established.**

I. Introduction

A concept of considerable importance in the geometric theory of linear multivariable systems is that of an (A, B)-invariant subspace [1]–[3]. Given a linear multivariable system Σ described by a set of state space equations

$$\dot{x} = Ax + Bu \tag{1a}$$

$$y = Cx \tag{1b}$$

where $A: \mathsf{X} \to \mathsf{X}$, $B: \mathsf{U} \to \mathsf{X}$, $C: \mathsf{X} \to \mathsf{Y}$, and X, U, Y are the state space, input, and output spaces, respectively, it is known that the family $L(A, B; \ker C)$ of (A, B)-invariant subspaces contained in $\ker C$ contains a unique supremal element which we denote by V^{max}. The solution of certain problems in linear multivariable control via the "geometric approach" necessitates the use of V^{max} (e.g., disturbance decoupling problem, output stabilization problem, etc. [3]).

Wonham and Morse described an algorithmic procedure for the computation of V^{max} and the problem of its numerical computation was investigated by Bengtson [4] and more recently by Moore and Laub [5].

In the past few years a number of papers have appeared with the objective of providing connections between the geometric and the polynomial matrix approaches to linear system theory (e.g., [6]–[8]). This paper has a similar objective and its primary purpose is to examine the structure of the maximal (A, B)-invariant subspace V^{max} via a polynomial matrix approach. In more precise terms we investigate the relation between the geometric concept of V^{max} and the algebraic concept of a numerator polynomial matrix $\tilde{C}(s)$ appearing in a special right coprime matrix fraction description (MFD): $\tilde{C}(s)\tilde{D}_o(s)^{-1}$ of the transfer function matrix $T_o(s) = C(sI - A)^{-1}B$ of Σ. We thus bring close together the state space concept of an (A, B)-invariant subspace and certain concepts associated with the polynomial matrix theory of linear systems. In the process, certain structural properties of linear multivariable systems are revealed, and links between the Wonham and Morse geometric theory and the Rosenbrock–Wolovich–Forney polynomial matrix approaches of linear systems are indicated.

In the following $\boldsymbol{m}$ denotes the set of integers $\{1, 2, \cdots, m\}$, $\mathbb{R}(s)$ is the field of rational functions and $\mathbb{R}[s]$ is the ring of polynomials both with coefficients in $\mathbb{R}$, the field of reals. $\text{Rank}_{\mathbb{F}}(\cdot)$ denotes the rank of $(\cdot)$ over the field $\mathbb{F}$ and $\dim_{\mathbb{F}}$ denotes the dimension of a vector space defined over $\mathbb{F}$. The subscript $\mathbb{F}$ will be dropped if the field is clear from the context.

II. Theoretical Background and Preliminary Results

We start by considering a linear multivariable system $\Sigma = (A, B, C)$ described by (1). We let $\dim \mathsf{X} = n$, $\dim \mathsf{U} = m$, $\dim \mathsf{Y} = r$, rank $B = m$, rank $C = r$, and assume that Σ is completely controllable and observable and that $\text{rank}_{\mathbb{R}(s)} T_o(s) = \min(m, r)$. Under the controllability assumption of (A, B) it is then well known that there exists a nonsingular coordinate transformation $\hat{x} = Tx$ such that $TAT^{-1} = \hat{A}$, $TB = \hat{B}$, $CT^{-1} = \hat{C}$, and the pair $(\hat{A}, \hat{B})$ is in the Luenberger controllable companion form [9]–[11]. If by $v_m \geq v_{m-1} \geq \cdots \geq v_1 \geq 1$ we denote the controllability indices[1] of (A, B), and $\hat{B}_m$ is the $m \times m$ (nonsingular) matrix consisting of the $\sigma_i = \sum_{j=1}^{i} v_{m+1-j}$, $i \in \boldsymbol{m}$, ordered rows of $\hat{B}$, then $\hat{B} = \tilde{B}\hat{B}_m$ [11] where $\tilde{B} = \text{block diag}(\tilde{b}_1, \tilde{b}_2, \cdots, \tilde{b}_m)$, $\tilde{b}_j = (0, 0, \cdots 0, 1)^T \in \mathbb{R}^{v_{m+1-j} \times 1}$, $j \in \boldsymbol{m}$ and, according to the Wolovich and Falb [10], [11] "structure theorem," the transfer function matrix $T_o(s)$ of Σ can be written as

$$T_o(s) = C(sI - A)^{-1}B = \hat{C}(sI - \hat{A})^{-1}\hat{B} = \tilde{C}S(s)\tilde{D}_o(s)^{-1} \tag{2}$$

where $\tilde{C} \equiv \hat{C}$, $S(s) = \text{block diag}\cdot[\hat{s}_1(s), \hat{s}_2(s), \cdots, \hat{s}_m(s)]$, $\hat{s}_j(s) = (1, s, s^2, \cdots, s^{v_{m+1-j}-1})^T$, $j \in \boldsymbol{m}$, $\tilde{D}_o(s) = \hat{B}_m^{-1}\delta_o(s)$, $\delta_o(s) = [s^v] - \tilde{A}_m S(s)$, $[s^v] = \text{diag}(s^{v_m}, s^{v_{m-1}}, \cdots, s^{v_1})$, and $\tilde{A}_m$ is the $m \times n$ matrix consisting of the m ordered σ_ith, $i \in \boldsymbol{m}$, rows of $\tilde{A} \equiv \hat{A}$.

Furthermore, if we consider (1) together with a linear state variable feedback control law (l.s.v.f.) given by

$$u = Fx - Gv \tag{3}$$

where $F: \mathsf{X} \to \mathsf{U}$, $G: \mathsf{U} \to \mathsf{U}$, $\det G \neq 0$, and take

$$F = \hat{B}_m^{-1}\tilde{F}T \tag{4}$$

$$G = \hat{B}_m^{-1}\tilde{G} \tag{5}$$

then it can be easily verified that the transfer function matrix $T_{F,G}(s)$ of the closed loop system $\Sigma_{F,G} = (A + BF, BG, C)$ is given by [11]

$$\begin{aligned} T_{F,G}(s) &= C(sI - A - BF)^{-1}BG = \tilde{C}(sI - \tilde{A} - \tilde{B}\tilde{F})^{-1}\tilde{B}\tilde{G} \\ &= \tilde{C}S(s)\delta_{\tilde{F}}(s)^{-1}\tilde{G} = T_{\tilde{F},\tilde{G}}(s) \end{aligned} \tag{6}$$

where $\delta_{\tilde{F}}(s) = [s^v] - (\tilde{A}_m + \tilde{F})S(s) = \delta_o(s) - \tilde{F}S(s)$.

The zeros of Σ have been defined in a number of equivalent ways, e.g., see [12]–[14]. Let $\bar{C}(s) = \tilde{C}S(s)$ be the $r \times m$ polynomial numerator matrix appearing in the right coprime MFD of $T_o(s)$ in (2), then our assumption

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[1] It is assumed that, if necessary, the columns of B have been rearranged so that the diagonal companion blocks $\hat{A}_{ii}$ of $\hat{A}$ have sizes $v_{m+1-i} \times v_{m+1-i}$ and are arranged in a decreasing order.

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that $T_o(s)$ has full rank (over $\mathbb{R}(s)$) implies that $\text{rank}_{\mathbb{R}(s)}\tilde{C}(s)=\min(m,r)$, and we have the following.

Definition 1 [14]: The zeros of Σ or, equivalently (as Σ is assumed to be completely controllable and observable), the zeros of $T_o(s)$ are equal to the zeros of the greatest common divisor (g.c.d.) of all the highest order minors of $\tilde{C}(s)$.

If $r>m$, then it is known [11] that a nonsingular (over $\mathbb{R}(s)$) greatest (common) right divisor (g.c.r.d.) of (the rows of) $\tilde{C}(s)$ can be obtained via elementary row operations, i.e., if $\text{rank}_{\mathbb{R}(s)}\tilde{C}(s)=m$, there exists an $r\times r$ unimodular matrix $U_L(s)$ such that

$$U_L(s)\tilde{C}(s)=\left[\begin{array}{c}\tilde{C}_R(s)\\ \hdashline 0\end{array}\right] \tag{7}$$

where $\tilde{C}_R(s)$ is an $m\times m$ nonsingular (over $\mathbb{R}(s)$) upper right triangular g.c.r.d. of (the rows of) $\tilde{C}(s)$ (see Theorems 2.5.11 and 2.5.16 in [11]).

Equation (7) implies in turn that

$$\tilde{C}(s)=U_L(s)^{-1}\left[\begin{array}{c}\tilde{C}_R(s)\\ \hdashline 0\end{array}\right]=\left[\begin{array}{c:c}L_1(s) & L_2(s)\\ \hdashline L_3(s) & L_4(s)\end{array}\right]\left[\begin{array}{c}I_m\\ \hdashline 0\end{array}\right]\tilde{C}_R(s)$$

$$=P(s)\tilde{C}_R(s) \tag{8}$$

where $P(s)=\begin{bmatrix}L_1(s)\\ L_3(s)\end{bmatrix}$, and whose r rows $p_i(s)^T$ are relatively right prime, i.e., the the g.c.d. of all m-order minors of $P(s)$ is 1. This implies that $\det\tilde{C}_R(s)$ is the g.c.d. of all m-order minors of $\tilde{C}(s)$, and, according to the above definition, the zeros of Σ (or of $T_o(s)$) are given in this case $(r>m)$ by the zeros of $\det\tilde{C}_R(s)$.

In order now to proceed, we will make the following two assumptions, (A1) and (A2), regarding the structure of Σ. These assumptions will allow us to state some preliminary results which will be needed later in order to prove the main results. (A1) We assume that Σ (or, equivalently, $T_o(s)$) has $q\geqslant 0$ *distinct* (finite) zeros: $z_i\in\Lambda_q$, $i\in\boldsymbol{q}$.

Now we define the polynomial matrix

$$K(s)\triangleq\tilde{C}(s)[x^{v_m-v}]\qquad \text{if } r\leqslant m \tag{9}$$

or

$$K(s)\triangleq\tilde{C}_R(s)[x^{v_m-v}]\qquad \text{if } r>m \tag{10}$$

where $[s^{v_m-v}]\triangleq\text{diag}(1,s^{v_m-v_{m-1}},\cdots,s^{v_m-v_1})$, and (A2) we assume that $K(s)$ is *row proper* [11], i.e., that 1) if $r\leqslant m$ the $r\times m$ (real) matrix $K=[\tilde{C}(s)[s^{v_m-v}]]_r^h$ has full (row) rank r; 2) if $r>m$ the $m\times m$ (real) matrix $K=[\tilde{C}_R(s)[s^{v_m-v}]]_r^h$ is nonsingular where $[\;]_r^h$ denotes the matrix with elements in $\mathbb{R}$ consisting of the coefficients of the highest degree s terms in each row of the expression inside the brackets.

Remark 1: If $r=m$ then (A2) 1) is just the decouplability condition[2] [15], [16]. Note also that, for SISO systems $(m=r=1)$, K is a scalar quantity and represents the highest coefficient c_q of the numerator $c(s)=c_o+c_1s+\cdots c_qs^q$ of $T_o(s)$.

Regarding assumption (A2), 1) or 2), it will be shown in the sequel that, even if it is not satisfied, we can still state and prove results similar to the ones presented in the next section.

Considering now the two cases $r\leqslant m$ or $r>m$ separately we define

Case 1: $r\leqslant m$.

1) $$f_i\triangleq\min_{j\in\boldsymbol{m}}\{v_{m-1-j}-1-\deg\cdot c_{ij}(s)\}\qquad i\in\boldsymbol{r} \tag{11}$$

where $c_{ij}(s)$ is the ijth polynomial element of the $r\times m$ matrix $\tilde{C}(s)$.

2) $$\Delta(s)\triangleq\text{diag}[\delta_1(s),\delta_2(s),\cdots,\delta_r(s)] \tag{12}$$

where $\delta_i(s)$, $i\in\boldsymbol{r}$ are arbitrary monic polynomials of $\deg\cdot\delta_i(s)=f_i+1$, $i\in\boldsymbol{r}$.

3) The $(f\times r)\times r$ $(f=\Sigma_{i=1}^r f_i)$ polynomial matrix

$$Z(s)\triangleq\begin{bmatrix}\begin{matrix}1\\ s\\ \vdots\\ s^{f_1}\end{matrix} & & \bigcirc\\ & \ddots & \\ \bigcirc & & \begin{matrix}1\\ s\\ \vdots\\ s^{f_r}\end{matrix}\end{bmatrix}. \tag{13}$$

4) The $(f+r)\times m$ polynomial matrix

$$\tilde{C}_E(s)\triangleq Z(s)\tilde{C}(s) \tag{14}$$

and 5) let $\tilde{C}_E$ be the unique $(f+r)\times n$ matrix satisfying

$$\tilde{C}_E(s)=\tilde{C}_E S(s). \tag{15}$$

Case 2: $r>m$,

1) $$f_i\triangleq\min_{j\in\boldsymbol{m}}\{v_{m+1-j}-1-\deg\cdot c_{Rij}(s)\}\qquad i\in\boldsymbol{m} \tag{16}$$

where $c_{Rij}(s)$ is the ijth polynomial element of the $m\times m$ matrix $\tilde{C}_R(s)$ in (8).[3]

2) $$\Delta(s)\triangleq\text{diag}[\delta_1(s),\delta_2(s),\cdots,\delta_m(s)] \tag{17}$$

where $\delta_i(s)$, $i\in\boldsymbol{m}$ are arbitrary monic polynomials of $\deg\cdot\delta_i(s)=f_i+1$, $i\in\boldsymbol{m}$.

3) The $(f+m)\times m$, $(f=\Sigma_{i=1}^m f_i)$ polynomial matrix

$$Z(s)\triangleq\begin{bmatrix}\begin{matrix}1\\ s\\ \vdots\\ s^{f_1}\end{matrix} & & \bigcirc\\ & \ddots & \\ \bigcirc & & \begin{matrix}1\\ s\\ \vdots\\ s^{f_m}\end{matrix}\end{bmatrix}. \tag{18}$$

4) The $(f+m)\times m$ polynomial matrix

$$\tilde{C}_{RE}(s)=Z(s)\tilde{C}_R(s) \tag{19}$$

and 5) let $\tilde{C}_{RE}$ be the unique $(f+m)\times n$ matrix satisfying

$$\tilde{C}_{RE}(s)=\tilde{C}_{RE}S(s). \tag{20}$$

Finally, in both Cases 1 and 2, by Λ_c, we denote the set of zeros of the $\delta_i(s)$, $i\in\boldsymbol{r}$ $(i\in\boldsymbol{m})$. With respect to the $(f+r)\times n$ matrix $\tilde{C}_E$ defined in 5), Case 1 $(r\leqslant m)$, and the $(f+m)\times n$ matrix $\tilde{C}_{RE}$ defined in 5), Case 2 $(r>m)$, we state the following.

[2] The decouplability condition is also expressed for $r=m$ by (A2), 2) because then in (8) $\tilde{C}_R(s)\equiv\tilde{C}(s)$, $P(s)\equiv I_m$.

[3] Note that if $\tilde{C}_R(s)$ is the g.c.r.d. of (the rows of) $\tilde{C}(s)$ obtained as described in [11] (Theorem 2.5.11), then $\tilde{C}_R(s)$ is an $m\times m$ upper right triangular polynomial matrix which, if $K(s)=\tilde{C}_R(s)[s^{v_m-v}]$ is row proper (i.e., (A2) 2) is satisfied), is such that the degree of its j-*th* column $\tilde{C}_{Rj}(s)$ satisfies $\deg\cdot\tilde{C}_{Rj}(s)\leqslant v_{m+1-j}-1$, $j\in\boldsymbol{m}$. So the f_i defined by (16) are $f_i\geqslant 0$, $i\in\boldsymbol{m}$. A proof of this fact is given in the Appendix.

Proposition 1: 1) if $r\leq m$, then $\operatorname{rank}_{\mathbf{R}(s)}\tilde{C}(s)=r \Leftrightarrow \operatorname{rank}\tilde{C}_E=f+r$, and 2) if $r>m$, then $\operatorname{rank}_{\mathbf{R}(s)}\tilde{C}_R(s)=m \Leftrightarrow \operatorname{rank}\tilde{C}_{RE}=f+m$.

We now give a proof for the case $r\leq m$. A similar proof can be given easily in the other case ($r>m$). Assume that $\operatorname{rank}_{\mathbf{R}(s)}\tilde{C}(s)=r$ but $\operatorname{rank}\tilde{C}_E<f+r$. Then there exists $\alpha^T\in\mathbb{R}^{1\times(f+r)}$

$$\alpha^T=(\alpha_{1o},\alpha_{11},\cdots,\alpha_{1f_1}|\alpha_{2o},\alpha_{21},\cdots,\alpha_{2f_2}|\cdots|\alpha_{ro},\alpha_{r1},\cdots,\alpha_{rf_r})\neq 0$$

such that $\alpha^T\tilde{C}_E=0$. Postmultiplying this equation by $S(s)$, and using (14) and (15) we have

$$\alpha^T\tilde{C}_ES(s)=\alpha^T\tilde{C}_E(s)=\alpha^TZ(s)\tilde{C}(s)=\alpha^T(s)\tilde{C}(s)=0$$

where $\alpha^T(s)=[\alpha_1(s),\alpha_2(s),\cdots,\alpha_r(s)]\neq 0$ and $\deg\cdot\alpha_i(s)\leq f_i$, $i\in\boldsymbol{r}$, i.e., $\operatorname{rank}_{\mathbf{R}(s)}\tilde{C}(s)<r$, contrary to the assumption. The same argument can be reversed, hence the proposition.

Remark 2: If $f_i=0$ for every $i\in\boldsymbol{r}$ ($i\in\boldsymbol{m}$), then $Z(s)=I_r$ ($=I_m$) and, therefore, $\tilde{C}_E\equiv\tilde{C}$ ($\tilde{C}_{RE}\equiv\tilde{C}_R$) where $\tilde{C}_R$ is the unique $m\times n$ matrix which satisfies

$$\tilde{C}_R(s)=\tilde{C}_RS(s). \tag{21}$$

Remark 3: If $r=m$, then it turns out [15], [16] that the f_i's defined via (11) coincide with the so-called "decoupling invariants" of Σ.

With $K(s)$ defined by (9) or (10), depending on whether $r\leq m$ or $r>m$, let $k_i^T(s)=[k_{i1}(s),k_{i2}(s),\cdots,k_{im}(s)]$ $i\in\boldsymbol{r}$ if $r\leq m$ ($i\in\boldsymbol{m}$ if $r>m$) be the ith row of $K(s)$ and $g_i=\deg\cdot k_i^T(s)=\max_{j\in\boldsymbol{m}}\{\deg\cdot k_{ij}(s)\}$, $i\in\boldsymbol{r}$ ($i\in\boldsymbol{m}$), then we have the following.

Lemma 1:

$$g_i=\deg\cdot k_i^T(s)=\upsilon_m-1-f_i \qquad i\in\boldsymbol{r}\text{ if }r\leq m,\ (i\in\boldsymbol{m}\text{ if }r>m). \tag{22}$$

Proof: Let $r\leq m$. From the definition of the f_i's via (11) it follows that $\forall i\in\boldsymbol{r}$ $\deg\cdot c_{ij}(s)=q_{ij}\leq\upsilon_{m+1-j}-1-f_i$ $j\in\boldsymbol{m}$ with equality holding for at least one $j\in\boldsymbol{m}$. Now, with $c_i^T(s)$, the ith row of $\tilde{C}(s)$, $k_i^T(s)=c_i^T(s)[s^{\upsilon_m-\upsilon}]=[c_{i1}(s),c_{i2}(s)s^{\upsilon_m-\upsilon_{m-1}},\cdots,c_{im}(s)s^{\upsilon_m-\upsilon_1}]$ so that

$$\deg\cdot k_{ij}(s)\leq\upsilon_m-1-f_i \qquad i\in\boldsymbol{r},\ j\in\boldsymbol{m} \tag{23}$$

with equality holding for at least one $j\in\boldsymbol{m}$, from which $\deg\cdot k_i^T(s)=\max_{j\in\boldsymbol{m}}\{\deg\cdot k_{ij}(s)\}=\upsilon_m-1-f_i$, $i\in\boldsymbol{r}$. Q.E.D.

A similar proof can be given in the case of $r>m$.

In view of Lemma 1, and if we write $k_i^T(s)=\sum_{j=0}^{g_i}k_{ij}^Ts^j$, $i\in\boldsymbol{p}$ (where $p=r$ if $r\leq m$ and $p=m$ if $r>m$), $K(s)$ can be written as

$$K(s)=\begin{bmatrix} s^{g_1} & & & \\ & s^{g_2} & & \\ & & \ddots & \\ & & & s^{g_p}\end{bmatrix}\begin{bmatrix}k_{1g_1}^T\\ k_{2g_2}^T\\ \vdots\\ k_{pg_p}^T\end{bmatrix}+K_R(s)$$

$$=[s^{\upsilon_m-1-f_i}]K+K_R(s) \tag{24}$$

where

$$[s^{\upsilon_m-1-f_i}]=\operatorname{diag}(s^{\upsilon_m-1-f_1},s^{\upsilon_m-1-f_2},\cdots,s^{\upsilon_m-1-f_p}]. \tag{25}$$

Also, with the notation introduced, we see that $\Delta(s)$ can always be written as

$$\Delta(s)=\operatorname{diag}[\delta_1(s),\cdots,\delta_p(s)]=[s^{f_i+1}]+\Delta Z(s) \tag{26}$$

where $[s^{f_i-1}]=\operatorname{diag}(s^{f_1-1},\cdots,s^{f_p-1})$ and Δ a $p\times(f+p)$ matrix composed from the coefficients of the $\delta_i(s)$, $i\in\boldsymbol{p}$.

Having introduced the above quantities, we are now able to state a number of results that reveal clearly the structure of the maximal $(\tilde{A},\tilde{B})$-invariant subspace $V^{\max}$ in $\ker\tilde{C}$. In the sequel we again distinguish between the two cases of $r\leq m$ and $r>m$ and treat them separately.

III. Structure of $V^{\max}$ Case 1: $r\leq m$

We start with the following.

Proposition 2: Let $r\leq m$ and let assumptions (A1), (A2) (1) be satisfied, then (a) there exists a family $\boldsymbol{F}$ of state feedback matrices $\tilde{F}^*$ such that

$$K\delta_{\tilde{F}^*}(s)=\Delta(s)\tilde{C}(s) \tag{27}$$

where $K=[K(s)]_r^h$, $\Delta(s)$ are defined by (9) and (12), and f_i by (11); (b) 1) for $z_i\in\Lambda_q$, $i\in\boldsymbol{q}$ we have: $\operatorname{rank}\delta_{\tilde{F}^*}(z_i)<m$., i.e., $z_i\in\Lambda_q$ (the zeros of Σ) are eigenvalues of $\tilde{A}+\tilde{B}\tilde{F}^*$, 2) also, for $\lambda_i\in\Lambda_c$, $i\in\boldsymbol{f+r}$, $\operatorname{rank}\delta_{\tilde{F}^*}(\lambda_i)<m$ and, similarly, $\lambda_i\in\Lambda_c$ are eigenvalues of $\tilde{A}+\tilde{B}\tilde{F}^*$.

Proof: (a) Postmultiplying both sides of (27) by $[s^{\upsilon_m-\upsilon}]=\operatorname{diag}[1,s^{\upsilon_m-\upsilon_{m-1}},\cdots,s^{\upsilon_m-\upsilon_1}]$, we see that the left-hand side of (27) becomes

$$\begin{aligned}K\delta_{\tilde{F}^*}(s)[s^{\upsilon_m-\upsilon}]&=K\big[[s^{\upsilon}]-(\tilde{A}_m+\tilde{F}^*)S(s)\big][s^{\upsilon_m-\upsilon}]\\&=K[s^{\upsilon}][s^{\upsilon_m-\upsilon}]-K(\tilde{A}_m+\tilde{F}^*)S(s)[s^{\upsilon_m-\upsilon}]\\&=s^{\upsilon_m}K-K(\tilde{A}_m+\tilde{F}^*)\hat{S}(s)\end{aligned}$$

where

$$\hat{S}(s)=S(s)[s^{\upsilon_m-\upsilon}]=\begin{bmatrix}1 & & & \\ s & & & \\ \vdots & & & \\ s^{\upsilon_m-1} & & & \\ & s^{\upsilon_m-\upsilon_{m-1}} & & \\ & \vdots & & \\ & s^{\upsilon_m-1} & & \\ & & \ddots & \\ & & & s^{\upsilon_m-\upsilon_1}\\ & & & \vdots\\ & & & s^{\upsilon_m-1}\end{bmatrix} \tag{28}$$

while the right-hand side of (27) gives

$$\begin{aligned}\Delta(s)\tilde{C}(s)[s^{\upsilon_m-\upsilon}]&=\big[[s^{f_i-1}]+\Delta Z(s)\big]\tilde{C}(s)[s^{\upsilon_m-\upsilon}]\\&=[s^{f_i+1}]K(s)+\Delta Z(s)\tilde{C}(s)[s^{\upsilon_m-\upsilon}]\\&=[s^{f_i+1}]\big[[s^{\upsilon_m-1-f_i}]K+K_R(s)\big]+\Delta\tilde{C}_E(s)[s^{\upsilon_m-\upsilon}]\\&=s^{\upsilon_m}K+[s^{f_i+1}]K_R(s)+\Delta\tilde{C}_ES(s)[s^{\upsilon_m-\upsilon}]\\&=s^{\upsilon_m}K+M\hat{S}(s)+\Delta\tilde{C}_E\hat{S}(s)\end{aligned}$$

and we made use of (26), (9), (24), Lemma 1, and M is the unique matrix satisfying $[s^{f_i+1}]K_R(s)=M\hat{S}(s)$. So, finally,

$$s^{\upsilon_m}K-K(\tilde{A}_m+\tilde{F}^*)\hat{S}(s)=s^{\upsilon_m}K+(M+\Delta\tilde{C}_E)\hat{S}(s) \tag{29}$$

i.e., the coefficients of s^{υ_m} are equal on both sides of (29) and, since, by assumption, $\operatorname{rank}K=r$, the remaining coefficients can be made equal by the choice of $\tilde{F}^*$. This then guarantees the existence of the family $\boldsymbol{F}$ of $\tilde{F}^*$ that satisfies (27).

(b) 1) Let $z_i\in\Lambda_q$, $i\in\boldsymbol{q}$, then $\operatorname{rank}\tilde{C}(z_i)<r$ and from (a) above: $\operatorname{rank}[K\delta_{\tilde{F}^*}(z_i)]=\operatorname{rank}[\Delta(z_i)\tilde{C}(z_i)]<r$. Now, as K has full (row) rank r, we must have: $\operatorname{rank}\delta_{\tilde{F}^*}(z_i)<m$. The rest follows from the definition of $\delta_{\tilde{F}}^*(s)$ and from the fact that $\det\delta_{\tilde{F}^*}(s)=\det(sI-\tilde{A}-\tilde{B}\tilde{F}^*)$, [10] [11]. 2) Again for $\lambda_i\in\Lambda_c$, $i\in\boldsymbol{f+r}$, $\operatorname{rank}\Delta(\lambda_i)<r$ and, therefore, from (a) above, $\operatorname{rank}\delta_{\tilde{F}^*}(\lambda_i)<m$. Q.E.D.

Remark 4: From (29) the family F of $\tilde{F}^*$ can be obtained via

$$K\tilde{F}^* = -\left[M+\Delta\tilde{C}_E+K\tilde{A}_m\right]. \tag{30}$$

If $L=-[M+\Delta\tilde{C}_E+K\tilde{A}_m]$, then the general solution of the equation $K\tilde{F}^*=L$ is

$$\tilde{F}^*=K^T(KK^T)^{-1}L+\left(I_m-K^T(KK^T)^{-1}K\right)N$$

where N is an arbitrary $m\times n$ matrix. Note that if $r=m$, then K is square and invertible, hence $\tilde{F}^*=K^{-1}L$ is unique. In the case $r<m$, then (30) represents m equations that the mn elements $\tilde{F}^*_{ij}$ of the matrix $\tilde{F}^*$ must satisfy. Hence the number of elements $\tilde{F}^*_{ij}$ of $\tilde{F}^*$ that can be specified arbitrarily and so generate the family F is $(m-r)n$. It can be proved that these $(m-r)n$ elements of $\tilde{F}^*$ can always be chosen so that the remaining $n-(q+f+r)$ eigenvalues of $\tilde{A}+\tilde{B}\tilde{F}^*$ can be assigned arbitrarily (see below).

Consider now $\tilde{C}(s)$ as an $\mathbb{R}(s)$-module morphism, $\tilde{C}(s):\mathbb{R}^m[s]\to\mathbb{R}^r[s]$, and let $B(s)=[\beta_1(s),\beta_2(s),\cdots,\beta_{m-r}(s)]$ be a polynomial basis for $\ker\tilde{C}(s)$, i.e., let $\tilde{C}(s)\beta_k(s)=0$, $k\in\boldsymbol{m-r}$ with $\beta_k\in\mathbb{R}^m[s]$ and independent over $\mathbb{R}(s)$. If we now set

$$\tilde{x}_k(s)=S(s)\beta_k(s) \qquad k\in\boldsymbol{m-r} \tag{31}$$

$$u_k(s)=\delta_o(s)\beta_k(s) \qquad k\in\boldsymbol{m-r} \tag{32}$$

then from the Wolovich and Falb "structure theorem" [10], [11] (see also [17])

$$[sI-\tilde{A},-\tilde{B}]\begin{bmatrix}\tilde{x}_k(s)\\u_k(s)\end{bmatrix}=0 \qquad k\in\boldsymbol{m-r} \tag{33}$$

and if we write: $\tilde{x}_k(s)=S(s)\beta_k(s)=\sum_{i=1}^{d_k}\tilde{x}_{k,i-1}s^{i-1}$, $\tilde{x}_{k,d_k-1}\neq 0$, $d_k-1=\deg\cdot\tilde{x}_k(s)$, $k\in\boldsymbol{m-r}$, then, from (33) and the Warren and Eckberg Lemma [6], it follows that $\mathsf{R}_k=\mathrm{span}\{\tilde{x}_{k,i-1}, i\in\boldsymbol{d_k}\}$, $k\in\boldsymbol{m-r}$ are controllability subspaces (c.s.) of $(\tilde{A},\tilde{B})$, [1]–[3], [6], [7] with $\dim\mathsf{R}_k\leqslant d_k$, $k\in\boldsymbol{m-r}$. Furthermore, from

$$\tilde{C}\tilde{x}_k(s)=\tilde{C}S(s)\beta_k(s)=\tilde{C}(s)\beta_k(s)=0 \qquad k\in\boldsymbol{m-r} \tag{34}$$

it follows that $\tilde{x}_{kj}\subset\ker\tilde{C}$, $k\in\boldsymbol{m-r}$, $j=0,1,\cdots,d_k-1$, i.e., $\mathsf{R}_k\subset\ker\tilde{C}$, $k\in\boldsymbol{m-r}$. The above analysis shows clearly that the $n\times(m-r)$ polynomial matrix $X^*(s)=S(s)B(s)$ with $\mathrm{rank}_{\mathbb{R}(s)}X^*(s)=m-r$, is a polynomial basis of a rational vector space $\mathsf{X}^*(s)$ [18] ($\dim_{\mathbb{R}(s)}\mathsf{X}^*(s)=m-r$), which has the property that every polynomial vector $\tilde{x}(s)\in\mathsf{X}^*(s)$ of $\deg\cdot\tilde{x}(s)=d-1$, if written as $\tilde{x}(s)=\tilde{x}_o+\tilde{x}_1s+\cdots+\tilde{x}_{d-1}s^{d-1}$, defines a c.s. of $(\tilde{A},\tilde{B})$: $\mathsf{R}=\mathrm{span}\{\tilde{x}_{i-1}, i\in\boldsymbol{d}\}$, which is contained in $\ker\tilde{C}$, i.e., the polynomial basis $X^*(s)$ characterizes the family of all c.s.'s R of $(\tilde{A},\tilde{B})$ in $\ker\tilde{C}$. Note that, if $r=m$, then $\dim_{\mathbb{R}(s)}\mathsf{X}^*(s)=0$, i.e., the family of c.s.'s of $(\tilde{A},\tilde{B})$ in $\ker\tilde{C}$ comprises only one element, which is the zero subspace.

It can be proved [7] that there always exists an $(m-r)$-polynomial vector $a(s)=[a_1(s),\cdots,a_{m-r}(s)]^T$ (with $a_i(s)$ nonzero and relatively prime polynomials), such that the polynomial vector

$$\tilde{x}^*(s)\triangleq X^*(s)a(s)=S(s)B(s)a(s)=\sum_{i=1}^{d}\tilde{x}^*_{i-1}s^{i-1} \tag{35}$$

satisfies (1) $\deg\cdot\tilde{x}^*(s)=d-1=n-(q+f+r)-1$ and (2) the vectors $\{\tilde{x}^*_{i-1}, i\in\boldsymbol{d}\}$ are independent. The subspace $\mathsf{R}^{\max}=\mathrm{span}\{\tilde{x}^*_{i-1}, i\in\boldsymbol{d}\}$ is then the maximal controllability subspace of $(\tilde{A},\tilde{B})$ contained in $\ker\tilde{C}$ [1]–[3].

Assume now that $\tilde{x}^*(s)$ satisfies (1) and (2) above, and let

$$u^*(s)\triangleq\delta_o(s)\beta^*(s)=\sum_{i=1}^{d+1}u^*_{i-1}s^{i-1} \tag{36}$$

where $\beta^*(s)=B(s)a(s)$, then, obviously, $\tilde{x}^*(s)$, $u^*(s)$ will satisfy

$$[sI-\tilde{A},-\tilde{B}]\begin{bmatrix}\tilde{x}^*(s)\\u^*(s)\end{bmatrix}=0. \tag{37}$$

If we now consider an arbitrary but symmetric set[4] of points in $\mathbb{C}$ $\Lambda_d=\{s_1,s_2,\cdots,s_d\}$, $d=n-(q+f+r)$, and let $\psi(s)=\Pi_{i=1}^d(s-s_i)=\psi_o+\psi_1s+\cdots+\psi_{d-1}s^{d-1}+s^d$, then we have

Proposition 3: There always exists an $\tilde{F}^*\in F$ such that

$$u^*(s)=\tilde{F}^*\tilde{x}^*(s)+u^*_d\psi(s) \tag{38}$$

[or equivalently such that

$$(u^*_o,u^*_1,\cdots,u^*_{d-1})=\tilde{F}^*(\tilde{x}^*_o,\tilde{x}^*_1,\cdots,\tilde{x}^*_{d-1})+u^*_d(\psi_o,\psi_1,\cdots,\psi_{d-1}).$$

For the proof we shall need the following.

Lemma 2: With $B(s)=[\beta_1(s),\cdots,\beta_{m-r}(s)]$ a polynomial basis for $\ker\tilde{C}(s)$ as above, if we define[5]

$$h_k\triangleq[[s^v]\beta_k(s)]^h_c, \qquad k\in\boldsymbol{m-r} \tag{39}$$

then $\mathrm{span}\{h_1,h_2,\cdots,h_{m-r}\}\subset\ker K$.

Proof: Consider the polynomial matrix $K(s)=\tilde{C}(s)[s^{v_m-v}]$ and note that $\mathrm{rank}_{\mathbb{R}(s)}K(s)=\mathrm{rank}_{\mathbb{R}(s)}\tilde{C}(s)=r$, since $[s^{v_m-v}]$ multiplies columns $1,2,\cdots,m$ of $\tilde{C}(s)$ by $1,s^{v_m-v_{m-1}},\cdots,s^{v_m-v_1}$, respectively. Hence $\dim_{\mathbb{R}(s)}\ker K(s)=m-r$. From

$$\begin{aligned}K(s)[s^v]\beta_k(s)&=\tilde{C}(s)[s^{v_m-v}][s^v]\beta_k(s)\\&=s^{v_m}\tilde{C}(s)\beta_k(s)=0, \qquad k\in\boldsymbol{m-r}\end{aligned} \tag{40}$$

it is easily seen that the $m\times(m-r)$ matrix $[s^v]B(s)$ is a polynomial basis for $\ker K(s)$. Now the result follows immediately by considering (24), (40), and the defining equation of h_k in (39).

Proof of Proposition 3: From (38), using (35) and (36), we easily obtain that existence of an $\tilde{F}^*\in F$ satisfying (38) is equivalent to the existence of an $\tilde{F}^*\in F$ such that

$$\delta_{\tilde{F}^*}(s)\beta^*(s)=u^*_d\psi(s). \tag{41}$$

Premultiplying (41) by K and noting

$$\begin{aligned}u^*_d&=[u^*(s)]^h_c=[\delta_o(s)\beta^*(s)]^h_c\\&=\left[\left([s^v]-\tilde{A}_mS(s)\right)\beta^*(s)\right]^h_c\\&=[[s^v]\beta^*(s)]^h_c=[[s^v]B(s)a(s)]^h_c\\&=\sum_{i=1}^{m-r}h_ia_i\subset\ker K \quad \text{(by Lemma 2)}\end{aligned}$$

we finally see that the existence of an $\tilde{F}^*\in F$ satisfying (38) is equivalent to the existence of an $\tilde{F}^*\in F$ such that

$$K\delta_{\tilde{F}^*}(s)\beta^*(s)=0 \tag{42}$$

which we know from Proposition 2 exists, since (42) can be obtained directly from (27) by postmultiplying it by $\beta^*(s)=B(s)a(s)\subset\ker\tilde{C}(s)$. Q.E.D.

Proposition 4: With $\tilde{F}^*\in F$ satisfying (38), then $s_i\in\Lambda_d$, $i\in\boldsymbol{d}$ are eigenvalues of $\tilde{A}+\tilde{B}\tilde{F}^*$, and the corresponding eigenvectors are $\tilde{x}^*(s_i)=S(s_i)\beta^*(s_i)$.

Proof: Combining (37) and (38), we obtain

[4] For simplicity of notation we assume that $s_i\in\Lambda_d$, $i\in\boldsymbol{d}$ are distinct points in $\mathbb{C}$ and that Λ_q, Λ_c, Λ_d are disjoint.

[5] $[\;]^h_c$ denotes the highest power of s coefficient column vector of the expression inside the brackets.

$$[sI-\tilde{A}-\tilde{B}\tilde{F}^*]\tilde{x}^*(s)=\tilde{B}u_d^*\psi(s)$$

which, for $s=s_i\in\Lambda_d$, $i\in\boldsymbol{d}$ gives: $[s_iI-\tilde{A}-\tilde{B}\tilde{F}^*]\tilde{x}^*(s_i)=0$. Q.E.D.

We can now state the main result of this section, which is the following theorem.

Theorem 1: Let Σ be completely controllable and observable, $r\leqslant m$, and assume that (A1), (A2) (1) are satisfied. Then, with l.s.v.f. control law defined by (4), (5), and $\tilde{G}=\tilde{G}^*=I_m$, $\tilde{F}=\tilde{F}^*\in\boldsymbol{F}$ that satisfies (27) and (38), the closed loop system $\Sigma_{\tilde{F}^*,\tilde{G}^*}=(\tilde{A}+\tilde{B}\tilde{F}^*,\tilde{B},\tilde{C})$ is controllable[6] and, if $f+r<n$, unobservable. The unobservable subspace is given by $\ker\tilde{C}_E$, and this is the maximal $(\tilde{A},\tilde{B})$-invariant subspace $V^{\max}$ contained in $\ker\tilde{C}$, i.e.,

$$V^{\max}=\ker\tilde{C}_E \tag{43}$$

and $\dim V^{\max}=\dim\ker\tilde{C}_E=n-(f+r)$. The transfer function matrix of the closed loop system $\Sigma_{\tilde{F}^*,\tilde{G}^*}$ is given by

$$T_{\tilde{F}^*,\tilde{G}^*}(s)=\Delta^{-1}(s)K \tag{44}$$

where $K=[K(s)]_r^h$, and $K(s)$, f_i, $\Delta(s)$, $\tilde{C}_E$ are defined by (9), (11), (12), and (15), respectively.

Proof: Let[7] $z_i\in\Lambda_q$ and $\beta_i\equiv\beta(z_i)=\ker\delta_{\tilde{F}^*}(z_i)$, $i\in\boldsymbol{q}$. From (27)

$$\beta_i\subseteq\ker\tilde{C}(z_i)\qquad i\in\boldsymbol{q}. \tag{45}$$

Now, from (6), we have the identity

$$\tilde{B}\delta_{\tilde{F}}(s)=[sI-\tilde{A}-\tilde{B}\tilde{F}]S(s). \tag{46}$$

Postmultiplying (46) by β_i and setting $\tilde{F}=\tilde{F}^*\in\boldsymbol{F}$ and $s=z_i\in\Lambda_q$, $i\in\boldsymbol{q}$

$$[z_iI-\tilde{A}-\tilde{B}\tilde{F}^*]S(z_i)\beta_i=0\qquad i\in\boldsymbol{q} \tag{47}$$

and if we define $\tilde{x}_i=S(z_i)\beta_i$, $i\in\boldsymbol{q}$, then (47) says that $\tilde{x}_i$ are q independent eigenvectors of $\tilde{A}+\tilde{B}\tilde{F}^*$ corresponding to its (distinct) eigenvalues $z_i\in\Lambda_q$ [see Proposition 2, (b), 1)]. Similarly for $\lambda_i\in\Lambda_c$ and $\beta_i'\equiv\beta_i'(\lambda_i)=\ker\delta_{\tilde{F}^*}(\lambda_i)$, $i\in\boldsymbol{f+r}$ the vectors $\tilde{x}_i'=S(\lambda_i)\beta_i'$, $i\in\boldsymbol{f+r}$ are $f+r$ independent eigenvectors of $\tilde{A}+\tilde{B}\tilde{F}^*$ corresponding to the eigenvalues $\lambda_i\in\Lambda_c$ (see Proposition 2, (b), (2)). Consider now the subspace

$$V_1=\operatorname{span}\{\tilde{x}_i,\ i\in\boldsymbol{q}\};$$

then obviously $\dim V_1=q$ and V_1 is an $\tilde{A}+\tilde{B}\tilde{F}^*$-invariant subspace or, simply, $(\tilde{A},\tilde{B})$-invariant for which we have that $V_1\subset\ker\tilde{C}$, since by (45)

$$\tilde{C}\tilde{x}_i=\tilde{C}S(z_i)\beta_i=\tilde{C}(z_i)\beta_i=0\qquad i\in\boldsymbol{q}. \tag{48}$$

Considering the subspace $V_2=\operatorname{span}\{\tilde{x}^*(s_i),\ s_i\in\Lambda_d,\ i\in\boldsymbol{d}\}$, we have from Proposition 4 that V_2 is also $\tilde{A}+\tilde{B}\tilde{F}^*$-invariant (or $(\tilde{A},\tilde{B})$-invariant) subspace with $\dim V_2=d=n-(q+f+r)$ and $V_2\subset\ker\tilde{C}$ since

$$\tilde{C}\tilde{x}^*(s_i)=\tilde{C}S(s_i)B(s_i)a(s_i)=\tilde{C}(s_i)B(s_i)a(s_i)=0\qquad\forall s_i\in\mathbb{C}.$$

From

$$[\tilde{x}^*(s_1),\cdots,\tilde{x}^*(s_d)]=[\tilde{x}_o^*,\tilde{x}_1^*,\cdots,\tilde{x}_{d-1}^*]\begin{bmatrix}1&\cdots&1\\ s_1&&s_d\\ \vdots&&\vdots\\ s_1^{d-1}&\cdots&s_d^{d-1}\end{bmatrix}$$

$V_2\equiv R^{\max}$: the maximal controllability subspace of $(\tilde{A},\tilde{B})$ contained in $\ker\tilde{C}$. So finally, $\tilde{A}+\tilde{B}\tilde{F}^*$ has $q+d=q+n-(q+f+r)=n-(f+r)$ eigenvectors in $\ker\tilde{C}$, i.e., if $n-(f+r)>0$, the pair $(\tilde{A}+\tilde{B}\tilde{F}^*,\tilde{C})$ is unobservable.

As V_1, $R^{\max}$ are independent subspaces

$$\dim(V_1\oplus R^{\max})=\dim V_1+\dim R^{\max}$$
$$=q+n-(q+f+r)=n-(f+r). \tag{49}$$

Since

$$\tilde{C}_E\tilde{x}_1=\tilde{C}_ES(z_i)\beta_i=\tilde{C}_E(z_i)\beta_i$$
$$=Z(z_i)\tilde{C}(z_i)\beta_i=0,\qquad z_i\in\Lambda_q,\ i\in\boldsymbol{q}$$
$$\tilde{C}_E\tilde{x}^*(s_i)=\tilde{C}_ES(s_i)B(s_i)a(s_i)=\tilde{C}_E(s_i)\beta^*(s_i)$$
$$=Z(s_i)\tilde{C}(s_i)\beta^*(s_i)=0,\qquad s_i\in\Lambda_d,\ i\in\boldsymbol{d}$$

we have

$$V_1\subset\ker\tilde{C}_E,\quad R^{\max}\subset\ker\tilde{C}_E.$$

From (49) and the fact that $\dim\ker\tilde{C}_E=n-(f+r)$ (see Proposition 1), we finally conclude that

$$V_1\oplus R^{\max}\triangleq V^{\max}=\ker\tilde{C}_E. \tag{50}$$

The last part of Theorem 1 follows simply from (6) and (27).

Corollary 1: If $r=m$ (and (A2), (1) is satisfied, i.e., the system is decouplable [15] [16]), then $R^{\max}=0$, $V^{\max}\equiv V_1$, and we have that the number q of the zeros of Σ is given by $q=n-(f+m)=\dim V^{\max}$ [16].

Case 2: $r>m$:

Proposition 5: Let $r>m$ and let assumptions (A1), (A2) (2) be satisfied, then (a) there exists a unique state feedback matrix $\tilde{F}^*$ such that

$$K\delta_{\tilde{F}^*}(s)=\Delta(s)\tilde{C}_R(s) \tag{51}$$

where $K=[K(s)]_r^h$ and $K(s)$ is now defined by (9), and $\Delta(s)$ by (17); and (b) for $z_i\in\Lambda_q$, $i\in\boldsymbol{q}$ we have rank $\delta_{\tilde{F}^*}(z_i)<m$, i.e., the zeros of Σ are eigenvalues of $\tilde{A}+\tilde{B}\tilde{F}^*$. Also for $\lambda_i\in\Lambda_c$, $i\in\boldsymbol{f+m}$, rank $\delta_{\tilde{F}^*}(\lambda_i)<m$ and, similarly, $\lambda_i\in\Lambda_c$ are eigenvalues of $\tilde{A}+\tilde{B}\tilde{F}^*$.

Proof: A proof can easily be given along the lines of the proof of Proposition 2. The unique state feedback matrix $\tilde{F}^*$ is then given by

$$\tilde{F}^*=-K^{-1}[M+\Delta\tilde{C}_{RE}]-\tilde{A}_m \tag{52}$$

where M is defined in an analogous manner as was done in Proposition 2.

Taking determinants and then degrees of both sides of (51) we have that $n=\sum_{i=1}^m(f_i+1)+q$, so we have the following.

Corollary 2: If $r>m$ [and (A2), (2) is satisfied] then the number q of the zeros of Σ is given by $q=n-(f+m)$, $(f=\sum_{i=1}^m f_i)$.[8]

Theorem 2: Let $r>m$. With l.s.v.f. control law defined by (4), (5), and $\tilde{G}=\tilde{G}^*=K^{-1}$, $\tilde{F}=\tilde{F}^*$ that satisfies (51), the closed loop system $\Sigma_{\tilde{F}^*,\tilde{G}^*}=(\tilde{A}+\tilde{B}\tilde{F}^*,\tilde{B}\tilde{G}^*,\tilde{C})$ is controllable and, if $q>0$, unobservable. The unobservable subspace is given by $\ker\tilde{C}_{RE}$ and this is the maximal $(\tilde{A},\tilde{B})$= invariant subspace $V^{\max}$ contained in $\ker\tilde{C}$, i.e.,

$$V^{\max}=\ker\tilde{C}_{RE} \tag{53}$$

and $\dim V^{\max}=n-(f+m)$ [f_i, defined by (16)]. The transfer function matrix of the closed loop system $\Sigma_{\tilde{F}^*,\tilde{G}^*}$ is given by

$$T_{\tilde{F}^*,\tilde{G}^*}(s)=P(s)\Delta^{-1}(s) \tag{54}$$

where $P(s)$ is defined in (8), and $\Delta(s)$, $\tilde{C}_{RE}$ are defined by (17) and (20).

Proof: We again consider the identity $[sI-\tilde{A}-\tilde{B}\tilde{F}^*]S(s)=\tilde{B}\delta_{\tilde{F}^*}(s)$, which, in view of (51), gives

$$[sI-\tilde{A}-\tilde{B}\tilde{F}^*]S(s)=\tilde{B}K^{-1}\Delta(s)\tilde{C}_R(s). \tag{55}$$

Let[9] $\bar{\beta}_i=\ker\tilde{C}_R(z_i)$, $z_i\in\Lambda_q$, $i\in\boldsymbol{q}$, then (55) for $s=z_i$ gives

$$[z_iI-\tilde{A}-\tilde{B}\tilde{F}^*]S(z_i)\bar{\beta}_i=\tilde{B}K^{-1}\Delta(z_i)\tilde{C}_R(z_i)\bar{\beta}_i=0 \tag{56}$$

[6] Controllability is invariant under state feedback.

[7] Note that the zeros $z_i\in\Lambda_q$, $i\in\boldsymbol{q}$ were assumed to be distinct; hence $\dim\ker\delta_{\tilde{F}^*}(z_i)=1$.

[8] Note that the f_i's are in this case defined by (16).

[9] Note that $\dim\ker\tilde{C}_R(z_i)=1$, since the zeros z_i of Σ are distinct.

and if we put $\tilde{x}_i = S(z_i)\bar{\beta}_i$, $i \in \boldsymbol{q}$, then, from (56), we have that the zeros z_i of Σ (i.e., the zeros of $\det \tilde{C}_R(s)$) are eigenvalues of $\tilde{A}+\tilde{B}\tilde{F}^*$ and the corresponding eigenvectors are $\tilde{x}_i$, $i\in\boldsymbol{q}$. Hence, the subspace $\mathsf{V}_1 = \mathrm{span}\{\tilde{x}_i, i \in \boldsymbol{q}\}$ is $(\tilde{A}, \tilde{B})$-invariant, has $\dim \mathsf{V}_1 = q$, and, since

$$\tilde{C}\tilde{x}_i = \tilde{C}S(z_i)\bar{\beta}_i = \tilde{C}(z_i)\bar{\beta}_i = P(z_i)\tilde{C}_R(z_i)\bar{\beta}_i = 0 \qquad i \in \boldsymbol{q}$$

$\mathsf{V}_1 \subset \ker \tilde{C}$. We also have

$$\tilde{C}_{RE}\tilde{x}_i = \tilde{C}_{RE}S(z_i)\bar{\beta}_i = \tilde{C}_{RE}(z_i)\bar{\beta}_i = Z(z_i)\tilde{C}_R(z_i)\bar{\beta}_i = 0 \qquad i \in \boldsymbol{q},$$

i.e., $\mathsf{V}_1 \subset \ker \tilde{C}_{RE}$. From $\dim \mathsf{V}_1 = q = n-(f+m) = \dim\ker \tilde{C}_{RE}$ (Proposition 1, (2)) we finally obtain

$$\mathsf{V}_1 = \mathsf{V}^{\max} = \ker \tilde{C}_{RE}. \tag{57}$$

Equation (54) follows immediately from (6), (8), and (51).

Example: Case 1, $r \leq m$.

Let $n=9$, $m=3$, $r=2$, $v_3=4$, $v_2=3$, $v_1=2$, and with $(\tilde{A},\tilde{B})$ any controllable pair in the Luenberger controllable companion form and the corresponding $\tilde{C}$ given by

$$\tilde{C} = \begin{bmatrix} 2 & 1 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

Then

$$S(s) = \begin{bmatrix} 1 & s & s^2 & s^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & s & s^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & s \end{bmatrix}^T,$$

$$\tilde{C}(s) = \tilde{C}S(s) = \begin{bmatrix} s+2 & s+2 & 0 \\ 1 & s+1 & 1 \end{bmatrix}$$

and the g.c.d. of 2-order minors of $\tilde{C}(s)$ is $s+2$, i.e., $q=1$, $\Lambda_q = \{-2\}$ (one zero) $f_1 = \min\{4-1-1, 3-1-1, 2-1-(-\infty)\} = 1$, $f_2 = \min\{4-1-0, 3-1-1, 2-1-0\} = 1$ $f = f_1 + f_2 = 2$.

$$K = \left[\tilde{C}(s)[s^{v_m - v}]\right]_r^h$$

$$= \left[\begin{bmatrix} s+2 & s+2 & 0 \\ 1 & s+1 & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ & s & \\ & & s^2 \end{bmatrix}\right]_r^h = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

i.e., $\mathrm{rank}\, K = 2$, so (A1) and (A2) (1) are both satisfied and Theorem 1 applies. From $\tilde{C}$ by inspection

$$\tilde{C}_E = \begin{bmatrix} 2 & 1 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 2 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$\mathrm{rank}\,\tilde{C}_E = f + r = 2+2 = 4$ and $\mathsf{V}^{\max} = \ker \tilde{C}_E$, $\dim \mathsf{V}^{\max} = \dim\ker\tilde{C}_E = 9-4 = 5$. From $\tilde{C}(s)$ a (minimal) basis for $\ker \tilde{C}(s)$ is $\beta_1(s) = (1, -1, s)^T$ and

$$\tilde{x}^*(s) = S(s)\beta_1(s) = \begin{bmatrix} 1 & 0 & 0 \\ s & 0 & 0 \\ s^2 & 0 & 0 \\ s^3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & s & 0 \\ 0 & s^2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & s \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ s \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ s \\ s^2 \\ s^3 \\ -1 \\ -s \\ -s^2 \\ s \\ s^2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} s^2 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} s^3$$

$$= \tilde{x}_o^* + \tilde{x}_1^* s + \tilde{x}_2^* s^2 + \tilde{x}_3^* s^3$$

and

$$\mathsf{R}^{\max} = \mathrm{span}\{\tilde{x}_o^*, \tilde{x}_1^*, \tilde{x}_2^*, \tilde{x}_3^*\}$$

IV. Structure of $\mathsf{V}^{\max}$ if $K(s)$ Is Not Row Proper

The results presented in §3 relied mainly on the assumption that $K(s)$ is a row proper matrix. We now relax this assumption and show that even in the case when (A2) [(1) or (2)] is not satisfied we can state and prove results similar to the ones presented above. In the following we consider the case $r \leq m$. The case $r > m$ can be treated in a similar way.

Let $r \leq m$ and assume that $K(s) = \tilde{C}(s)[s^{v_m - v}]$ is not row proper, i.e., let $\mathrm{rank}\, K < r$. Then it is known [19], [20] that there always exists a unimodular $(r\times r)$ matrix $Q(s)$ such that $K'(s) = Q(s)K(s) = Q(s)\tilde{C}(s)[s^{v_m - v}] = \Gamma(s)[s^{v_m - v}]$ is 1) row proper and 2) $\deg \cdot \{j\text{-th}$ column of $\Gamma(s)\} \leq v_{m+1-j} - 1$, $j \in \boldsymbol{m}$, where

$$\Gamma(s) = Q(s)\tilde{C}(s) = [\gamma_{ij}(s)], \qquad i \in \boldsymbol{r},\ j \in \boldsymbol{m}. \tag{58}$$

Proposition 6: The polynomial matrices $\Gamma(s)$, $D_o(s) = \hat{B}_m^{-1}\delta_o(s)$ are relatively (right) prime (r.r.p.).

Proof: Assume the contrary. Then there will be some $s_o \in \mathbb{C}$ such that

$$\mathrm{rank}_{\mathbb{C}} \begin{bmatrix} D_o(s_o) \\ \Gamma(s_o) \end{bmatrix} < m.$$

But

$$\mathrm{rank}_{\mathbb{C}} \begin{bmatrix} D_o(s_o) \\ \Gamma(s_o) \end{bmatrix} = \mathrm{rank}_{\mathbb{C}} \begin{bmatrix} D_o(s_o) \\ Q(s_o)\tilde{C}(s_o) \end{bmatrix} = m,$$

for every s_o since $Q(s)$ is unimodular and $D_o(s)$, $\tilde{C}(s)$ relatively (right) prime, i.e., a contradiction. Hence, the proposition.

Define now the $r \times n$ matrix Γ via $\Gamma(s) = \Gamma S(s)$, then, from Proposition 6, the triple $(\tilde{A}, \tilde{B}, \Gamma)$ is completely controllable and observable and, by analogy to Case 1 in Section II, we can define

$$f_i^{\Gamma} = \min_{j \in \boldsymbol{m}} \{v_{m+1-j} - 1 - \deg \cdot \gamma_{ij}(s)\} \qquad i \in \boldsymbol{r} \tag{59}$$

$$\Delta^{\Gamma}(s) = \mathrm{diag}\left[\delta_1^{\Gamma}(s), \cdots, \delta_r^{\Gamma}(s)\right] \tag{60}$$

where $\delta_i^{\Gamma}(s)$ arbitrary monic polynomials of $\deg \cdot \delta_i^{\Gamma}(s) = f_i^{\Gamma} + 1$, $i \in \boldsymbol{r}$

$$Z^{\Gamma}(s) = \begin{bmatrix} \begin{matrix} 1 \\ s \\ \vdots \\ s^{f_1^{\Gamma}} \end{matrix} & & \bigcirc \\ & \ddots & \\ \bigcirc & & \begin{matrix} 1 \\ s \\ \vdots \\ s^{f_r^{\Gamma}} \end{matrix} \end{bmatrix} \tag{61}$$

$$\Gamma_E(s) = Z^{\Gamma}(s)\Gamma(s) = \Gamma_E S(s). \tag{62}$$

Then, from Theorem 1: $\ker \Gamma_E = \mathsf{V}_{\Gamma}^{\max}$ is the maximal $(\tilde{A}, \tilde{B})$-invariant subspace in $\ker \Gamma$ and we can state the following.

Proposition 7: $\mathsf{V}_{\Gamma}^{\max} = \ker \Gamma_E$ is also the maximal $(\tilde{A}, \tilde{B})$-invariant subspace contained in $\ker \tilde{C}$.

Proof: From (58) $\Gamma(s)$ has the same Smith form as $\tilde{C}(s)$ so that, if $\Lambda_q = \{z_i, i \in \boldsymbol{q}\}$ is the set of zeros of $(\tilde{A}, \tilde{B}, \tilde{C})$, then Λ_q is also the set of zeros of $(\tilde{A}, \tilde{B}, \Gamma)$. According to Theorem 1: $\mathsf{V}_{\Gamma}^{\max} = \mathsf{V}_{1\Gamma} \oplus \mathsf{R}_{\Gamma}^{\max}$ with (1) $\mathsf{V}_{1\Gamma} = \mathrm{span}\{\tilde{x}_i^{\Gamma}, i \in \boldsymbol{q}\}$, $\tilde{x}_i^{\Gamma} = S(z_i)\beta_i^{\Gamma}$, $\beta_i^{\Gamma} \equiv \beta^{\Gamma}(z_i) = \ker \delta_{\tilde{F}^*}(z_i)$, $i \in \boldsymbol{q}$ and $\tilde{F}^*$ belonging to the family $\boldsymbol{F}$ of state feedback matrices that satisfy

$$K'\delta_{\tilde{F}^*}(s) = \Delta^{\Gamma}(s)\Gamma(s) \tag{63}$$

where $K' = [K'(s)]_r^h$ and $\mathrm{rank}\, K' = r$.

2) $R_\Gamma^{\max} = \text{span}\{\tilde{x}_{i-1}^*, i \in \boldsymbol{d}\}$, where $\tilde{x}_{i-1}^*$ are defined from

$$\tilde{x}^*(s) = S(s)B(s)a(s) = \sum_{i=1}^{d} \tilde{x}_{i-1}^* s^{i-1}$$

$B(s) = \ker \Gamma(s)$ and $a(s)$ such that (1) $\deg \cdot \tilde{x}^*(s) = d-1$, $d = n-(q+f^\Gamma + r)$, $f^\Gamma = \Sigma_{i=1}^r f_i^\Gamma$, (2) the vectors $\tilde{x}_{i-1}^*$, $i \in \boldsymbol{d}$ are independent.

Now from (63): $\beta_i^\Gamma \subset \ker \Gamma(z_i)$, $i \in \boldsymbol{q}$, so

$$\tilde{C}\tilde{x}_i^\Gamma = \tilde{C}S(z_i)\beta_i^\Gamma = \tilde{C}(z_i)\beta_i^\Gamma = Q^{-1}(z_i)\Gamma(z_i)\beta_i^\Gamma = 0, \qquad i \in \boldsymbol{q}$$

i.e., $V_{1\Gamma} \subset \ker \tilde{C}$. Also from (58): $B(s) = \ker \Gamma(s) \equiv \ker \tilde{C}(s)$, i.e., $R_\Gamma^{\max} \subset \ker \tilde{C}$. So, finally, $V_\Gamma^{\max}$ is an $(\tilde{A}, \tilde{B})$-invariant subspace in $\ker \tilde{C}$. Let $V^{\max} = V_1 + R^{\max}$ be the maximal $(\tilde{A}, \tilde{B})$-invariant subspace in $\ker \tilde{C}$. Then, by construction, $R_\Gamma^{\max} \equiv R^{\max}$, $\dim R_\Gamma^{\max} = n-(q+f^\Gamma + r)$. Also, $\dim V_1 = q$, $\dim V_{1\Gamma} = q$ and we must have that $V_1 \equiv V_{1\Gamma}$ because otherwise $\ker \tilde{C}$ will contain $V^{\max}$ of $\dim V^{\max} = q + n - (q + f^\Gamma + r) = n - (f^\Gamma + r)$, and $V_\Gamma^{\max}$ of the same dimension but different subspace, i.e., a contradiction, since the maximal $(\tilde{A}, \tilde{B})$-invariant subspace in $\ker \tilde{C}$ is unique.

Corollary 3: If $r = m$ but the system $\Sigma = (A, B, C)$ is not decouplable [i.e., $K(s)$ is not row proper], then $R^{\max} = 0$, $V^{\max} = V_1 = V_{1\Gamma} = \ker \Gamma_E$, and the number q of the zeros of Σ is given by $q = n - (f^\Gamma + m) = \dim V^{\max}$.

V. Conclusions

This paper has examined in some detail the intimate relation existing between the maximal $(\tilde{A}, \tilde{B})$-invariant subspace $V^{\max}$ in $\ker \tilde{C}$ and a special right coprime MFD: $\tilde{C}(s)\tilde{D}_o(s)^{-1}$ of the transfer function matrix $T_o(s)$ of $\Sigma = (A, B, C)$ in which $\tilde{D}_o(s) = [d_1(s), \cdots, d_m(s)]$ is column proper and its column degrees: $v_{m+1-j} = \deg \cdot d_j(s)$, $j \in \boldsymbol{m}$ are ordered so that $\deg \cdot d_j(s) \geq \deg \cdot d_{j+1}(s)$, $j \in \boldsymbol{m} - 1$. By using the l.s.v.f. form of the Wolovich–Falb "structure theorem", the structure of $V^{\max}$ has been examined and expressions for the exact dimension of $V^{\max}$ in terms of certain system parameters have been obtained. It must be mentioned that the transformation matrix that brings (A, B) to the controllable companion form $(\hat{A}, \hat{B})$ with the stated properties is not unique. If now $r \leq m$ and $(\bar{A}, \bar{B}, \bar{C})$ is any other minimal realization of $T_o(s)$ with $(\bar{A}, \bar{B})$ in controllable companion form,[10] giving rise to the right coprime MFD of $T_o(s): \bar{C}(s)\bar{D}_o(s)^{-1}$ (with $\bar{D}_o(s)$ satisfying the conditions mentioned above), then $\bar{C}(s) = \tilde{C}(s)U_R(s)$, $\bar{D}_o(s) = \tilde{D}_o(s)U_R(s)$ for some unimodular matrix $U_R(s)$ [14], and it can be shown that if both $\tilde{C}(s)$ and $\bar{C}(s)$ satisfy condition (A2) (1), then (11) defines the f_is uniquely. Furthermore, if $\tilde{C}(s)$ and $\bar{C}(s)$ give rise via (15) to the matrices $\tilde{C}_E$ and $\bar{C}_E$, respectively, then $\tilde{C}_E = \bar{C}_E T_1$ where T_1 is the nonsingular coordinate transformation: $(\hat{A}, \hat{B}) \to (\bar{A}, \bar{B})$. Similar results hold in the case $r > m$.

Finally, the facts that, if $r = m$ and (A2) (1) is satisfied, Σ is decouplable [15], and the f_is coincide with the "decoupling invariants" of Σ (see Remarks 1 and 2), show that there is much in common between our results and those given by Bhattacharyya [21].

Appendix

Consider the case when $r > m$ and let $\tilde{C}_R(s)$ be the $m \times m$ upper right triangular g.c.r.d. of (the rows of) $\tilde{C}(s)$ which is obtained from $\tilde{C}(s)$ by the procedure described in [11, Theorems 2.5.11, 2.5.16]. Then, if

$$\tilde{C}_{Rj}(s) = [c_{R1j}(s), c_{R2j}(s), \cdots, c_{Rj-1,j}(s), c_{Rjj}(s), 0, \cdots, 0]^T$$

is the j-th column of $\tilde{C}_R(s)$, the polynomials $c_{R1j}(s), \cdots, c_{Rj-1,j}(s)$ are of lower degree than $c_{Rjj}(s)$ for all $j \in \boldsymbol{m}$ if $\deg \cdot c_{Rjj}(s) > 0$ and are all zero if $c_{Rjj}(s)$ is a nonzero scalar in $\mathbb{R}$, i.e., by consraction $\tilde{C}_R(s)$ is *column proper* and its column degrees are equal to the degrees of its diagonal elements $c_{Rjj}(s)$, $j \in \boldsymbol{m}$.

Proposition: If $K(s) = \tilde{C}_R(s)[s^{v_m - v}]$ is *row proper*, then $\deg \cdot \tilde{C}_{Rj}(s) \leq \deg \cdot \tilde{C}_j(s) \leq v_{m+1-j} - 1$, $j \in \boldsymbol{m}$, where $\tilde{C}_j(s)$ is the jth column of $\tilde{C}(s)$.

Proof: Notice that, due to the upper right triangular structure of $\tilde{C}_R(s)$, the assumption that $K(s)$ is row proper implies that the degrees of the diagonal elements $c_{Rjj}(s)$ of $\tilde{C}_R(s)$ are greater than or equal to the degrees of all other elements $c_{Rj,j+1}(s), \cdots, c_{Rjm}(s)$ in the same row, i.e., if $K(s)$ is row proper then $\tilde{C}_R(s)$ is also row proper, and the row degrees are also equal to the degrees of the diagonal elements $c_{Rjj}(s)$.

Let now $p_j = \deg \cdot c_{Rjj}(s)$, $j \in \boldsymbol{m}$ and write

$$\tilde{C}_R(s) = [s^{p_j}]\Lambda + L(s)$$

where $[s^{p_j}] = \text{diag}(s^{p_1}, \cdots, s^{p_m})$, Λ the diagonal matrix of the coefficients of highest degree (p_j) of the diagonal elements $c_{Rjj}(s)$ of $\tilde{C}_R(s)$, and $L(s)$ is an $m \times m$ matrix of row degrees which are equal or less than p_j, $j \in \boldsymbol{m}$. Also, let $P_j(s)$ be the jth column of $P(s)$ in (8), $k_j = \deg \cdot P_j(s)$, $j \in \boldsymbol{m}$, and write

$$P(s) = [P(s)]_c^h [s^{k_j}] + N(s)$$

where $[s^{k_j}] = \text{diag} \cdot (s^{k_1}, \cdots, s^{k_m})$ and $N(s)$ an $r \times m$ matrix of column degrees less than k_j. Finally, let $q_j = \deg \cdot \tilde{C}_j(s) \leq v_{m-1-j} - 1$, $j \in \boldsymbol{m}$, and write

$$\tilde{C}(s) = [\tilde{C}(s)]_c^h [s^{q_j}] + V(s). \tag{A1.1}$$

Then

$$\begin{aligned} P(s)\tilde{C}_R(s) &= \big[[P(s)]_c^h [s^{k_j}] + N(s)\big]\big[[s^{p_j}]\Lambda + L(s)\big] \\ &= \big[[P(s)]_c^h \Lambda [s^{k_j + p_j}] \\ &\quad + \text{lower column degree terms}. \end{aligned} \tag{A1.2}$$

From (A1.1), (A1.2), and (8) it follows that $q_j = k_j + p_j$, i.e., $p_j = \deg \cdot \tilde{C}_{Rj}(s) \leq q_j \leq v_{m+1-j} - 1$, $j \in \boldsymbol{m}$. Q.E.D.

[10] And the v_{m+1-j}, $j \in \boldsymbol{m}$, arranged in the given order.

References

[1] W. M. Wonham and A. S. Morse, "Decoupling and pole assignment in multivariable systems: A geometric approach," *SIAM J. Contr.*, vol. 8, pp. 1–18, 1970.
[2] A. S. Morse and W. M. Wonham, "Status of noninteracting control," *IEEE Trans. Automat. Contr.*, vol. AC-16, pp. 568–580, Dec. 1971.
[3] W. M. Wonham, *Linear Multivariable Control, A Geometric Approach*. New York: Springer, 1974.
[4] G. Bengtsson, "A computational algebra for geometric concepts in linear systems theory," Rep. 7501, Lund Inst. of Technology, Dept. Automat. Contr., 1975.
[5] B. C. Moore and A. J. Laub, "Computation of supremal (A, B)-invariant and Controllability subspaces," *IEEE Trans. Automat. Contr.*, vol. AC-23, pp. 783–792, Oct. 1978.
[6] M. E. Warren and A. E. Eckberg, "On the dimensions of controllability subspaces: A characterization via polynomial matrices and Kronecker invariants," *SIAM J. Contr.*, vol. 13, pp. 435–445, 1975.
[7] P. N. R. Stoyle and A. I. G. Vardulakis, "On certain connections between the geometric and the polynomial matrix approaches to linear system theory," *Int. J. Contr.*, vol. 29, pp. 565–588, 1979.
[8] P. A. Fuhrmann and J. C. Willems, "A study of (A, B)-invariant subspaces via polynomial models," in Math. Rep. 218 NATO Advanced Study Inst. Algebraic Geometric Methods Linear Syst. Theory, May 1979.
[9] D. G. Luenberger, "Canonical forms for linear multivariable systems," *IEEE Trans. Automat. Contr.*, vol. AC-12, pp. 290–293, June 1967.
[10] W. A. Wolovich and P. L. Falb, "On the structure of multivariable systems," *SIAM J. Contr.*, vol. 7, pp. 437–451, 1969.
[11] W. A. Wolovich, *Linear Multivariable Systems*. New York: Springer, 1974.
[12] H. H. Rosenbrock, *State Space and Multivariable Theory*. London: Nelson, 1970.
[13] ——, "The zeros of a system," *Int. J. Contr.*, vol. 18, pp. 297–299, 1973; correction to "The zeros of a system," *Int. J. Contr.*, vol. 20, pp. 525–527, 1974.
[14] W. A. Wolovich, "On the numerators and zeros of rational transfer matrices," *IEEE Trans. Automat. Contr.*, vol. AC-18, pp. 544–546, Oct. 1973.
[15] P. N. R. Stoyle and A. I. G. Vardulakis, "The mechanism of decoupling," *Int. J. Contr.*, vol. 29, pp. 589–605, 1979.
[16] A. I. G. Vardulakis and P. N. R. Stoyle, "An algorithm for decoupling and maximal pole assignment in multivariable systems by the use of state feedback," *IEEE Trans. Automat. Contr.*, vol. AC 24, pp. 362–365, Apr. 1979.
[17] A. I. G. Vardulakis, "On the structure of the bases of all possible controllability subspaces of a controllable pair (A, B) in canonical form," *Int. J. Contr.*, vol. 28, pp. 393–409, 1978.
[18] D. G. Forney, "Minimal bases of rational vector spaces, with applications to multivariable linear systems," *SIAM J. Contr.*, vol. 13, pp. 493–520, 1975.
[19] M. K. Sain, "A free-modular algorithm for minimal design of linear multivariable systems," Paper 9.1, in *Preprints IFAC 1975*, Part IB, pp. 1-7.
[20] W. A. Wolovich, "The determination of state space representations for linear multivariable systems," *Automatica*, vol. 9, pp. 97–106, 1973.
[21] S. P. Bhattacharyya, "On calculating maximal (A, B) invariant subspaces," *IEEE Trans. Automat. Contr.*, vol. AC-20, pp. 264–265, Apr. 1975.