III. Conclusions

It is well known that linear operators can have no cross-causal part. Actually this is a misstatement of the theorem which, more precisely, is as follows.

Theorem: Let $L: (U, E) \rightarrow (Y, P)$ where $I = E_t + \delta E(t) + E^t$ and $I = P_t + \delta P(t) + P^t$. If L is linear then its cross-causal part must be the null operator.

The results of this paper show that a linear operator on a space U' with a more complicated time structure, i.e., one in which $(I - \mathcal{P}_t - \delta\mathcal{P}(t) - \mathcal{P}^t) \neq 0$ can have a nontrivial cross-causal part. There is, then, an interplay between nonlinearity of the operator and the complexity of the time structure on the input space. This is a very desirable feature, especially if one is trying to use a Stone–Weierstrass type theorem [4] to approximate a nonlinear operator by a polynomic operator. This paper guarantees that operators can be so approximated without losing their fundamental cross causality.

References

[1] A. Feintuch and R. Saeks, *Systems Theory—A Hilbert Space Approach*. New York: Academic, to be published.

[2] R. M. DeSantis, "Causality structure of engineering systems," Ph.D. dissertation, Univ. Michigan, Ann Arbor, 1972.

[3] W. A. Porter, "Multiple signal extraction by polynomic filtering," *Math. Syst. Theory*, vol. 13, no. 4, 1980.

[4] ——, "Causality structure and the Weierstrass theorem," *J. Math. Anal. Appl.*, vol. 52, Nov. 1975.

Relations Between Strict Equivalence Invariants and Structure at Infinity of Matrix Pencils

A. I. G. VARDULAKIS AND N. KARCANIAS

Abstract—**The structure at infinity of matrix pencils is investigated via their Smith–MacMillan forms at infinity [1]. Relations between some strict equivalence invariants and pole-zero structure at infinity are established.**

I. Introduction

Matrix pencils are intimately related to the study of systems of linear first-order differential equations. Various problems in linear systems theory may be reduced to the study of the invariant structure of appropriate matrix pencils under strict equivalence [5]. The set of finite (infinite) elementary divisors of a matrix pencil $A + sB$ may be defined by the Smith form of $A + sB$ over $\mathbb{R}[s]$ [4] ($wA + B$ over $\mathbb{R}[w]$ [5]). The aim of the present correspondence is to provide an alternative treatment of the structure of matrix pencils at infinity by employing the theory of the Smith–MacMillan form of a rational matrix at $s = \infty$ [1], [7]; relations between some strict equivalence invariants and pole-zero structure at infinity of matrix pencils are established.

II. Background

Let $\mathbb{R}$ be the field of reals, $\mathbb{R}[s]$ be the ring of polynomials, and $\mathbb{R}(s)$ be the field of rational functions $t(s) = n(s)/d(s)$, $n(s), d(s) \in \mathbb{R}[s]$, both with coefficients in $\mathbb{R}$.

Define the map $\delta_\infty: \mathbb{R}(s) \rightarrow \mathbb{Z} \cup \{+\infty\}$ [1] via

$$\delta_\infty(t(s)) = \begin{cases} \deg d(s) - \deg n(s), & t(s) \not\equiv 0 \\ +\infty, & t(s) \equiv 0.^1 \end{cases} \tag{1}$$

Manuscript received June 7, 1982; revised September 20, 1982.

A. I. G. Vardulakis is with the Control and Management Systems Division, Department of Engineering, Cambridge University, Cambridge, England.

N. Karcanias is with the Department of Systems Science, City University, London, England.

[1] $\deg(0) = -\infty$.

The map $\delta_\infty(\cdot)$ is a discrete valuation on $\mathbb{R}(s)$ [2] and every $t(s) \in \mathbb{R}(s)$ can be factored as

$$t(s) = \left(\frac{1}{s}\right)^{q_\infty} \frac{n_1(s)}{d_1(s)} \tag{2}$$

where $q_\infty = \delta_\infty(t(s))$ and $\deg n_1(s) = \deg d_1(s)$. If $q_\infty > 0$ we say that $t(s)$ has a zero at $s = \infty$ of order q_∞ and if $q_\infty < 0$, then we say that $t(s)$ has a pole of order $|q_\infty|$ at $s = \infty$. If $t(s) \in \mathbb{R}(s)$ and $\delta_\infty(t(s)) \geqslant 0$, then $t(s)$ is called a proper rational function. Thus, proper rational functions have no poles at $s = \infty$. It is easily verified that the set of all proper rational functions, which we denote by $\mathbb{R}_{pr}(s)$, is an integral domain. The units $u(s) \in \mathbb{R}_{pr}(s)$ also have no zeros at $s = \infty$ and thus, if $u(s) = n(s)/d(s) \in \mathbb{R}_{pr}(s)$ is a unit, $\delta_\infty(u(s)) = 0$, i.e., $\deg n(s) = \deg d(s)$. Now consider the following.

Lemma 1 [1]: Given $t_1(s), t_2(s) \in \mathbb{R}(s)$ with $t_2(s) \not\equiv 0$ then there exist a $p(s) \in \mathbb{R}_{pr}(s)$ and a $r(s) \in \mathbb{R}(s)$ such that

$$t_1(s) = t_2(s)p(s) + r(s) \tag{3}$$

and either $r(s) = 0$ or else $\delta_\infty(r(s)) < \delta_\infty(t_2(s))$.

Since for any $t(s) \in \mathbb{R}_{pr}(s)$, $\delta_\infty(t(s)) \geqslant 0$, $\delta_\infty(\cdot)$ serves as a "stathm" or "degree" [3] for the elements of $\mathbb{R}_{pr}(s)$ and so for $t_1(s), t_2(s) \in \mathbb{R}_{pr}(s)$ Lemma 1 describes a Euclidean division algorithm. Thus, $\mathbb{R}_{pr}(s)$ is a Euclidean ring and therefore a principal ideal domain [1]. Denote by $\mathbb{R}_{pr}^{p \times m}(s)$ the set of $p \times m$ matrices with elements in $\mathbb{R}_{pr}(s)$. Such matrices are called *proper* rational matrices. $T(s) \in \mathbb{R}_{pr}^{p \times p}(s)$ is called $\mathbb{R}_{pr}(s)$-*unimodular* or *biproper* if there exists a $\overline{T}(s) \in \mathbb{R}_{pr}^{p \times p}(s)$ such that $T(s)\overline{T}(s) = I_p$. "Elementary row and column operations" on a $T(s) \in \mathbb{R}^{p \times m}(s)$ are defined in the following usual way [3]: 1) interchange any two rows (columns) of $T(s)$, 2) multiply row (column) i of $T(s)$ by a unit $u(s) \in \mathbb{R}_{pr}(s)$, and 3) add to row (column) i a multiple by a $t(s) \in \mathbb{R}_{pr}(s)$ of row (column) j. These elementary operations can be accomplished by multiplying the given $T(s)$ on the left (right) by "elementary" biproper matrices obtained by performing the above elementary operations on the identity matrix $I_p(m)$.

Definition 1 [1]: $T_1(s) \in \mathbb{R}^{p \times m}(s)$, $T_2(s) \in \mathbb{R}^{p \times m}(s)$ are called equivalent at $s = \infty$ if there exist biproper rational matrices $T_L(s) \in \mathbb{R}_{pr}^{p \times p}(s)$, $T_R(s) \in \mathbb{R}_{pr}^{m \times m}(s)$ such that

$$T_L(s)T_1(s)T_R(s) = T_2(s). \tag{4}$$

We have the following.

Theorem 1 [1], [7] (Smith–MacMillan form of a rational matrix at $s = \infty$): Let $T(s) \in \mathbb{R}^{p \times m}(s)$ with $\operatorname{rank}_{\mathbb{R}(s)} T(s) = r$. Then $T(s)$ is equivalent at $s = \infty$ to a diagonal matrix $S_{T(s)}^\infty$ having the following form:

$$S_{T(s)}^\infty = \operatorname{diag}\left\{ s^{q_\infty^1}, s^{q_\infty^2}, \cdots, s^{q_\infty^j}, \frac{1}{s^{\hat{q}_\infty^{j+1}}}, \frac{1}{s^{\hat{q}_\infty^{j+2}}}, \cdots, \frac{1}{s^{\hat{q}_\infty^r}}, 0_{p-r, m-r} \right\} \tag{5}$$

where

$$q_\infty^1 \geqslant q_\infty^2 \geqslant \cdots \geqslant q_\infty^j \geqslant 0 \tag{6}$$

$$\hat{q}_\infty^r \geqslant \hat{q}_\infty^{r-1} \geqslant \cdots \geqslant \hat{q}_\infty^{j+1} \geqslant 0. \tag{7}$$

Proof: (see [1]).

Remark: If $T(s) \in \mathbb{R}_{pr}^{p \times m}(s)$ then $q_\infty^i = 0$, $i = 1, 2, \cdots, j$, i.e., $S_{T(s)}^\infty$ is a proper rational matrix ($\in \mathbb{R}_{pr}^{p \times m}(s)$) and it is called the Smith form of $T(s)$ at $s = \infty$. Otherwise, i.e., if $T(s)$ is nonproper, then $S_{T(s)}^\infty$ is also nonproper and it is called the MacMillan form of $T(s)$ at $s = \infty$. If p_∞ is the number of q_∞^i's in (6) with $q_\infty^i > 0$, $i \in j$, then we say that $T(s)$ has p_∞ poles at infinity, each one of order $q_\infty^i > 0$. Also if z_∞ is the number of $\hat{q}_\infty^i$'s with $\hat{q}_\infty^i > 0$, $i = j+1, \cdots, r$, then we say that $T(s)$ has z_∞ zeros at infinity, each one of order $\hat{q}_\infty^i > 0$.

III. Structure at Infinity of Matrix Pencils

A polynomial matrix $P(s) = A + sB \in \mathbb{R}^{p \times m}[s]$, $A, B \in \mathbb{R}^{p \times m}$ is called a (matrix) *pencil*. A pencil $P(s)$ is called *regular* if 1) $A \in \mathbb{R}^{p \times p}$, $B \in \mathbb{R}^{p \times p}$, and 2) $\det P(s) \not\equiv 0$. In all other cases the pencil $P(s)$ is called *singular* [4]. Two pencils $P_i(s) = A_i + sB_i \in \mathbb{R}^{p\times m}[s]$ $i = 1,2$ are called *strictly equivalent* if there exist constant nonsingular matrices $M \in \mathbb{R}^{p\times p}$ and $N \in \mathbb{R}^{m\times m}$ such that [4]

$$MP_1(s)N = P_2(s). \tag{8}$$

We examine now the structure of pencils at $s = \infty$.

A. Regular Pencils

Consider a regular pencil $P(s) = A + sB$, $A \in \mathbb{R}^{p\times p}$, $B \in \mathbb{R}^{p\times p}$ and let $\deg|P(s)| =: n \geq 0$. It is known [4] that $P(s)$ is strictly equivalent to its Weierstrass normal form $\tilde{P}(s)$, i.e., there exist nonsingular matrices $M \in \mathbb{R}^{p\times p}$, $N \in \mathbb{R}^{p\times p}$ such that

$$M[A+sB]N = \begin{bmatrix} sJ_\infty + I_{p-n} & 0 \\ 0 & sI_n + J \end{bmatrix} = \begin{bmatrix} I_{p-n} & 0 \\ 0 & J \end{bmatrix} + s\begin{bmatrix} J_\infty & 0 \\ 0 & I_n \end{bmatrix} = \tilde{A} + s\tilde{B} = \tilde{P}(s) \tag{9}$$

where $J_\infty \in \mathbb{R}^{(p-n)\times(p-n)}$ and nilpotent, $J \in \mathbb{R}^{n\times n}$ and

$$J_\infty = \text{block diag}[J_{\infty 1}, \cdots, J_{\infty k}] \tag{10}$$

$$J_{\infty i} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}}_{v_i+1} \quad v_i + 1 = \mu_i \quad v_i \geq 0 \quad i \in \boldsymbol{k} \tag{11}$$

$$k = \text{rank defect of } B = p - \text{rank } B \geq 0 \tag{12}$$

$$J = \text{block diag}[J_1, \cdots, J_f] \tag{13}$$

$$J_i = \underbrace{\begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & & \cdots & 1 \\ 0 & 0 & & \cdots & \lambda_i \end{bmatrix}}_{\sigma_i} \quad \sigma_i \quad \sigma_i > 0 \quad i \in \boldsymbol{f}. \tag{14}$$

$f \geq 0$ is the number of finite elementary divisors (FED) of $P(s)$, assumed here (for notational simplicity) to have the form $(s - \lambda_i)^{\sigma_i}$ (i.e., $\lambda_i \in \mathbb{R}$, $i \in \boldsymbol{f}$) and $\Sigma_{i=1}^f \sigma_i = n$. The infinite elementary divisors (IED) of $P(s)$ are given by

$$w^{\mu_1}, w^{\mu_2}, \cdots, w^{\mu_k} \tag{15}$$

where $\mu_i = v_i + 1$, $i \in \boldsymbol{k}$ are the sizes of the Jordan blocks $J_{\infty i}$, $i \in \boldsymbol{k}$ of J_∞ and they can be defined as the FED's of the "dual" pencil [5]

$$w\tilde{A} + \tilde{B} = w\begin{bmatrix} I_{p-n} & 0 \\ 0 & J \end{bmatrix} + \begin{bmatrix} J_\infty & 0 \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} wI_{r-n} + J_\infty & 0 \\ 0 & wJ + I_n \end{bmatrix} =: \tilde{P}(w) \tag{16}$$

at $w = 0$.

Let now that

$$S^\infty_{P(s)} = \text{diag}\left(s^{q^1_\infty}, s^{q^2_\infty}, \cdots, s^{q^j_\infty}, \frac{1}{s^{\hat{q}^{j+1}_\infty}}, \cdots, \frac{1}{s^{\hat{q}^p_\infty}}\right) \tag{17}$$

is the MacMillan form of $P(s)$ at $s = \infty$ where $q^1_\infty \geq q^2_\infty \geq \cdots \geq q^j_\infty > 0$ are the orders of its infinite poles and $\hat{q}^p_\infty \geq \hat{q}^{p-1}_\infty \geq \cdots \geq \hat{q}^{j+1}_\infty \geq 0$ are the orders of its infinite zeros. Then we have the following.

Proposition 1: For a regular pencil $P(s) = A + sB \in \mathbb{R}^{p\times p}[s]$.

i) The number j of its poles at $s = \infty$ is given by $j = \text{rank } B$, and their orders satisfy $q^i_\infty = 1$, $i \in \boldsymbol{j}$.

ii) The degrees μ_i of its IED's $w^{\mu_i}(\mu_i > 0)$, $i \in \boldsymbol{k}$, $k = p - \text{rank } B$, satisfy $\mu_i = \hat{q}^i_\infty + 1$, $i \in \boldsymbol{k}$, where $\hat{q}^i_\infty$ are the orders of the zeros at $s = \infty$ of $P(s)$.

Proof: The pole-zero structure of a rational matrix $T(s) \in \mathbb{R}^{p\times m}(s)$ at $s = \infty$ can also be obtained using the transformation $s = 1/w$, from the pole-zero structure of $P(1/w) \in \mathbb{R}^{p\times m}(w)$ at $w = 0$ [6]. Hence the pole-zero structure of $P(s) = A + sB \in \mathbb{R}^{p\times p}[s]$ at $s = \infty$ can be obtained from the pole-zero structure of $P(1/w) = A + 1/wB \in \mathbb{R}^{p\times p}(w)$ at $w = 0$. From (17) the MacMillan form of $P(1/w)$ at $w = 0$ is

$$S^0_{P(1/w)} = \text{diag}\left[\frac{1}{w^{q^1_\infty}}, \cdots, \frac{1}{w^{q^j_\infty}}, w^{\hat{q}^{j+1}_\infty}, \cdots, w^{\hat{q}^p_\infty}\right]. \tag{18}$$

Hence, the Smith form $S^0_{wP(1/w)}$ of $wP(1/w) = wA + B \in \mathbb{R}^{p\times p}[w]$ at $w = 0$ is given by

$$S^0_{wP(1/w)} = wS^0_{P(1/w)} = \text{diag}\left[\frac{1}{w^{q^1_\infty - 1}}, \cdots, \frac{1}{w^{q^j_\infty - 1}}, w^{\hat{q}^{j-1}_\infty + 1}, \cdots, w^{\hat{q}^p_\infty + 1}\right]. \tag{19}$$

Since $wP(1/w)$ is polynomial (i.e., (19) is a Smith form at $w = 0$ of $wA + B$), it follows that $q^i_\infty - 1 = 0$, $i \in \boldsymbol{j}$, or

$$q^i_\infty = 1 \qquad i \in \boldsymbol{j} \tag{20}$$

which implies that the Smith form of $wA + B$ at $w = 0$ is

$$S^0_{wA+B} = wS^0_{P(1/w)} = \text{diag}\left[1, 1, \cdots, 1, w^{\hat{q}^{j+1}_\infty + 1}, \cdots, w^{\hat{q}^p_\infty + 1}\right] \tag{21}$$

and therefore the IED's w^{μ_i} of $A + sB$ are given by the terms $w^{\hat{q}^i_\infty + 1}$, $i = j + 1, \cdots, p$, i.e.,

$$\mu_i = \hat{q}^i_\infty + 1. \tag{22}$$

Since there are $k = p - \text{rank } B$ IED's w^{μ_i}, $i \in \boldsymbol{k}$, from (17) and (20)

$$S^\infty_{P(s)} = \text{diag}\left[\overbrace{s, s, \cdots, s}^{p-k = \text{rank } B}, \overbrace{\frac{1}{s^{\hat{q}^{j+1}_\infty}}, \cdots, \frac{1}{s^{\hat{q}^p_\infty}}}^{k}\right] \tag{23}$$

$$= \text{diag}\left[sI_j, \frac{1}{s^{\hat{q}^{j+1}_\infty}}, \cdots, \frac{1}{s^{\hat{q}^p_\infty}}\right], \tag{24}$$

i.e., the number j of the infinite poles of $P(s)$ is $j = p - k = \text{rank } B$. □

Corollary 1: The MacMillan form $S^\infty_{P(s)}$ of a regular pencil $P(s)$ at $s = \infty$ uniquely determines the structure of the matrix J_∞ in its Weierstrass form $\tilde{P}(s)$, i.e., $v_i \equiv \hat{q}^i_\infty$, $i \in \boldsymbol{k}$.

Given two regular pencils $P_i(s) = A_i + sB_i \in \mathbb{R}^{p\times p}[s]$, $i = 1,2$, then it is known [4] that they are strictly equivalent if and only if they have the same FED's and the same IED's [4]. In view of the above we can rephrase this result by stating the following.

Theorem 2: Two regular pencils $P_i(s) = A_i + sB_i \in \mathbb{R}^{p\times p}[s]$, $i = 1,2$ are strictly equivalent if and only if they have the same (finite) Smith form and the same MacMillan form at $s = \infty$, i.e., $\exists M, N \in \mathbb{R}^{p\times p}$ $|M| \neq 0$, $|N| \neq 0$ such that $MP_1(s)N = P_2(s) \Leftrightarrow S^{\mathbb{C}}_{P_1(s)} \equiv S^{\mathbb{C}}_{P_2(s)}$,[2] and $S^\infty_{P_1(s)} \equiv S^\infty_{P_2(s)}$.

Proof:

(if) Let $P_1(s), P_2(s)$ be strictly equivalent. Then $S^{\mathbb{C}}_{P_1(s)} \equiv S^{\mathbb{C}}_{P_2(s)}$, and $S^\infty_{P_1(s)} \equiv S^\infty_{P_2(s)}$ since M and N are both unimodular and biproper matrices.

(only if) $S^{\mathbb{C}}_{P_1(s)} \equiv S^{\mathbb{C}}_{P_2(s)}$ implies that $P_1(s)$ and $P_2(s)$ have the same FED's, i.e., their Weierstrass normal forms have the matrices J identical. Also from $S^\infty_{P_1(s)} = S^\infty_{P_2(s)}$, and Corollary 1 the Weierstrass normal forms have also identical J_∞ matrices. Thus, $P_1(s), P_2(s)$ have the same Weierstrass normal form, i.e., they are strictly equivalent. □

B. Singular Pencils

Consider now a singular pencil $P(s) = A + sB \in \mathbb{R}^{p\times m}[s]$ and let $P_K(s) = A_K + sB_k$ be its Kronecker form [4]

$$P_K(s) = \text{block diag}\{0_{h,g}, L_\epsilon(s), L_\eta(s), \tilde{A} + s\tilde{B}\} \tag{25}$$

[2] $S^{\mathbb{C}}_{P(s)}$ denotes the (finite) Smith form of $P(s) \in \mathbb{R}^{p\times m}[s]$.

where $\tilde{A} + s\tilde{B}$ is a regular pencil in its Weierstrass form,

$$L_\epsilon(s) = \text{block diag}\left[L_{\epsilon_{g+1}}(s), \cdots, L_{\epsilon_l}(s)\right] \tag{26}$$

$$L_{\epsilon_i}(s) = \begin{bmatrix} s & 1 & & \cdots & 0 \\ 0 & s & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & s \quad 1 \end{bmatrix} \in \mathbb{R}^{\epsilon_i \times (\epsilon_i + 1)}[s] \tag{27}$$

$$L_\eta(s) = \text{block diag}\left[L_{\eta_{h+1}}(s), \cdots, L_{\eta_t}(s)\right] \tag{28}$$

$$L_{\eta_i}(s) = \begin{bmatrix} s & 0 & \cdots & 0 \\ 1 & s & & 0 \\ 0 & 1 & & \vdots \\ \vdots & \vdots & & s \\ 0 & 0 & \cdots & 1 \end{bmatrix} \in \mathbb{R}^{(\eta_i + 1) \times \eta_i}[s] \tag{29}$$

and $\epsilon_1 = \cdots = \epsilon_g = 0 < \epsilon_{g+1} \leqslant \cdots \leqslant \epsilon_l$ are the column minimal indexes and $\eta_1 = \cdots = \eta_h = 0 < \eta_{h+1} \leqslant \cdots \leqslant \eta_t$ are the row minimal indexes [4]. Then we have the following.

Proposition 2: For a singular pencil $P(s) = A + sB \in \mathbb{R}^{p \times m}[s]$:

i) the number j of its poles at $s = \infty$ is given by $j = n_\epsilon + n_\eta + r_w = \text{rank}\, B$, $n_\epsilon = \Sigma\epsilon_i$, $n_\eta = \Sigma\eta_i$, $r_w = \text{rank}\, \tilde{B}$ and their orders satisfy $q^i_\infty = 1$, $i \in \boldsymbol{j}$,

ii) (α) the number k of its IED's w^{μ_i} ($\mu_i > 0$), $i \in \boldsymbol{k}$, is given by $k = \text{rank}_{\mathbf{R}(s)}\,(A + sB) - \text{rank}_{\mathbf{R}}\, B$ and (β); we have $\mu_i = \hat{q}^i_\infty + 1$, $i \in \boldsymbol{k}$, where $\hat{q}^i_\infty$ are the orders of its zeros at $s = \infty$, and

iii)

$$\begin{aligned} S^\infty_{P(s)} &= \text{block diag}\left\{sI_{n_\epsilon + n_\eta}, S^\infty_{\tilde{A} + s\tilde{B}}, 0_{t,l}\right\} \\ &= \text{block diag}\left\{sI_j, \frac{1}{s^{\hat{q}^{j+1}_\infty}}, \cdots, \frac{1}{s^{\hat{q}^{j+k}_\infty}}, 0_{t,l}\right\}. \end{aligned} \tag{30}$$

Proof: As $L_\epsilon(s)$, $L_\eta(s)$ are both row and column proper, from Proposition 4 in [1] we have:

$$S^\infty_{L_\epsilon(s)} = \left[sI_{n_\epsilon}, 0_{n_\epsilon, l-g}\right] \text{ and } S^\infty_{L_\eta(s)} = \begin{bmatrix} sI_{n_\eta} \\ 0_{t-h, n_\eta} \end{bmatrix}.$$

Therefore, the number j of the poles at $s = \infty$ of $P_k(s)$ is given by the total number of poles at $s = \infty$ of $L_\epsilon(s)$, $L_\eta(s)$ and $\tilde{A} + s\tilde{B}$, i.e., $j = n_\epsilon + n_\eta + r_w = \text{rank}\, B_k = \text{rank}\, B$. Also there exist biproper matrices $U_L(s) \in \mathbb{R}^{p \times p}_{pr}(s)$, $U_R(s) \in \mathbb{R}^{m \times m}_{pr}(s)$ such that

$$U_L(s)\,P(s)\,U_R(s) = \text{block diag}\left\{0_{h,g}, S^\infty_{L_\epsilon(s)}, S^\infty_{L_\eta(s)}, S^\infty_{\tilde{A} + s\tilde{B}}\right\}$$

which can be further reduced to (30) by pre- and postmultiplication by appropriate permutation (i.e., biproper) matrices. Equating the minimum sizes in (30) we have

$$j + k + \min(t, l) = \min(p, m), \tag{31}$$

but $\text{rank}_{\mathbf{R}(s)}(A + sB) = \text{rank}_{\mathbf{R}(s)} S^\infty_{A+sB} = \min(p, m) - \min(t, l)$ and ii) (α) follows by substitution, while ii) (β) follows from Proposition 1.

Concluding Remarks: Note that for singular pencils the two conditions of Theorem 2 are again necessary, but not sufficient for strict equivalence. This is because elementary divisors do not form a complete set of invariants for singular pencils and that row and column minimal indexes also have to be considered. A result similar to that of Theorem 2 may however be stated if results on *proper minimal MacMillan degree bases of rational vector spaces* [8] are deployed. In the latter case column, row minimal indexes can be interpreted as MacMillan degrees of proper minimal MacMillan degree bases [8] of the right, left null (rational vector) space of the given pencil.

References

[1] A. I. G. Vardulakis, D. J. N. Limebeer, and N. Karcanias, "Structure and Smith-MacMillan form of a rational matrix at infinity," *Int. J. Contr.*, vol. 35, pp. 701–725, Apr. 1982.

[2] M. F. Atiyah and I. G. MacDonald, *Introduction to Commutative Algebra*. Reading, MA: Addison-Wesley, 1969.

[3] C. C. MacDuffee, *The Theory of Matrices*. Berlin: Springer-Verlag, 1933 (New York: Chelsea, reprinted 1946).

[4] F. R. Gantmacher, *The Theory of Matrices*. New York: Chelsea, 1960.

[5] N. Karcanias and G. E. Hayton, "Generalized autonomous dynamical systems, algebraic duality and geometric theory," in *Proc. 8th IFAC World Congress*, Kyoto Japan, 1981.

[6] A. C. Pugh and P. A. Ratcliffe, "On the zeros and poles of a rational matrix," *Int. J. Contr.*, vol. 30, no. 2, pp. 213–226.

[7] T. Kailath, *Linear Systems*. Englewood Cliffs, NJ: Prentice-Hall, 1980.

[8] A. I. G. Vardulakis and N. Karcanias, "Proper, minimal MacMillan degree, bases of rational vector spaces," Dep. Eng., Cambridge Univ., Cambridge, England, Research Rep. CUED/F-CAMS/TR-226.

A New Method on Computation of the Characteristic Polynomial for a Class of Square Matrices

DA-ZHONG ZHENG

Abstract—**In this correspondence, a new method is proposed to construct the characteristic polynomial for a class of square matrices with some restrictions. An algorithm is also given. The proposed method and algorithm have recursive forms and are therefore suitable for calculation by computer.**

I. Introduction

As is well known, the computation of the characteristic polynomial of a square matrix is essential for the analysis and design of linear time-invariant control systems. For example, it is often required in transforming the system matrix pair (A, B) into its controllable and observable canonical forms which are usually used in the design of state controllers and observers. Hence, there have been some methods and algorithms developed in relation to this problem [1]–[5]. In this note, we introduce a new method and also give an algorithm, both of which are completely different from the previously cited methods and algorithms. The proposed method and algorithm have recursive forms and, as a result, are convenient to compute by computer.

II. The Main Results

Suppose we consider the square constant matrix given by

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \tag{1}$$

and use

$$\Delta(s) = \det(sI - A) \tag{2}$$

to denote its characteristic polynomial.

The following notation will be used. Let B'_i denote the following row vector:

$$B'_i = (0 \cdots 0, \underset{i\text{th}}{1}, 0 \cdots 0). \tag{3}$$

Let $K_i^{(R)}$ and $K_i^{(L)}$ denote the row vectors with the respective forms

$$K_i^{(R)} = (0 \cdots 0, a_{i,i-1} \cdots a_{in}) \tag{4}$$

and

Manuscript received June 2, 1982; revised August 9, 1982.

The author is with the Department of Electrical Engineering, State University of New York, Stony Brook, NY 11794, on leave from the Department of Automation, Tsinghua University, Peking, China.