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Structure, Smith-MacMillan form and coprime MFDs of a rational matrix inside a region $P = \omega U\{\infty\}$

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Structure, Smith–MacMillan form and coprime MFDs of a rational matrix inside a region $\mathscr{P} = \Omega \cup \{\infty\}$

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The structure of the Smith–MacMillan form of a rational matrix $T(s)$ inside a region $\mathscr{P} = \Omega \cup \{\infty\}$ (where Ω is a symmetric with respect to the real axis subset of the finite complex plane $\mathbb{C}$) is determined. Algorithmic procedures based on elementary row and column operations over the euclidean ring $\mathbb{R}_{\mathscr{P}}(s)$ consisting of all rational functions with no poles in $\mathscr{P}$ are given. Coprimeness in $\mathscr{P}$ of a pair of rational matrices is studied in detail. These results lead to constructive procedures for determining the coprime in $\mathscr{P}$ matrix fraction descriptions of $T(s)$.

1. Introduction

The algebraic approaches to linear systems theory as developed by Kalman (1965, 1966), Rosenbrock (1970), Wolovich (1974), Forney (1975) and many other researchers rely heavily on the study of the fine structure of polynomial matrices. Polynomial matrix theory is a special case of the theory of *integral matrices* (Newman 1972), i.e. matrices having elements in a principal ideal domain (PID) (MacDuffee 1946).

Problems of realizability, stability and performance of linear multivariable control systems, motivate the study of matrices having elements in special rings that describe in an algebraic sense these properties. Thus realizability of a rational transfer function $T(s)$ can be characterized by its property of having no poles at $s = \infty$ (properness of $T(s)$) and stability (and to a certain extent the performance of a control system) can be characterized by the absence of poles of its transfer function matrix from a prescribed symmetric (with respect to the real axis) region Ω of the finite complex plane.

The algebraic structure of the set $\mathbb{R}_{\mathscr{P}}(s)$ of proper rational functions which have no poles inside a region $\mathscr{P} := \Omega \cup \{\infty\}$, $(\Omega \subset \mathbb{C})$ was initially examined by Morse (1975). Subsequently, Hung and Anderson (1978) showed that, with an appropriately defined ' degree ' function, the set $\mathbb{R}_{\mathscr{P}}(s)$ has the structure of a euclidean ring. This important result has been the basis for the subsequent work of Vidyasagar (1978), Francis and Vidyasagar (1980), Desoer *et al.* (1980), Saeks and Murray (1981), Vidyasagar *et al.* (1982), Vidyasagar and Viswanadham (1982), Francis and Vidyasagar (1983), Saeks and Murray (1982), on what these authors term as ' fractional representations ' of proper rational matrices and their use to analysis and synthesis problems.

The algebraic aspects of $\mathbb{R}_{\mathscr{P}}(s)$ related to computational procedures such as factorizations, euclidean division, determination of greatest common divisors and least common multiples, determination of the structure and the derivation

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of the Smith–MacMillan form of a rational matrix inside the set $\mathscr{P}$ as well as procedures for constructing coprime in $\mathscr{P}$ matrix fraction descriptions of a rational matrix $T(s)$ have nòt been systematically considered so far.

The primary purpose of this paper is to give a detailed account of the fine structure of the ring $\mathbb{R}_{\mathscr{P}}(s)$ and emphasize those aspects mentioned above which are related to computations. This study forms the basis for the determination of the structure of the Smith–MacMillan form of a rational matrix in $\mathscr{P}$. The notion of coprimeness of rational matrices with elements in $\mathbb{R}_{\mathscr{P}}(s)$ is examined in detail and various equivalent criteria for coprimeness are given which form generalizations of the standard results known for polynomial matrices. These results lead to systematic procedures for constructing coprime (in $\mathscr{P}$) rational matrix fraction descriptions ($\mathbb{R}_{\mathscr{P}}(s)$-coprime MFDs).

The paper is structured as follows. In § 2, background results concerning the ring $\mathbb{R}_{\mathscr{P}}(s)$ of proper rational matrices having no poles inside a region $\mathscr{P}:=\Omega\cup\{\infty\}$ is examined. Specifically, aspects related to 'degree', euclidean division, factorizations, structure of primes, coprimeness conditions and the study of the set of rational functions $\mathbb{R}(s)$ as a quotient field of $\mathbb{R}_{\mathscr{P}}(s)$ are discussed. Section 3 deals with the detailed study of the Smith–MacMillan form in $\mathscr{P}$: $S_{T(s)}{}^{\mathscr{P}}$ of a general rational matrix $T(s)$, and examines the reduction of a $T(s)$ to $S_{T(s)}{}^{\mathscr{P}}$ via elementary row and column operations over $\mathbb{R}_{\mathscr{P}}(s)$. In § 3, the results are used for the characterization of coprimeness in $\mathscr{P}$ of rational matrices and for constructing the family of coprime in $\mathscr{P}$ rational matrix fraction descriptions of a rational matrix $T(s)$. Finally, in § 4, the definition of the 'degree in $\mathscr{P}=\Omega\cup\{\infty\}$' : $\delta_{\mathscr{P}}(t(s))$ of a scalar $t(s)\in\mathbb{R}_{\mathscr{P}}(s)$ is generalized for the matrix case and a number of properties of this 'degree function' are derived.

2. Background

Let $\mathbb{R}$ be the field of reals, $\mathbb{R}[s]$ the ring of polynomials with coefficients in $\mathbb{R}$ and $\mathbb{R}(s)$ the field of rational functions $t(s)=n(s)/d(s)$, $n(s), d(s)\in\mathbb{R}[s]$, $d(s)\not\equiv 0$. Given a rational function $t(s)=n(s)/d(s)$ (where $n(s), d(s)$ are coprime), then the *finite zeros and poles* of $t(s)$ are respectively the zeros of $n(s)$ and $d(s)$ in the finite complex plane $\mathbb{C}$.

Regarding the point at infinity, we write

$$t(s)=\left(\frac{1}{s}\right)^{q_\infty}\frac{n_1(s)}{d_1(s)}$$

where $q_\infty=\deg d(s)-\deg n(s)\in\mathbb{Z}$ (where $\mathbb{Z}$ is the ring of integers), $\deg n_1(s)=\deg d_1(s)$. If $q_\infty>0$, we say that $t(s)$ has a zero at $s=\infty$ of order q_∞, while if $q_\infty<0$, we say that $t(s)$ has a pole at $s=\infty$ of order $|q_\infty|$.

If $t(s)=n(s)/d(s)\in\mathbb{R}(s)$ has $\deg d(s)\geqslant\deg n(s)$, then $t(s)$ is called a *proper* rational function and if the inequality is strict then $t(s)$ is called *strictly proper*. It can be easily verified (Vardulakis *et al.* 1982) that (just as $\mathbb{R}[s]$) the set of all proper rational functions, denoted by $\mathbb{R}_{\mathrm{pr}}(s)$, is a euclidean ring with 'degree' defined by the map $\delta_\infty:\ \mathbb{R}(s)\to\mathbb{Z}\cup\{+\infty\}$; $\delta_\infty(t(s))=\deg d(s)-\deg n(s)=q_\infty\in\mathbb{Z}$, i.e. that $t(s)\in\mathbb{R}_{\mathrm{pr}}(s)\Leftrightarrow\delta_\infty(t(s))=q_\infty\geqslant 0$. Thus $\mathbb{R}_{\mathrm{pr}}(s)$ is a principal ideal ring and both $\mathbb{R}[s]$ and $\mathbb{R}_{\mathrm{pr}}(s)$ are subrings of $\mathbb{R}(s)$.

The elements of $\mathbb{R}[s]$, i.e. the polynomials, can be regarded as rational functions with no poles in $\mathbb{C}$, while writing every $t(s)=n(s)/d(s)\in\mathbb{R}_{\mathrm{pr}}(s)$ as

$$t(s)=\left(\frac{1}{s}\right)^{q_\infty}\frac{n_1(s)}{d_1(s)},\quad q_\infty := \delta_\infty(t(s))\geqslant 0$$

with $\deg n_1(s)=\deg d_1(s)$, we see that the elements of $\mathbb{R}_{\mathrm{pr}}(s)$ can be regarded as rational functions with *no poles at* $s=\infty$.

The *units* $u(s)$ in $\mathbb{R}[s]$ (in $\mathbb{R}_{\mathrm{pr}}(s)$) are those elements for which there exists an element $\hat{u}(s)\in\mathbb{R}[s]$ $(\in\mathbb{R}_{\mathrm{pr}}(s))$ such that $u(s)\hat{u}(s)=1$ (the *unity* element in $\mathbb{R}[s]$ $(\mathbb{R}_{\mathrm{pr}}(s))$). Thus the units in $\mathbb{R}[s]$ are the (non-zero) elements in $\mathbb{R}$ (constants) and the units in $\mathbb{R}_{\mathrm{pr}}(s)$ are the proper rational functions $t(s)=n(s)/d(s)$ with $\deg n(s)=\deg d(s)$, i.e. also having no zeros at $s=\infty$. The units in $\mathbb{R}_{\mathrm{pr}}(s)$ we call *biproper* rational functions.

Let us now consider matrices. Denote by $\mathbb{R}^{p\times m}(s)$ the set of $p\times m$ matrices with elements in $\mathbb{R}(s)$. These matrices are called (real) rational matrices. By $\mathbb{R}^{p\times m}[s]$ and $\mathbb{R}_{\mathrm{pr}}{}^{p\times m}(s)$ we denote the subset of $\mathbb{R}^{p\times m}(s)$ consisting of all $p\times m$ matrices with elements respectively in $\mathbb{R}[s]$ and $\mathbb{R}_{\mathrm{pr}}(s)$. A polynomial matrix $T(s)\in\mathbb{R}^{p\times p}[s]$ is called $\mathbb{R}[s]$-unimodular or simply unimodular if there exist a $\hat{T}(s)\in\mathbb{R}^{p\times p}[s]$ such that $T(s)\hat{T}(s)=I_p$, and similarly a proper rational matrix $T(s)\in\mathbb{R}_{\mathrm{pr}}{}^{p\times p}(s)$ is called $\mathbb{R}_{\mathrm{pr}}(s)$-unimodular or *biproper* if there exist a $\hat{T}(s)\in\mathbb{R}_{\mathrm{pr}}{}^{p\times p}(s)$ such that $T(s)\hat{T}(s)=I_p$ (see Vardulakis *et al.* 1982, Vardulakis and Karcanias 1983).

Now, in order to generalize the aforegoing, let Ω be a region in the finite complex plane $\mathbb{C}$, symmetrically located with respect to the real axis and which excludes at least one point on the real axis $\mathbb{R}$ and let Ω_c be the complement of Ω with respect to $\mathbb{C}$, i.e. let $\mathbb{C}=\Omega\cup\Omega_c$. Let $t(s)\in\mathbb{R}(s)$ and factorize it as

$$t(s)=t_\Omega(s)\hat{t}(s) \tag{2.1}$$

where $t_\Omega(s)=n_\Omega(s)/d_\Omega(s)$, $n_\Omega(s), d_\Omega(s)$ are coprime polynomials *with all their zeros not outside* Ω and $\hat{t}(s)=\hat{n}(s)/\hat{d}(s)$, $\hat{n}(s)$, $\hat{d}(s)$ are coprime polynomials *with all their zeros outside* Ω.

We define now the map $\delta_\Omega:\ \mathbb{R}(s)\to\mathbb{Z}\cup\{\infty\}$ via

$$\delta_\Omega(t(s))=\begin{cases}\deg\hat{d}(s)-\deg\hat{n}(s)\in\mathbb{Z}, & t(s)\not\equiv 0\\ +\infty, & t(s)\equiv 0\dagger\end{cases} \tag{2.2}$$

From the above definition of the map $\delta_\Omega(\cdot)$ we can easily obtain the following properties

$$\delta_\Omega(t_1(s)t_2(s))=\delta_\Omega(t_1(s))+\delta_\Omega(t_2(s)) \tag{2.3}$$
$$\delta_\Omega(t_1(s)t_2(s)^{-1}=\delta_\Omega(t_1(s))-\delta_\Omega(t_2(s))$$
$$t_1(s),\ t_2(s)\not\equiv 0\in\mathbb{R}(s)$$

From the definition of the ' valuation at $s=\infty$ ' of a $t(s)\in\mathbb{R}(s)$ (Vardulakis *et al.* 1982), we also easily obtain the following proposition.

† Notice that if $t(s)=n(s)/d(s)$, $d(s)\not\equiv 0$ is the identically equal to zero rational functions then $n(s)=n_\Omega(s)\hat{n}(s)\equiv 0$ if $d(s)=d_\Omega(s)\hat{d}(s)$ then $\delta_\Omega(0)=\deg\hat{d}(s)-\deg(0)=\deg\hat{d}(s)-(-\infty)=+\infty$.

Proposition 2.1

Let $t(s)=n(s)/d(s)\in\mathbb{R}(s)$ $(\not\equiv 0)$ and factorize it as in (2.1). Then

$$\delta_\Omega(t(s))=q_\infty+\deg n_\Omega(s)-\deg d_\Omega(s) \tag{2.4}$$

where $q_\infty := \delta_\infty(t(s))=\deg d(s)-\deg n(s)$ is the valuation at $s=\infty$ of $t(s)$ (see Vardulakis *et al.* 1982).

If $\deg n_\Omega(s)-\deg d_\Omega(s) =: \gamma_\Omega(t(s))$, then $\gamma_\Omega(t(s))$ is the number of encirclements of the origin in the $t(s)$-plane by the mapping of the boundary $\partial\Omega$ of Ω under $t(s)$ when $\partial\Omega$ is traversed in the clockwise direction (i.e. $\gamma_\Omega(\cdot)$ is a winding number).

Consider now the subset of $\mathbb{R}(s)$ consisting of all rational functions that satisfy the following two requirements : (i) are proper ; and (ii) have no poles in Ω (such rational functions we will call Ω-stable rational functions) and denote this set by $\mathbb{R}_{\mathscr{P}}(s)$, i.e. let

$$\mathbb{R}_{\mathscr{P}}(s) := \{t(s)\in\mathbb{R}(s)\,|\,t(s) \text{ has no poles in } \mathscr{P} := \Omega\cup\{\infty\}\}$$

It can be easily verified that the set $\mathbb{R}_{\mathscr{P}}(s)$ endowed with the operations of addition and multiplication forms a commutative ring with unity element (the real number 1) and no zero divisors. $\mathbb{R}_{\mathscr{P}}(s)$ is therefore an *integral domain.*

Now let $t(s)\in\mathbb{R}_{\mathscr{P}}(s)$ $(\not\equiv 0)$; since $t(s)$ has no poles in Ω, it can be written (according to (2.1)) as

$$t(s)=n_\Omega(s)\frac{\hat{n}(s)}{\hat{d}(s)} \tag{2.5}$$

Also, since $t(s)$ has no poles at $s=\infty$, i.e. $t(s)\in\mathbb{R}_{\mathrm{pr}}(s)$ we have

$$\deg(n_\Omega(s)\hat{n}(s))\leqslant\deg\hat{d}(s)\Rightarrow\deg n_\Omega(s)+\deg\hat{n}(s)\leqslant\deg\hat{d}(s)$$

which implies that $\deg\hat{n}(s)\leqslant\deg\hat{d}(s)$. Hence for every $t(s)\in\mathbb{R}_{\mathscr{P}}(s)$ $(\not\equiv 0)$ we have

$$\delta_\Omega(t(s)) := \deg\hat{d}(s)-\deg\hat{n}(s)\geqslant 0 \tag{2.6}$$

Thus, $\delta_\Omega(\cdot)$ for the non-zero elements of $\mathbb{R}_{\mathscr{P}}(s)$ may serve as a ' degree ' function. The algebraic structure of $\mathbb{R}_{\mathscr{P}}(s)$ has been examined initially by Morse (1976) and subsequently by Hung and Anderson (1978), and it has been shown that, with $\delta_\Omega(\cdot)$ as degree, it is that of a *euclidean ring* and therefore of a principal ideal domain (PID).

The algebraic aspects of $\mathbb{R}_{\mathscr{P}}(s)$ related to computational procedures, such as unique factorizations, determination of greatest common divisors and least common multiples and algorithmic procedures for euclidean division, have not been systematically considered so far. In the following these aspects of $\mathbb{R}_{\mathscr{P}}(s)$ are closely examined.

We first note that the function $\delta_\Omega : \mathbb{R}(s)\to\mathbb{Z}\cup\{\infty\}$ when restricted to the subdomain $\mathbb{R}_{\mathscr{P}}(s)\subset\mathbb{R}(s)$ will be denoted by $\delta_{\mathscr{P}} : \mathbb{R}_{\mathscr{P}}(s)\to\mathbb{Z}\cup\{\infty\}$. The *units* in $\mathbb{R}_{\mathscr{P}}(s)$ are those $t(s)\in\mathbb{R}_{\mathscr{P}}(s)$ for which there exist a $t'(s)\in\mathbb{R}_{\mathscr{P}}(s)$ such that $t(s)t'(s)=1$. This implies that $t(s)\in\mathbb{R}_{\mathscr{P}}(s)$ is a *unit* if and only if $t(s)$ has no poles and no zeros in $\mathscr{P}=\Omega\cup\{\infty\}$, and so the units in $\mathbb{R}_{\mathscr{P}}(s)$ are biproper rational functions $t(s)=\hat{n}(s)/\hat{d}(s)$, where $\hat{n}(s)$, $\hat{d}(s)$ are coprime polynomials with no zeros in Ω and satisfy $\deg\hat{n}(s)=\deg\hat{d}(s)$. Thus $t(s)\in\mathbb{R}_{\mathscr{P}}(s)$ is a unit if and only if $\delta_{\mathscr{P}}(t(s))=0$.

Remark 2.1

If Ω coincides with the closed righthalf plane $\mathbb{C}_+ := \{s \in \mathbb{C}, \text{Re}(s) \geq 0\}$ then $\mathscr{P} \equiv \mathbb{C}_+ \cup \{\infty\} = \overline{\mathbb{C}}_+$, and $\mathbb{R}_{\overline{\mathbb{C}}_+}(s)$ is the euclidean ring of ' proper and stable ' rational functions. The units in $\mathbb{R}_{\overline{\mathbb{C}}_+}(s)$ are biproper, stable and ' minimum phase ' rational functions.

From Proposition (2.1) it follows that if $t(s) \in \mathbb{R}_{\mathscr{P}}(s)$ then

$$q := \delta_{\mathscr{P}}(t(s)) = q_\infty + \deg n_\Omega(s) \tag{2.7}$$

where $q_\infty \geq 0$ gives the order of the *zero at* $s = \infty$ of $t(s) \in \mathbb{R}_{\mathscr{P}}(s)$ and $\deg n_\Omega(s)$ gives the number of *finite zeros* of $t(s)$ inside Ω.

2.1. *Factorizations in* $\mathbb{R}(s)$ *and* $\mathbb{R}_{\mathscr{P}}(s)$

We start this section by starting a result concerning the factorization of a general (non-zero) rational function.

Proposition 2.2

Every $t(s) \in \mathbb{R}(s)$ can be written as

$$t(s) = \frac{n_\Omega(s)}{d_\Omega(s)} \frac{1}{(s+\alpha)^q} \frac{\hat{n}(s)(s+\alpha)^q}{\hat{d}(s)} \tag{2.8}$$

where $-\alpha \in \mathbb{R}$, is outside Ω and otherwise arbitrary, $q := \delta_\Omega(t(s)) = \deg \hat{d}(s) - \deg \hat{n}(s) \in \mathbb{Z}$ and $\hat{n}(s)(s+\alpha)^q / \hat{d}(s)$ is a unit of $\mathbb{R}_{\mathscr{P}}(s)$.

Proof

From Proposition 2.2 it is clear why the region Ω must ' exclude at least one point $-\alpha \in \mathbb{R}$ '.

Remark 2.2

The term

$$\frac{n_\Omega(s)}{d_\Omega(s)} \left(\frac{1}{(s+\alpha)} \right)^q$$

gives the pole-zero structure of $t(s)$ in $\mathscr{P} = \Omega \cup \{\infty\}$. Thus the zeros of $n_\Omega(s)$ give the (finite) zeros of $t(s)$ in Ω and the zeros of $d_\Omega(s)$ give the (finite) poles of $t(s)$ in Ω. Furthermore, if

$$q_\infty := q + \deg d_\Omega(s) - \deg n_\Omega(s) > 0 \tag{2.9}$$

then $t(s)$ has a *zero at* $s = \infty$ of order q_∞ while if $q_\infty < 0$ then $t(s)$ has a *pole at* $s = \infty$ of order $|q_\infty|$ (see Vardulakis *et al.* 1982).

From Proposition 2.2 we now obtain the following corollary.

Corollary 2.1

Every $t(s) \in \mathbb{R}_{\mathscr{P}}(s)$ can be factorized as

$$t(s) = \frac{n_\Omega(s)}{(s+\alpha)^q} u(s) = \frac{n_\Omega(s)}{(s+\alpha)^{d'}} \frac{1}{(s+\alpha)^{q_\infty}} u(s) \tag{2.10}$$

where $n_\Omega(s)\in\mathbb{R}[s]$ has no zeros outside Ω, $d' := \deg n_\Omega(s)$, $-\alpha\in\mathbb{R}$ is outside Ω and otherwise arbitrary, $q := \delta_{\mathscr{P}}(t(s))$, and $u(s)$ is a unit in $\mathbb{R}_{\mathscr{P}}(s)$.

By factorizing $n_\Omega(s)$ into irreducible factors over $\mathbb{R}[s]$ as

$$n_\Omega(s)=k(s+\lambda_1)^{v_1}\dots(s+\lambda_\nu)^{v_\nu}(s^2+a_1s+b_1)^{v'_1}\dots(s+a_\rho s+b_\rho)^{v'_\rho}$$

$k\in\mathbb{R}$, we have that

$$\begin{aligned} t(s)&=\left[\frac{s+\lambda_1}{s+\alpha}\right]^{v_1}\dots\left[\frac{s+\lambda_\nu}{s+\alpha}\right]^{v_\nu}\left[\frac{s^2+a_1s+b_1}{(s+\alpha)^2}\right]^{v'_1}\\ &\qquad\dots\left[\frac{s^2+a_\rho s+b_\rho}{(s+\alpha)^2}\right]^{v'_\rho}\frac{1}{(s+\alpha)^{q_\infty}}u(s)\\ &=p_1{}^{v_1}(s)\dots p_\nu{}^{v_\nu}(s)\,p'_1(s)^{v'_1}\dots p_\rho(s)^{v'_\rho}\,p^*(s)^{q_\infty}u(s) \end{aligned}\tag{2.11}$$

The uniqueness of the factorization of $n_\Omega(s)$ implies that (2.11) is a *unique factorization* of $t(s)\in\mathbb{R}_{\mathscr{P}}(s)$ modulo α. (Of course the fact that $-\alpha\in\mathbb{R}$ (and $-\alpha\notin\Omega$) is arbitrary explains the term ' modulo α '.) The elements $p_i(s)$, $p'_j(s)$, $p^*(s)$, $i\in\mathbf{v}$, $j\in\boldsymbol{\rho}$ are the *primes* of $t(s)\in\mathbb{R}_{\mathscr{P}}(s)$ modulo α.

The set $\mathscr{D}=\{p_i(s)^{v_i}, p'_j(s)^{v'_j}, p^*(s)^{q_\infty}, i\in\mathbf{v}, j\in\boldsymbol{\rho}, v_i, v'_j, q_\infty\geqslant 0\}$ is defined as the set of elementary divisors (mod α) of $t(s)\in\mathbb{R}_{\mathscr{P}}(s)$.

Remark 2.3

The primes of $\mathbb{R}_{\mathscr{P}}(s)$ which have no zeros at $s=\infty$ are biproper rational functions of the following two types: $p(s)=(s+\lambda)/(s+\alpha)$ or $p'(s)=(s^2+as+b)/(s+\alpha)^2$; the only strictly proper prime of $\mathbb{R}_{\mathscr{P}}(s)$ is of the type $p^*(s)=1/(s+\alpha)$, which has a zero at $s=\infty$. Those three types of primes are the only possible (mod α) for the ring $\mathbb{R}_{\mathscr{P}}(s)$.

The unique (mod α) factorization of a $t(s)\in\mathbb{R}_{\mathscr{P}}(s)$ given in (2.11) allows us to connect $\delta_{\mathscr{P}}(t(s))$ with the $\delta_{\mathscr{P}}(\cdot)$ of the elementary divisors of $t(s)$. Thus for all $-\alpha\in\mathbb{R}$ and $-\alpha\notin\Omega$ we have $\delta_{\mathscr{P}}(p(s))=1$, $\delta_{\mathscr{P}}(p'(s))=2$ and $\delta_{\mathscr{P}}(p^*(s))=1$, and so if $t(s)\in\mathbb{R}_{\mathscr{P}}(s)$ is factorized as in (2.11) we have

$$\delta_{\mathscr{P}}(t(s))=\sum_{i=1}^{\nu}v_i+\sum_{j=1}^{\rho}2v'_j+q_\infty$$

which once more verifies that $\delta_{\mathscr{P}}(t(s))$ expresses the total number of zeros of $t(s)$ in $\mathscr{P}=\Omega\cup\{\infty\}$. Let now $t(s)\in\mathbb{R}_{\mathscr{P}}(s)$ and let

$$t(s)=p_1(s)^{v_1}\dots p_\mu(s)^{v_\mu}u(s)\tag{2.12}$$

be its (mod α) factorization, where $p_i(s)\in\mathbb{R}_{\mathscr{P}}(s)$, $i\in\boldsymbol{\mu}$ are distinct primes of the types $p(s)$, $p'(s)$ and $p^*(s)$ and $u(s)$ is a unit in $\mathbb{R}_{\mathscr{P}}(s)$. If we now denote by

$$v_{p_i(s)}(t) := v_i=\delta_{\mathscr{P}}(p_i(s)^{v_i}),\quad i\in\boldsymbol{\mu}$$

the $\delta_{\mathscr{P}}(\cdot)$ of a prime $p_i(s)$ of $t(s)$ in (2.12), by $\mathscr{P}_{t(s)}$ the set of primes $p_i(s)$ of $t(s)$, $\mathscr{Z}_{t(s)}$ ($\mathscr{Z}_{t(s)}\subset\mathscr{P}$) the set of zeros $z_i\in\mathscr{P}=\Omega\cup\{\infty\}$ of the elements $p_i(s)$ of $\mathscr{P}_{t(s)}$ and by $v_{z_i}(t)$ the multiplicity of the zero $z_i\subset\mathscr{Z}_{t(s)}$ of $t(s)$ (clearly, $v_{z_i}(t)=v_{p_i(s)}(t)$), then the divisibility of $t_1(s)$ by $t_2(s)$ ($\in\mathbb{R}_{\mathscr{P}}(s)$) may be stated as in the following proposition.

Proposition 2.3

Let $t_1(s)$, $t_2(s) \in \mathbb{R}_{\mathscr{P}}(s) - \{0\}$, then the following statements are equivalent :

(i) $t_1(s)$ divides $t_2(s)$;

(ii) $\mathscr{P}_{t_1(s)} \subseteq \mathscr{P}_{t_2(s)}$ and $v_{p_i(s)}(t_1) \leqslant v_{p_i(s)}(t_2)$ for all $p_i(s) \in \mathscr{P}_{t_1(s)}$; and

(iii) $\mathscr{Z}_{t_1(s)} \subseteq \mathscr{Z}_{t_2(s)}$ and $v_{z_i}(t_1) \leqslant v_{z_i}(t_2)$ for all $z_i \in \mathscr{Z}_{t_i(s)}$, $i = 1, 2$.

This proposition gives rise to the following interpretation of coprimeness in $\mathscr{P}$ of a set of (non-zero) elements of $\mathbb{R}_{\mathscr{P}}(s)$.

Proposition 2.4

Let $t_i(s) \in \mathbb{R}_{\mathscr{P}}(s)$, $i \in \mathbf{k}$ and let $\mathscr{P}_{t_i(s)}$, $\mathscr{Z}_{t_i(s)}$ be the sets associated with each $t_i(s)$. Then the following statements are equivalent.

(i) $t_i(s)$, $i \in \mathbf{k}$ are coprime in $\mathscr{P}$.

(ii)
$$\bigcap_{i=1}^{k} \mathscr{P}_{t_i(s)} = \phi$$

(iii)
$$\bigcap_{i=1}^{k} \mathscr{Z}_{t_i(s)} = \phi$$

(iv) (*a*) At least one of the $t_i(s)$ is biproper (i.e. $q_\infty{}^i = 0$ for at least one $i \in \mathbf{k}$) and (*b*) the polynomials $n_{i\Omega}(s)$, $i \in \mathbf{k}$ are coprime (in Ω).

(v) There exist elements $t'_i(s) \in \mathbb{R}_{\mathscr{P}}(s)$, $i \in \mathbf{k}$ such that
$$\sum_{i=1}^{k} t_i(s) t'_i(s) = 1$$

These results and the unique (mod α) factorization of a $t(s) \in \mathbb{R}_{\mathscr{P}}(s)$ given in (2.12) provides the means for constructing the (mod α) greatest common divisor (g.c.d.) and (mod α) least common multiple (l.c.m.) of elements of $\mathbb{R}_{\mathscr{P}}(s)$.

Example

Let $\Omega \equiv \mathbb{C}_+$, $\alpha > 0$ and let
$$t_1(s) = \frac{(s^2 - s + 1)(s-1)^2(s+2)}{(s+3)^2(s+4)^3(s+1)}, \quad t_2(s) = \frac{(s-1)^3(s+1)}{(s^2+s+1)(s+3)} \in \mathbb{R}_{\mathscr{P}}(s)$$
then
$$t_1(s) = \left\{ \frac{(s+2)(s+\alpha)^6}{(s+3)^2(s+4)^3(s+1)} \right\} \left[\frac{s-1}{s+\alpha} \right]^2 \left[\frac{s^2 - s + 1}{(s+\alpha)^2} \right] \left[\frac{1}{s+\alpha} \right]^2$$
$$t_2(s) = \left\{ \frac{(s+1)(s+\alpha)^6}{(s^2+s+1)^3(s+3)} \right\} \left[\frac{s-1}{s+\alpha} \right]^3 \left[\frac{1}{s+\alpha} \right]^3$$
Thus the (mod α) g.c.d. and l.c.m. of $t_1(s)$ and $t_2(s)$ are
$$\text{g.c.d.} \quad \{t_1(s), t_2(s)\} = \left[\frac{s-1}{s+\alpha} \right]^2 \left[\frac{1}{s+\alpha} \right]^2 = \frac{(s-1)^2}{(s+\alpha)^4}$$
$$\text{l.c.m.} \quad \{t_1(s), t_2(s)\} = \left[\frac{s-1}{s+\alpha} \right]^3 \left[\frac{s^2 - s + 1}{(s+\alpha)^2} \right] \left[\frac{1}{s+\alpha} \right]^3 = \frac{(s-1)^3(s^2 - s + 1)}{(s+\alpha)^8}$$

The algorithmic procedure for carrying out euclidean division between two elements of $\mathbb{R}_{\mathscr{P}}(s)$ may be reduced to a standard division of polynomials. This is due to the following result.

Proposition 2.5

Let $t(s)\in\mathbb{R}_{\mathscr{P}}(s)$, $-\alpha\in\mathbb{R}$, $-\alpha\notin\Omega$ and let us denote by $w=1/(s+\alpha)$. Then $t(s)$ may be expressed as

$$t(s)=u_\alpha(s)\tilde{t}_\alpha(w) \tag{2.13}$$

where $u_\alpha(s)$ is a unit in $\mathbb{R}_{\mathscr{P}}(s)$ and $\tilde{t}_\alpha(w)$ is a polynomial in $\mathbb{R}[w]$ such that $\deg \tilde{t}_\alpha(w)=\delta_{\mathscr{P}}(t(s))$.

Proof

For any α, such that $\alpha\in\mathbb{R}$, $-\alpha\notin\Omega$, we may write

$$t(s)=u_\alpha(s)\frac{n_\Omega(s)}{(s+\alpha)^{d'}}\frac{1}{(s+\alpha)^{q_\infty}}=u_\alpha(s)\tilde{t}_\alpha(s) \tag{2.14}$$

Given that $w=1/(s+\alpha)$, then $s=(1-\alpha w)/w$, substituting s by $(1-\alpha w)/w$ in $\tilde{t}_\alpha(s)$ we have

$$\tilde{t}_\alpha(s)=\tilde{t}_\alpha\left(\frac{1-\alpha w}{w}\right)=w^{d'}n_\Omega\left(\frac{1-\alpha w}{w}\right)w^{q_\infty} \tag{2.15}$$

If $n_\Omega(s)=a_{d'}s^{d'}+a_{d'-1}s^{d'-1}+\ \dots\ +a_1s+a_0$, then

$$\begin{aligned} n_\Omega\left(\frac{1-\alpha w}{w}\right)&=\frac{1}{w^{d'}}\{a_{d'}(1-\alpha w)^{d'}+a_{d'-1}w(1-\alpha w)^{d'-1}+\ \dots\ +a_0w^{d'}\}\\ &=\frac{1}{w^{d'}}\tilde{n}_\Omega(w) \end{aligned} \tag{2.16}$$

where $\tilde{n}_\Omega(w)\in\mathbb{R}[w]$ and $\deg\tilde{n}_\Omega(w)=d'$. By (2.14), (2.15) and (2.16), we have

$$t(s)=u_\alpha(s)w^{d'}\frac{1}{w_{d'}}\tilde{n}_\Omega(w)w^{q_\infty}=u_\alpha(s)\tilde{n}_\Omega(w)w^{q_\infty}=:u_\alpha(s)\tilde{t}_\alpha(w)\qquad\square$$

The transformation $w=1/(s+\alpha)$, $\alpha>0$, $\alpha\notin\Omega^c$ is a bilinear transformation which maps the region Ω of the s-plane onto a region Ω_w of the w-plane. The set $\mathscr{P}:=\Omega\cup\{\infty\}$ is mapped under w to the set $\mathscr{P}_w:=\Omega_w\cup\{0\}$. In the case where $\Omega\equiv\mathbb{C}_+$, Ω_w is the region of the w-plane enclosed by the circle with its centre at $(1/2\alpha, 0)$ and radius $1/2\alpha$ (Ω_w includes the circle boundary but excludes the origin $w=0$) and $\mathscr{P}_w$ is the above disc *together* with the origin $w=0$ of the w-plane (see Fig. 1).

Remark 2.4

The mod α primes of $\mathbb{R}_{\mathscr{P}}(s)$ are transformed under the transformation $s=(1-\alpha w)/w$ into irreducible polynomials in $\mathbb{R}[w]$, with zeros inside $\mathscr{P}_w$. Thus

$$\begin{aligned} p^*(s)&=\frac{1}{s+\alpha}=w\\ p(s)&=\frac{s+\lambda}{s+\alpha}=(\lambda-\alpha)\left[\frac{1}{s+\alpha}\right]+1=(\lambda-\alpha)w+1\\ p'(s)&=\frac{s^2+as+b}{(s+\alpha)^2}=(\alpha^2-\alpha a+b)\left[\frac{1}{s+\alpha}\right]^2+(a-2\alpha)\left[\frac{1}{s+\alpha}\right]+1\\ &=(\alpha^2-\alpha a+b)w^2+(\alpha-2\alpha)w+1 \end{aligned}$$

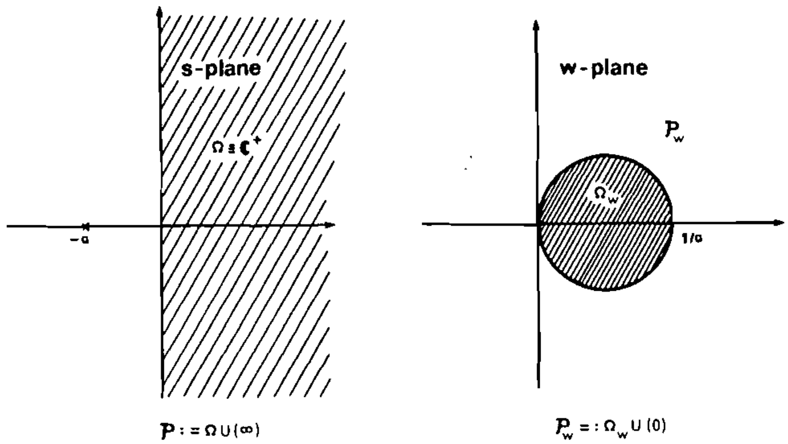

Figure 1.

Proposition 2.6

Let $\bar{l}(w)\in\mathbb{R}[w]$, $-\alpha\in\mathbb{R}$, $-\alpha\notin\Omega$ and let $\mathscr{P}_w$ be the region of the w-plane defined as the mapping of $\mathscr{P}=\Omega\cup\{\infty\}$ under $w=1/(s+\alpha)$. The rational function defined as $t(s):=\bar{l}(1/(s+\alpha))$ belongs to $\mathbb{R}_{\mathscr{P}}(s)$ and $\delta_{\mathscr{P}}(t(s))\leqslant \deg \bar{l}(w)$. Furthermore, $\delta_{\mathscr{P}}(t(s))$ is equal to the total number of zeros of $\bar{l}(w)$ in $\mathscr{P}_w$.

Proof

Let $\bar{l}(w)=a_d w^d+\ \dots\ +a_1 s+a_0$. Then

$$t(s):=\bar{l}\left(\frac{1}{s+\alpha}\right)=\frac{1}{(s+\alpha)^d}\{a_d+\ \dots\ +a_1(s+\alpha)^{d-1}+a_0(s+\alpha)^d\}=\frac{n(s)}{(s+\alpha)^d}$$

and thus $t(s)\in\mathbb{R}_{\mathscr{P}}(s)$. The maximum number of zeros of $n(s)$ is d; given that $n(s)$ might have zeros in Ω^c, it follows that $\delta_{\mathscr{P}}(t(s))\leqslant d$. By factorizing $t(s)$ as $t(s)=u(s)n_{\Omega}(s)/(s+\alpha)^d$, where $u(s)$ is a unit, $d=\delta_{\mathscr{P}}(t(s))$ and $n_{\Omega}(s)$ has no zeros outside Ω, then by Proposition 2.5 and Remark 2.4, $n_{\Omega}(s)/(s+\alpha)^d$ yields under the transformation $s=(1-\alpha w)/w$ a polynomial $\bar{l}'(w)$ with all its zeros in $\mathscr{P}_w$ and of degree $\delta_{\mathscr{P}}(t(s))$. □

Example

Let $\Omega=\mathbb{C}_+$ and $\bar{l}(w)=(w-\frac{1}{3})(w-\frac{1}{5})$.

(i) $\alpha=1$ or $w=1/(s+1)$. Then both roots of $\bar{l}(w)$ are in $\mathscr{P}_{1w}$ and $\bar{l}(s):=t(1/(s+1))$ should have $\delta_{\mathscr{P}}(t(s))=2$. In fact

$$t(s)=\left[\frac{1}{s+1}-\tfrac{1}{3}\right]\left[\frac{1}{s+1}-\tfrac{1}{5}\right]=\frac{(s-2)(s-4)}{15(s+1)^2}$$

and

$$\delta_{\mathscr{P}}(t(s))=2=\deg \bar{l}(w)$$

(ii) $\alpha=4$ or $w=1/(s+4)$. Then only one zero of $\bar{l}(w)$ is in $\mathscr{P}_{4w}$

$$t(s)=\left[\frac{1}{s+4}-\tfrac{1}{3}\right]\left[\frac{1}{s+4}-\tfrac{1}{5}\right]=\frac{(s+1)(s-1)}{15(s+4)^2}$$

and

$$\delta_{\mathscr{P}}(t(s)) = 1 < \deg \tilde{t}(w)$$

which is equal to the number of zeros of $\tilde{t}(w)$ in $\mathscr{P}_{4w}$.

(iii) $\alpha = 6$ or $w = 1/(s+6)$. Then $\mathscr{P}_{6w}$ contains none of the roots of $\tilde{t}(w)$. In fact

$$t(s) = \left[\frac{1}{s+6} - \tfrac{1}{3}\right]\left[\frac{1}{s+6} - \tfrac{1}{5}\right] = \frac{(s+1)(s+3)}{15(s+6)^2}$$

and

$$\delta_{\mathscr{P}}(t(s)) = 0$$

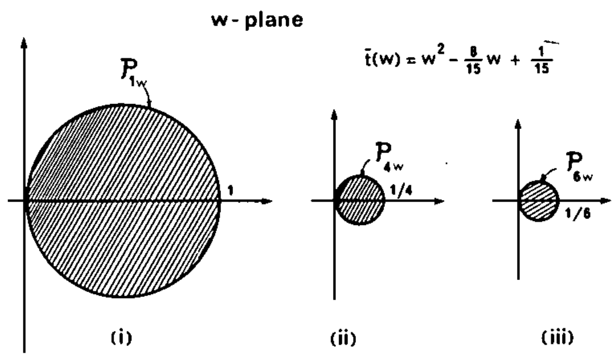

Figure 2.

2.2. *Euclidean division in* $\mathbb{R}_{\mathscr{P}}(s)$

The existence of a euclidean division in the ring $\mathbb{R}_{\mathscr{P}}(s)$ has been established by Hung and Anderson (1978). The strong links between $\mathbb{R}_{\mathscr{P}}(s)$ and $\mathbb{R}[w]$, where $w = 1/(s+\alpha)$, allow the reduction of a euclidean division in $\mathbb{R}_{\mathscr{P}}(s)$, to a standard division of polynomials in $\mathbb{R}[w]$. This approach is more suitable for computations and will be discussed in the following. (Note that the $\delta_{\mathscr{P}}(t(s))$ function serves as a degree.)

Theorem 2.1

Let $t_1(s), t_2(s) \in \mathbb{R}_{\mathscr{P}}(s)$, $t_2(s) \not\equiv 0$ and let $w = 1/(s+\alpha)$, $-\alpha \in \mathbb{R}$, $-\alpha \notin \Omega$. If $t_i(s) = \tilde{t}_{i\alpha}(w)u_{i\alpha}(s)$, $i = 1, 2$ are (mod α) factorizations of $t_1(s)$, $t_2(s)$ where $\tilde{t}_{1\alpha}(w), \tilde{t}_{2\alpha}(w) \in \mathbb{R}[w]$, $u_{1\alpha}(s)$, $u_{2\alpha}(s)$ are units in $\mathbb{R}_{\mathscr{P}}(s)$ and $\delta_{\mathscr{P}}(t_i(s)) = \deg \tilde{t}_{1\alpha}(w)$ then :

(i) There exist polynomials $\tilde{q}_\alpha(w)$, $\tilde{r}_\alpha(w) \in \mathbb{R}[w]$ such that

$$\tilde{t}_{1\alpha}(w) = \tilde{t}_{2\alpha}(w)\tilde{q}_\alpha(w) + \tilde{r}_\alpha(w) \tag{2.16}$$

and either $\tilde{r}_\alpha(w) = 0$ or else $\deg \tilde{r}_\alpha(w) < \deg \tilde{t}_{2\alpha}(w)$.

(ii) The rational functions $q_\alpha(s)$, $r_\alpha(s) \in \mathbb{R}_{\mathscr{P}}(s)$, defined by

$$q_\alpha(s) := u_{1\alpha}(s)u_{2\alpha}(s)^{-1}\tilde{q}_\alpha\left(\frac{1}{s+\alpha}\right) \tag{2.17}$$

$$r_\alpha(s) := u_{1\alpha}(s)\tilde{r}_\alpha\left(\frac{1}{s+\alpha}\right) \tag{2.18}$$

satisfy the euclidean division conditions for $t_1(s)$, $t_2(s)$, i.e.

$$t_1(s) = t_2(s)q_\alpha(s) + r_\alpha(s) \tag{2.19}$$

and either $r_\alpha(s) = 0$ or else $\delta_{\mathscr{P}}(r_\alpha(s)) < \delta_{\mathscr{P}}(t_2(s))$.

Proof

The (mod α) factorizations of $t_1(s)$, $t_2(s)$ has been established by Proposition 2.5 and for the polynomials $\tilde{t}_{1\alpha}(w)$, $\tilde{t}_{2\alpha}(w)$ part (i) of the theorem is obvious. By multiplying both sides of (2.16) by $u_{1\alpha}(s)$ and by setting $w = 1/(s+\alpha)$ we have the following identity

$$t_1(s) = u_{1\alpha}(s)\tilde{t}_{1\alpha}(w)$$

$$= \left\{u_{1\alpha}(s)u_{2\alpha}(s)^{-1}\tilde{q}_\alpha\left(\frac{1}{s+\alpha}\right)\right\}\left\{u_{2\alpha}(s)\tilde{t}_{2\alpha}\left(\frac{1}{s+\alpha}\right)\right\} + u_{1\alpha}(s)\tilde{r}_\alpha\left(\frac{1}{s+\alpha}\right)$$

or

$$t_1(s) = q_\alpha(s)t_2(s) + r_\alpha(s)$$

By Proposition 2.6, $q_\alpha(s), r_\alpha(s) \in \mathbb{R}_{\mathscr{P}}(s)$ and $\delta_{\mathscr{P}}(r_\alpha) \leqslant \deg \tilde{r}_\alpha(w)$. Given that $\deg \tilde{r}_\alpha(w) < \deg \tilde{t}_{2\alpha}(w) = \delta_{\mathscr{P}}(t_2(s))$, it follows that $\delta_{\mathscr{P}}(r_\alpha(s)) < \delta_{\mathscr{P}}(t_2(s))$. □

The above result defines the mod α euclidean division of two elements of $\mathbb{R}_{\mathscr{P}}(s)$ and provides an alternative proof that $\mathbb{R}_{\mathscr{P}}(s)$ is a euclidean ring. For a given pair $\tilde{t}_{1\alpha}(s)$, $\tilde{t}_{2\alpha}(w) \in \mathbb{R}[w]$ the pair $\tilde{q}\ (w), \tilde{r}\ (w) \in \mathbb{R}[w]$ is uniquely defined; thus $q_\alpha(s)$ and $r_\alpha(s)$ are also uniquely defined for a mod α euclidean division. However, different choices of α yield different pairs of $q_\alpha(s)$, $r_\alpha(s)$ and thus, the euclidean division in $\mathbb{R}_{\mathscr{P}}(s)$ does not possess the uniqueness property for the quotient and the remainder.

The family of $(q_\alpha(s), r_\alpha(s))$ obtained for various values of α is not characterized by a uniquely defined 'degree' remainder; this is further emphasized by Proposition 2.6 where it is stated that $\deg \tilde{r}(w) \geqslant \delta_{\mathscr{P}}(r(s))$. The non-uniqueness of $\delta_{\mathscr{P}}(r_\alpha(s))$ for the family of remainders has been noticed by Vidyasagar and Viswanadham (1982) and this has motivated them to consider the pair $(q(s), r(s))$ with $\delta_{\mathscr{P}}(r)$ minimum. However, it is not clear how one should go about choosing such a pair. In the following, the mod α euclidean division should be used; the fact that $\delta_{\mathscr{P}}(r_\alpha(s))$ might not be minimal does not affect our analysis in the subsequent sections.

Example

Let $\Omega \equiv \mathbb{C}_+$ and let $t_1(s) = (s-1)/(s+2)^3$, $t_2(s) = (s-2)/(s+2)^2$.

(i) Choose $\alpha = 2$, or $w = 1/(s+2)$. Then by setting $s = (1-2w)/w$, $\tilde{t}_{1\alpha}(w) = w^2(1-3w) = -3w^3 + w^2$, $\tilde{t}_{2\alpha}(w) = w(1-4w) = -4w^2 + w$. The pair $\tilde{q}_\alpha(w) = \frac{3}{4}w - \frac{1}{16}$, $\tilde{r}_\alpha(w) = \frac{1}{16}w$ satisfies $\tilde{t}_{1\alpha}(w) = \tilde{t}_{2\alpha}(w)\tilde{q}_\alpha(w) + \tilde{r}_\alpha(w)$ and thus

$$\frac{s-1}{(s+2)^3} = \frac{s-2}{(s+2)^2}\,\frac{-s+10}{16(s+2)} + \frac{1}{16(s+2)}$$

i.e.

$$q_\alpha(s) = \frac{-s+10}{(s+2)^2}, \quad r_\alpha(s) = \frac{1}{16(s+2)}$$

(ii) Choose $\alpha = 1$, or $w = 1/(s+1)$. Then

$$\tilde{t}_{1\alpha}(s) = \frac{s-1}{(s+1)^3}\,\frac{(s+1)^3}{(s+2)^3} = \tilde{t}_{1\alpha}(s)u_{1\alpha}(s)$$

$$\tilde{t}_{2\alpha}(s) = \frac{s-2}{(s+1)^2}\,\frac{(s+1)^2}{(s+2)^2} = \tilde{t}_{2\alpha}(s)u_{2\alpha}(s)$$

and by setting $s=(1-w)/w$ then $\tilde{t}_{1\alpha}(w)=-2w^3+w^2$, $\tilde{t}_{2\alpha}(w)=-3w^2+w$. The pair $\tilde{q}_\alpha(w)=\frac{2}{3}w-\frac{1}{9}$, $\tilde{r}_\alpha(w)=\frac{1}{9}$ satisfies $\tilde{t}_{1\alpha}(w)=\tilde{t}_{2\alpha}(w)\tilde{q}_\alpha(w)+\tilde{r}_\alpha(w)$ and thus

$$\frac{s-1}{(s+2)^2}=\frac{s-2}{(s+2)^2}\,\frac{-s+5}{9(s+2)}+\frac{(s+1)^2}{9(s+2)^3}$$

i.e.

$$q_\alpha(s)=\frac{-s+5}{9(s+2)},\quad r_\alpha(s)=\frac{(s+1)^2}{9(s+2)^3}$$

2.3. $\mathbb{R}(s)$ *as a quotient field of* $\mathbb{R}_{\mathscr{P}}(s)$

It is well known that corresponding to any integral domain D there is a unique field K, called the *quotient field of* D, such that:

(i) $D\subset K$; and
(ii) Any element of K is the quotient of two elements of D.

The following proposition states this fact for the particular case of $\mathbb{R}_{\mathscr{P}}(s)\subset\mathbb{R}(s)$, i.e. that every rational function can be written as a quotient of (coprime in $\mathscr{P}$) proper and Ω-stable rational functions. This representation of a rational function appeared previously in the works of Desoer *et al.* (1980), Saeks and Muray (1980) and Francis and Vidyasagar (1980), but no constructive procedure to achieve such a representation has yet been presented.

Proposition 2.7

Let $t(s)\in\mathbb{R}(s)$ ($\not\equiv 0$). Then $t(s)$ can be written (non-uniquely) as

$$t(s)=t_1(s)t_2(s)^{-1} \tag{2.20}$$

where $t_1(s)$, $t_2(s)\in\mathbb{R}_{\mathscr{P}}(s)$ are coprime (in $\mathscr{P}$).

Proof

Factorize $t(s)$ as in (2.1), i.e. let

$$t(s)=\frac{n_\Omega(s)}{d_\Omega(s)}\cdot\frac{\hat{n}(s)}{\hat{d}(s)}$$

and define $\bar{p}:=\deg\hat{d}(s)-\deg n_\Omega(s)\in\mathbb{Z}$, $\bar{q}:=\deg\hat{n}(s)-\deg d_\Omega(s)\in\mathbb{Z}$, $l=\min(\bar{p},\bar{q})$. Let $-\alpha\in\mathbb{R}$, be outside Ω but otherwise arbitrary. Then

$$\begin{aligned}t(s)&=\frac{n_\Omega(s)}{d_\Omega(s)}\cdot\frac{\hat{n}(s)}{\hat{d}(s)}=\frac{n_\Omega(s)}{\hat{d}(s)}(s+\alpha)^l\cdot\frac{\hat{n}(s)}{d_\Omega(s)}(s+\alpha)^{-l}\\&=\frac{n_\Omega(s)}{\hat{d}(s)}(s+\alpha)^l\left[\frac{d_\Omega(s)}{\hat{n}(s)}(s+\alpha)^l\right]^{-1}\\&=t_1(s)t_2(s)^{-1}\end{aligned}$$

Assume that $t(s)\in\mathbb{R}_{\mathrm{pr}}(s)$. Then
$\deg d_\Omega(s)+\deg\hat{d}(s)-\deg n_\Omega(s)-\deg\hat{n}(s)=\bar{p}-\bar{q}\geqslant 0\Rightarrow l=\min(\bar{p},\bar{q})=\bar{q}$.

If $\bar{q} \geqslant 0$, then

$$t_1(s) = \frac{n_\Omega(s)}{\hat{d}(s)}(s+\alpha)^{\bar{q}}$$

and

$$\delta_\infty(t_1(s)) = \deg \hat{d}(s) - \deg n_\Omega(s) - \bar{q} = \bar{p} - \bar{q} \geqslant 0$$

hence $t_1(s) \in \mathbb{R}_{\mathscr{P}}(s)$. Also

$$t_2(s) = \frac{d_\Omega(s)}{\hat{n}(s)}(s+\alpha)^{\bar{q}}$$

and

$$\delta_\infty(t_2(s)) = \deg \hat{n}(s) - \deg d_\Omega(s) - \bar{q} = \bar{q} - \bar{q} = 0$$

i.e. $t_2(s) \in \mathbb{R}_{\mathscr{P}}(s)$ and it is biproper. If $\bar{q} < 0$ then

$$t_1(s) = \frac{n_\Omega(s)}{\hat{d}(s)} \frac{1}{(s+\alpha)^{-\bar{q}}}$$

and

$$\delta_\infty(t_1(s)) = \deg \hat{d}(s) - \bar{q} - \deg n_\Omega(s) = \bar{p} - \bar{q} \geqslant 0$$

hence $t_1(s) \in \mathbb{R}_{\mathscr{P}}(s)$.

Also

$$t_2(s) = \frac{d_\Omega(s)}{\hat{n}(s)} \frac{1}{(s+\alpha)^{-\bar{q}}}$$

and

$$\delta_\infty(t_2(s)) = \deg \hat{n}(s) - \bar{q} - \deg d_\Omega(s) = \bar{q} - \bar{q} = 0$$

i.e. $t_2(s) \in \mathbb{R}_{\mathscr{P}}(s)$ is biproper. Using similar arguments we can prove that if $t(s) \notin \mathbb{R}_{\mathrm{pr}}(s)$ then again it can be written as in (2.20) where $t_1(s), t_2(s) \in \mathbb{R}_{\mathscr{P}}(s)$ and at least one is biproper. The fact that $t_1(s)$, $t_2(s)$ are coprime in $\mathscr{P}$ follows from the fact that at least one of them is biproper and $n_\Omega(s)$, $d_\Omega(s)$ are coprime polynomials (see (iv) in Proposition 2.4). Of course the non-uniqueness of the representation is due only to the arbitrariness in the choice of $\alpha \in \mathbb{R}$ and outside Ω. □

Example

Let $t(s) = 1/(s-1)$ and $\Omega \equiv \mathbb{C}_+$. Then $\bar{p} = \deg 1 - \deg 1 = 0$, $\bar{q} = \deg 1 - \deg (s-1) = -1$ and $l = \min(0, -1) = -1$.

$$t_1(s) = \frac{n_\Omega(s)}{\hat{d}(s)}(s+\alpha)^l = \frac{1}{s+\alpha}, \quad t_2(s) = \frac{d_\Omega(s)}{\hat{n}(s)}(s+\alpha)^l = \frac{s-1}{s+\alpha}$$

so that

$$\frac{1}{s-1} = \frac{1}{s+\alpha}\left[\frac{s-1}{s+\alpha}\right]^{-1}.$$

Corollary 2.2

Let $t(s) \in \mathbb{R}(s)\,(\not\equiv 0)$ and write $t(s) = t_1(s)t_2(s)^{-1}$ where $t_1(s), t_2(s) \in \mathbb{R}_{\mathscr{P}}(s)$ are coprime in $\mathscr{P}$. Then $t(s) \in \mathbb{R}_{\mathrm{pr}}$ if and only if $t_2(s)$ is biproper (i.e. a unit in $\mathbb{R}_{\mathrm{pr}}(s)$).

Corollary 2.3

Let $t(s) \in \mathbb{R}_{\mathrm{pr}}(s)$. Then $t(s) \in \mathbb{R}_{\mathscr{P}}(s)$ if and only if $t_2(s) \in \mathbb{R}_{\mathscr{P}}(s)$ in (2.20) is a *unit* in $\mathbb{R}_{\mathscr{P}}(s)$.

Proof

From Corollary 2.1 $t(s) \in \mathbb{R}_{\mathscr{P}}(s) \Leftrightarrow$ that

$$t(s) = \frac{n_\Omega(s)}{(s+\alpha)^q}\, u(s) = t_1(s)t_2(s)^{-1}$$

where

$$t_1(s) = \frac{n_\Omega(s)}{(s+\alpha)^q} \in \mathbb{R}_{\mathscr{P}}(s)$$

and $t_2(s) = u(s)^{-1}$: $\mathbb{R}_{\mathscr{P}}(s)$ is a unit in $\mathbb{R}_{\mathscr{P}}(s)$. □

3. Smith–MacMillan form of a rational matrix in $\mathscr{P} = \Omega \cup \{\infty\}$

Denote now by $\mathbb{R}_{\mathscr{P}}{}^{p \times m}(s)$ the set of all $p \times m$ matrices with elements in $\mathbb{R}_{\mathscr{P}}(s)$. These matrices we call proper and ' Ω-stable ' rational matrices (in the sense that they have no poles at infinity (proper) and also no finite poles inside Ω). If $\Omega \equiv \mathbb{C}_+ = \{s \in \mathbb{C} \mid \mathrm{Re}\,(s) \geqslant 0\}$ then $\mathbb{R}_{\mathscr{P}}{}^{p \times m}(s)$ represents the set of proper and ' stable ' rational matrices. A matrix $T(s) \in \mathbb{R}_{\mathscr{P}}{}^{p \times p}(s)$ is called $\mathbb{R}_{\mathscr{P}}(s)$-unimodular if there exists a $\hat{T}(s) \in \mathbb{R}_{\mathscr{P}}{}^{p \times p}(s)$ such that $\hat{T}(s)T(s) = I_p$. A direct implication of the above is that $\hat{T}(s) \in \mathbb{R}_{\mathscr{P}}{}^{p \times p}(s)$ is $\mathbb{R}_{\mathscr{P}}$-unimodular if it has also no zeros at $s = \infty$ and no finite zeros in Ω (i.e. $T(s)$ has no poles or zeros in $\mathscr{P} = \Omega \cup \{\infty\}$). A system theoretic interpretation of an $\mathbb{R}_{\mathscr{P}}(s)$-unimodular matrix $T(s) \in \mathbb{R}_{\mathscr{P}}{}^{p \times p}(s)$ is the following. Let $T(s) \in \mathbb{R}_{\mathscr{P}}{}^{p \times p}(s)$ be $\mathbb{R}_{\mathscr{P}}(s)$-unimodular and let $n := \delta_M(T(s)) \geqslant 0$ be the MacMillan degree of $T(s)$. If now $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times p}$, $C \in \mathbb{R}^{p \times n}$, $E \in \mathbb{R}^{p \times p}$ is a canonical (minimal) realization of $T(s)$ then we have the following points.

(i) Spectrum (A) is outside Ω, i.e. if $n > 0$ then spectrum $(A) \subset \Omega^C$ ($T(s)$ has no finite poles in Ω).
(ii) $\mathrm{rank}_{\mathbb{R}}\, E = p$ ($T(s)$ has no zeros at $s = \infty$).
(iii) Spectrum $(A - BE^{-1}C)$ is outside Ω, i.e. if $n > 0$ then spectrum $(A - BE^{-1}C) \subset \Omega^C$ ($T(s)$ has no finite zeros in Ω) where Ω^C is the complement of Ω with respect to $\mathbb{C}$, i.e. $\Omega \cup \Omega^C = \mathbb{C}$.

In the particular case where $\Omega \equiv \mathbb{C}_+$, an $\mathbb{R}_{\mathscr{P}}(s)$-unimodular matrix $T(s) \in \mathbb{R}_{\mathscr{P}}{}^{p \times p}(s)$ represents a square, biproper (i.e. having also no zeros at $s = \infty$), stable and ' minimum phase ' transfer function matrix.

Elementary row and column operations on a $T(s) \in \mathbb{R}^{p\times m}(s)$ are now defined as follows.

(i) Interchange any two rows (columns) of $T(s)$.
(ii) Multiply row (column) i of $T(s)$ by a *unit* $u(s) \in \mathbb{R}_{\mathscr{P}}(s)$.
(iii) Add to row (column) i of $T(s)$ a multiple by $t(s) \in \mathbb{R}_{\mathscr{P}}(s)$ of row (column) j.

These elementary operations can be accomplished by multiplying the given $T(s)$ on the left (right) by 'elementary' $\mathbb{R}_{\mathscr{P}}(s)$-unimodular matrices obtained by performing the above operations on the identity matrix $I_{p(m)}$. It can also be shown that every $\mathbb{R}_{\mathscr{P}}(s)$-unimodular matrix may be represented as the product of a finite number of elementary $\mathbb{R}_{\mathscr{P}}(s)$-unimodular matrices (Gantmacher 1960).

Definition 3.1

Let $T_1(s) \in \mathbb{R}^{p\times m}(s)$, $T_2(s) \in \mathbb{R}^{p\times m}(s)$. Then $T_1(s)$ and $T_2(s)$ are called *equivalent in* $\mathscr{P}$ if there exist $\mathbb{R}_{\mathscr{P}}(s)$-unimodular matrices $T_L(s) \in \mathbb{R}_{\mathscr{P}}{}^{p\times p}(s)$, $T_R(s) \in \mathbb{R}_{\mathscr{P}}{}^{m\times m}(s)$ such that

$$T_L(s) T_1(s) T_R(s) = T_2(s) \tag{3.1}$$

If $T_L(s) \equiv I_p \in \mathbb{R}_{\mathscr{P}}{}^{p\times p}(s)$ ($\equiv I_m \in \mathbb{R}_{\mathscr{P}}{}^{m\times m}(s)$) then $T_1(s)$, $T_2(s)$ are called *column* (*row*) *equivalent in* $\mathscr{P}$. Equation (3.1) defines an equivalence relation on $\mathbb{R}^{p\times m}(s)$, which we denote by $\mathscr{E}^{\mathscr{P}}$, and if $T_1(s)$, $T_2(s)$ are equivalent in $\mathscr{P}$ we denote this fact by writing $(T_1(s), T_2(s)) \in \mathscr{E}^{\mathscr{P}}$. The $\mathscr{E}^{\mathscr{P}}$ equivalence class or the 'orbit' of a fixed $T(s) \in \mathbb{R}^{p\times m}(s)$ we denote by $[T(s)]_{\mathscr{E}}{}^{\mathscr{P}}$. Let $T(s) \in \mathbb{R}^{p\times m}(s)$ with $\operatorname{rank}_{\mathbb{R}(s)} T(s) = r$ and consider the quotient of $\mathbb{R}^{p\times m}(s)$ by $\mathscr{E}^{\mathscr{P}}$, i.e. the set (denoted by) $\mathbb{R}^{p\times m}(s)/\mathscr{E}^{\mathscr{P}}$ of $\mathscr{E}^{\mathscr{P}}$-equivalence classes $[T(s)]_{\mathscr{E}}{}^{\mathscr{P}}$ when $T(s)$ runs through the elements of $\mathbb{R}^{p\times m}(s)$. We can characterize these equivalence classes by determining complete sets of invariants and canonical forms. We now have the following theorem.

Theorem 3.1 (*Smith–MacMillan form of a rational matrix in* $\mathscr{P} := \Omega \cup \{\infty\}$)

Let $T(s) \in \mathbb{R}^{p\times m}(s)$ with $\operatorname{rank}_{\mathbb{R}(s)} T(s) = r$. Then $T(s)$ is equivalent in $\mathscr{P}$ to a diagonal matrix $S_{T(s)}{}^{\mathscr{P}}$ having the form

$$S_{T(s)}{}^{\mathscr{P}} = \operatorname{diag}\{\epsilon_1(s)\psi_1(s)^{-1}, \epsilon_2(s)\psi_2(s)^{-1}, \ldots, \epsilon_r(s)\psi_r(s)^{-1}, 0, \ldots, 0\} \tag{3.2}$$

where

$$\epsilon_i(s) = \frac{\epsilon_{i\Omega}(s)}{(s+\alpha)^{p_i}} \in \mathbb{R}_{\mathscr{P}}(s) \quad \text{and} \quad \psi_i(s) = \frac{\psi_{i\Omega}(s)}{(s+\alpha)^{l_i}} \in \mathbb{R}_{\mathscr{P}}(s)$$

are coprime in $\mathscr{P}$, $i \in \mathbf{r}$; $\epsilon_{i\Omega}(s), \psi_{i\Omega}(s) \in \mathbb{R}[s]$ have their zeros not outside Ω, $-\alpha \in \mathbb{R}$ is outside Ω and otherwise arbitrary, and

$$0 \leqslant \delta_\Omega(\epsilon_i(s)) := p_i \leqslant p_{i+1} =: \delta_\Omega(\epsilon_{i+1}(s)),\ \epsilon_i(s) \mid \epsilon_{i+1}(s), \quad i \in \mathbf{r}-1$$

$$0 \leqslant \delta_\Omega(\psi_{i+1}(s)) := l_{i+1} \leqslant l_i =: \delta_\Omega(\psi_i(s)),\ \psi_{i+1}(s) \mid \psi_i(s), \quad i \in \mathbf{r}-1$$

The matrix $S_{T(s)}{}^{\mathscr{P}}$ can also be written as

$$S_{T(s)}{}^{\mathscr{P}} = E^{\mathscr{P}}(s) \Psi^{\mathscr{P}}{}_R(s)^{-1} = \Psi_L{}^{\mathscr{P}}(s)^{-1} E^{\mathscr{P}}(s) \tag{3.3}$$

where

$$E^{\mathscr{P}}(s)=\left[\begin{array}{ccc|c} \epsilon_1(s) & & \bigcirc & \\ & \epsilon_2(s) & & 0_{r,m-r} \\ \bigcirc & & \epsilon_r(s) & \\ \hline & 0_{p-r,r} & & 0_{p-r,m-r} \end{array}\right]\in\mathbb{R}_{\mathscr{P}}{}^{p\times m}(s) \tag{3.4}$$

$$\Psi_R{}^{\mathscr{P}}(s)=\operatorname{diag}\,(\psi_1(s),\,\psi_2(s),\,\ldots,\,\psi_r(s),\,I_{m-r})\in\mathbb{R}_{\mathscr{P}}{}^{m\times m}(s) \tag{3.5}$$

$$\Psi_L{}^{\mathscr{P}}(r)=\operatorname{diag}\,(\psi_1,\,\psi_2,\,\ldots,\,\psi_r(s),\,I_{p-r})\in\mathbb{R}_{\mathscr{P}}{}^{p\times p}(s) \tag{3.6}$$

and as

$$S_{T(s)}{}^{\mathscr{P}}=\operatorname{diag}\left\{\frac{\epsilon_{1\Omega}(s)}{\psi_{1\Omega}(s)}\,(s+\alpha)^{q_1},\,\ldots,\,\frac{\epsilon_{r\Omega}(s)}{\psi_{r\Omega}(s)}\,(s+\alpha)^{q_r},\,0\,\ldots\,0\right\} \tag{3.7}$$

where $q_i := p_i - l_i \in \mathbb{Z}$.

Proof

We firstly write every element $t_{ij}(s)$ of $T(s)$ as a quotient (mod α) : $t_{ij}(s) = n_{\alpha ij}(s)d_{\alpha ij}(s)^{-1}$ where $n_{\alpha ij}(s),\, d_{\alpha ij}(s)\in\mathbb{R}_{\mathscr{P}}(s)$. Let $d_\alpha(s)\in\mathbb{R}_{\mathscr{P}}(s)$ be the l.c.m. of the 'denominators' $d_{\alpha ij}(s)$, $i\in\mathbf{p}$, $j\in\mathbf{m}$, and write

$$T(s)=d_\alpha(s)^{-1}N_\alpha(s) \tag{2.8}$$

where $N_\alpha(s)\in\mathbb{R}_{\mathscr{P}}{}^{p\times m}(s)$. Following the steps used in order to reduce a polynomial matrix to its (finite) Smith form (Rosenbrock 1970) and replacing : (i) the degree in $\mathbb{R}[s]$ by $\delta_{\mathscr{P}}(\cdot)$ in $\mathbb{R}_{\mathscr{P}}(s)$; and (ii) euclidean division in $\mathbb{R}[s]$ by the mod α euclidean division in $\mathbb{R}_{\mathscr{P}}(s)$, it follows that $N_\alpha(s)$ may be reduced by elementary $\mathbb{R}_{\mathscr{P}}(s)$-unimodular column and row operations to its Smith form in $\mathscr{P}$: $E_{N_\alpha(s)}{}^{\mathscr{P}}$, i.e. for $\mathbb{R}_{\mathscr{P}}(s)$-unimodular matrices $U_{\alpha L}(s)$, $U_{\alpha R}(s)$

$$U_{\alpha L}(s)N_\alpha(s)U_{\alpha R}(s)=E_{N_\alpha(s)}{}^{\mathscr{P}}=\left[\begin{array}{c|c} \bar{E}^{\mathscr{P}}(s) & 0_{r\times m-r} \\ \hline 0_{p-r,r} & 0_{p-r,m-r}\end{array}\right] \tag{3.9}$$

where

$$\bar{E}^{\mathscr{P}}(s)=\operatorname{diag}\,\{\bar{\epsilon}_1(s),\,\ldots,\,\bar{\epsilon}_r(s)\}$$

$$\bar{\epsilon}_i(s)=\frac{\bar{\epsilon}_{i\Omega}(s)}{(s+\alpha)^{p_i}}\in\mathbb{R}_{\mathscr{P}}(s) \quad\text{and}\quad \bar{\epsilon}_i(s)\,|\,\bar{\epsilon}_{i+1}(s),\quad i\in\mathbf{r}-1$$

From (2.8) and (2.9) we have

$$U_{\alpha L}(s)T(s)U_{\alpha R}(s)=d_\alpha(s)^{-1}E_{N_\alpha(s)}{}^{\mathscr{P}}$$

By factorizing (mod α) $d_\alpha(s)$ and $\bar{\epsilon}_i(s)$ into irreducible over $\mathbb{R}_{\mathscr{P}}(s)$ factors and carrying out all possible cancellations in $\mathbb{R}_{\mathscr{P}}(s)$ the result follows. □

Corollary 3.1

The rational functions $\epsilon_i(s)\psi_i(s)^{-1}\in\mathbb{R}(s)$ form a complete (mod α) set of invariants for $\mathscr{E}^{\mathscr{P}}$ on $\mathbb{R}^{p\times m}(s)$ and they will be referred to as the *invariant rational functions of* $T(s)$ *in* $\mathscr{P}$. Also $S_{T(s)}{}^{\mathscr{P}}$ is a (unique mod α) canonical form for $\mathscr{E}^{\mathscr{P}}$ on $\mathbb{R}^{p\times m}(s)$.

Remark 3.1

The finite zeros of

$$\epsilon_i(s)=\frac{\epsilon_{i\Omega}(s)}{(s+\alpha)^{p_i}}\in\mathbb{R}_{\mathscr{P}}(s)$$

i.e. the (finite) zeros of $\epsilon_{i\Omega}(s)\in\mathbb{R}[s]$, $i\in\mathbf{r}$, give the (finite) zeros of $T(s)$ in Ω while the $q_{z\infty}{}^i>0$ where

$$q_{z\infty}{}^i:=p_i-\deg\epsilon_{i\Omega}(s)\geqslant 0 \tag{3.10}$$

give the orders of the zeros at $s=\infty$ of $T(s)$. Also the finite zeros of

$$\psi_i(s)=\frac{\psi_{i\Omega}(s)}{(s+\alpha)^{l_i}}\in\mathbb{R}_{\mathscr{P}}(s)$$

i.e. the (finite) zeros of $\psi_{i\Omega}(s)\in\mathbb{R}[s]$, $i\in\mathbf{r}$, give the (finite) poles of $T(s)$ in Ω while the $q_{p\infty}{}^i>0$ where

$$q_{p\infty}{}^i:=l_i-\deg\psi_{i\Omega}(s)\geqslant 0 \tag{3.11}$$

give the orders of the poles at $s=\infty$ of $T(s)$.

Corollary 3.2

Let $T(s)\in\mathbb{R}^{p\times m}(s)$. Then $T(s)$ is proper, i.e. $T(s)\in\mathbb{R}_{\mathrm{pr}}{}^{p\times m}(s)$ if and only if $\Psi_R{}^{\mathscr{P}}(s)$, $\Psi_L{}^{\mathscr{P}}(s)$ are *biproper* rational matrices (see Vardulakis and Karcanias 1982).

Proof

$T(s)\in\mathbb{R}_{\mathrm{pr}}{}^{p\times m}(s)$ means that $T(s)$ *has no poles at* $s=\infty$ and from (3.11) this implies that $q_{p\infty}{}^i=l_i-\deg\psi_{i\Omega}(s)=0$, $\forall i\in\mathbf{r}$, or that $l_i=\deg\psi_{i\Omega}(s)$, $\forall i\in\mathbf{r}$, i.e. that

$$\psi_i(s)=\frac{\psi_{i\Omega}(s)}{(s+\alpha)^{l_i}}\in\mathbb{R}_{\mathrm{pr}}(s)$$

are biproper rational functions (units in $\mathbb{R}_{\mathrm{pr}}(s)$) and hence both $\Psi_R(s)$ and $\Psi_L(s)$ are biproper rational matrices (see Vardulakis and Karcanias 1983). □

Corollary 3.3

Let $T(s)\in\mathbb{R}_{\mathrm{pr}}{}^{p\times m}(s)$. Then $T(s)$ is left (right) biproper (i.e. it consists of the m (p) columns (rows) of a biproper rational matrix (see Vardulakis and Karcanias 1982) if and only if $E^{\mathscr{P}}(s)$ is left (right) biproper.

Proof

$T(s)\in\mathbb{R}_{\mathrm{pr}}{}^{p\times m}(s)$ being left (right) biproper means that $p\geqslant m$ ($p\leqslant m$), $\lim_{s\to\infty}T(s)=E\in\mathbb{R}^{p\times m}$ and $\mathrm{rank}_{\mathbb{R}}\,E=m$ $(=p)$, i.e. that $T(s)$ has also no zeros at $s=\infty$ (see Vardulakis *et al.* 1982, Vardulakis and Karcanias 1983). Thus from (3.10) we have $q_{z\infty}=0$, $\forall i\in\mathbf{r}$ ($r=\min(p,m)$), or equivalently that $p_i=\deg\epsilon_{i\Omega}(s)$, $\forall i\in\mathbf{r}$, or that $\epsilon_i(s)=\epsilon_{i\Omega}(s)/(s+\alpha)^{p_i}$ are biproper functions, i.e. that $\mathscr{E}^{\mathscr{P}}(s)$ is left (right) biproper rational matrix. □

Corollary 3.4

Let $T(s)\in\mathbb{R}^{p\times m}(s)$. Then $T(s)\in\mathbb{R}_{\mathscr{P}}{}^{p\times m}(s)$, i.e. $T(s)$ is proper and Ω-stable if and only if $\Psi_R{}^{\mathscr{P}}(s)=I_m$, $\Psi_L{}^{\mathscr{P}}(s)=I_p$, i.e. $\Psi_R{}^{\mathscr{P}}(s)$, $\Psi_L{}^{\mathscr{P}}(s)$ are $\mathbb{R}_{\mathscr{P}}(s)$-unimodular matrices.

Proof

If $T(s)\in\mathbb{R}_{\mathscr{P}}{}^{p\times m}(s)$, i.e. it is proper and Ω-stable or equivalently has no poles in $\mathscr{P}:=\Omega\cup\{\infty\}$, then $\psi_{i\Omega}(s)=1$, $l_i=0$, $i\in\mathbf{r}$, i.e. $\psi_i(s)=1$ and $p_i\equiv q_i$, $i\in\mathbf{r}$. Thus $\Psi_R{}^{\mathscr{P}}(s)=I_m$, $\Psi_L{}^{\mathscr{P}}(s)=I_p$. □

Remark 3.1

If $T(s)\in\mathbb{R}_{\mathscr{P}}{}^{p\times m}(s)$ then we have

$$S_{T(s)}{}^{\mathscr{P}}=b\ \mathrm{diag}\left[\frac{\epsilon_{1\Omega}(s)}{(s+\alpha)^{q_1}},\ \ldots,\ \frac{\epsilon_{r\Omega}(s)}{(s+\alpha)^{q_r}},\ 0,\ \ldots,\ 0\right]\equiv E^{\mathscr{P}}(s)\in\mathbb{R}_{\mathscr{P}}{}^{p\times m}(s) \quad (3.12)$$

and then $S_{T(s)}{}^{\mathscr{P}}$ is called the *Smith form of* $T(s)$ *in* $\mathscr{P}$. Otherwise, some of the $\psi_i(s)$s will be different than unity and then $S_{T(s)}{}^{\mathscr{P}}$ is called the *MacMillan form of* $T(s)$ *in* $\mathscr{P}$.

If $T(s)\in\mathbb{R}_{\mathscr{P}}{}^{p\times m}(s)$ then the invariant rational functions of $T(s)$ in

$$\mathscr{P}:\ \epsilon_i(s)=\frac{\epsilon_{i\Omega}(s)}{(s+\alpha)^{q_i}}\in\mathbb{R}_{\mathscr{P}}(s),\quad i\in\mathbf{r}$$

can be computed directly using the standard 'Smith algorithm':

$$\epsilon_i(s)=m_i(s)m_{i-1}(s)^{-1},\quad i\in\mathbf{r} \quad (2.13)$$

with $m_0(s):=1$ and where $m_i(s)\in\mathbb{R}_{\mathscr{P}}(s)$ is the (mod α) g.c.d. of the ith order minors of $T(s)$.

Remark 3.2

If Ω coincides with the whole $\mathbb{C}$ plane except of a *unique* point $-\alpha\in\mathbb{R}$ then $S_{T(s)}{}^{\mathscr{P}}$ will give the Smith–MacMillan form of $T(s)$ in all, except one (i.e. at $s=-\alpha$), points of the 'extended' complex plane $\mathbb{C}^\infty=\mathbb{C}\cup\{\infty\}$.

Example

Let $\Omega\equiv\mathbb{C}_+$. Then $\mathbb{R}_{\mathscr{P}}(s)$ is the set

$$\mathbb{R}_{\mathscr{P}}(s)=\{t(s)\in\mathbb{R}(s)\,|\,t(s)\text{ has no poles in }\mathbb{C}_+\cup\{\infty\}\}$$

i.e. $\mathbb{R}_{\mathscr{P}}(s)$ is the euclidean ring of proper and 'stable' rational functions. Consider the proper and stable rational matrix

$$T(s)=\begin{bmatrix}\dfrac{1-s}{(s+1)^2} & \dfrac{1}{2s+1}\\[2ex] 0 & \dfrac{1}{s+1}\end{bmatrix}\in\mathbb{R}_{\mathscr{P}}{}^{2\times 2}(s)$$

and find its Smith form in $\mathscr{P}=\mathbb{C}_+\cup\{\infty\}$

$$\delta_{\mathscr{P}}(t_{11})=2,\quad \delta_{\mathscr{P}}(t_{12})=1,\quad \delta_{\mathscr{P}}(t_{21})=+\infty,\quad \delta_{\mathscr{P}}(t_{22})=1$$

(i) By column interchange, bring element (1, 2), which has least $\delta_{\mathscr{P}}(\cdot)$, to position (1, 1)

$$\begin{bmatrix} \dfrac{1-s}{(s+1)^2} & \dfrac{1}{2s+1} \\ 0 & \dfrac{1}{s+1} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{2s+1} & \dfrac{1-s}{(s+1)^2} \\ \dfrac{1}{s+1} & 0 \end{bmatrix}$$

(ii) Write

$$\frac{1-s}{(s+1)^2} = \frac{1}{2s+1} \cdot \frac{(2s+1)(1-s)}{(s+1)^2}$$

where

$$\frac{(2s+1)(1-s)}{(s+1)^2} \in \mathbb{R}_{\mathscr{P}}(s)$$

Now multiply column 1 by $-(2s+1)(1-s)/(s+1)$ and add it to column 2

$$\begin{bmatrix} \dfrac{1}{2s+1} & \dfrac{1-s}{(s+1)^2} \\ \dfrac{1}{s+1} & 0 \end{bmatrix} \begin{bmatrix} 1 & -\dfrac{(2s+1)(1-s)}{(s+1)^2} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{2s+1} & 0 \\ \dfrac{1}{s+1} & -\dfrac{(2s+1)(1-s)}{(s+1)^3} \end{bmatrix}$$

(iii) Write

$$\frac{1}{s+1} = \frac{1}{2s+1} \cdot \frac{2s+1}{s+1}$$

and multiply row 1 by $-(2s+1)/(s+1) \in \mathbb{R}_{\mathscr{P}}(s)$ and add it to row 2

$$\begin{bmatrix} 1 & 0 \\ -\dfrac{2s+1}{s+1} & 1 \end{bmatrix} \begin{bmatrix} \dfrac{1}{2s+1} & 0 \\ \dfrac{1}{s+1} & -\dfrac{(2s+1)(1-s)}{(s+1)^3} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{2s+1} & 0 \\ 0 & -\dfrac{(2s+1)(1-s)}{(s+1)^3} \end{bmatrix}$$

(iv) Now write

$$\frac{1}{2s+1} = \frac{1}{s+\alpha} \cdot \left(\frac{s+\alpha}{2s+1} \right)$$

and

$$-\frac{(2s+1)(1-s)}{(s+1)^3} = \frac{1-s}{(s+\alpha)^2} \cdot \left[-\frac{(s+\alpha)^2(2s+1)}{(s+1)^3} \right]$$

where $\alpha > 0$ but is otherwise arbitrary, and multiply the resulting matrix in (iii) by

$$\begin{bmatrix} \frac{2s+1}{s+\alpha} & 0 \\ 0 & -\frac{(s+1)^3}{(s+\alpha)^2(2s+1)} \end{bmatrix} \begin{bmatrix} \frac{1}{2s+1} & 0 \\ 0 & -\frac{(2s+1)(1-s)}{(s+1)^3} \end{bmatrix} = \begin{bmatrix} \frac{1}{s+\alpha} & 0 \\ 0 & \frac{1-s}{(s+\alpha)^2} \end{bmatrix}$$

So the Smith form of $T(s)$ in $\mathscr{P} = \mathbb{C}_+ \cup \{\infty\}$ is

$$S_{T(s)}{}^{\mathscr{P}} = \begin{bmatrix} \frac{1}{s+\alpha} & 0 \\ 0 & \frac{1-s}{(s+\alpha)^2} \end{bmatrix} = \begin{bmatrix} \epsilon_1(s) & 0 \\ 0 & \epsilon_2(s) \end{bmatrix}, \quad \alpha > 0 \text{ and arbitrary}$$

and $T(s)$ has no (finite) poles in $\mathbb{C}_+$ and one (finite) zero in $\mathbb{C}_+$: $s=1$. Also $p_1 = 1$, $p_2 = 2$. Therefore from (3.10)

$$q_{z\infty}{}^1 = p_1 - \deg \epsilon_{1\Omega}(s) = 1 - 0 = 1$$

$$q_{z\infty}{}^2 = p_2 - \deg \epsilon_{2\Omega}(s) = 2 - 1 = 1$$

$T(s)$ has two zeros at $s = \infty$ each one of order $q_{z\infty}{}^i = 1$, $i = 1, 2$.

If we are not interested in the elementary $\mathbb{R}_{\mathscr{P}}(s)$-unimodular row and column operations that bring $T(s)$ to $S_{T(s)}{}^{\mathscr{P}}$, we can also proceed as follows : the (non-zero) minors of $T(s)$ of order one (i.e. its elements) are $(1-s)/(s+1)^2$, $1/(2s+1)$, $1/(s+1)$ and the minor of $T(s)$ of order two, i.e. its determinant is $(1-s)/(s+1)^3$. Now write

$$\frac{1-s}{(s+1)^2} = \frac{1-s}{(s+\alpha)^2}\frac{(s+\alpha)^2}{(s+1)^2}, \quad \frac{1}{2s+1} = \frac{1}{s+\alpha}\frac{s+\alpha}{2s+1}, \quad \frac{1}{s+1} = \frac{1}{s+\alpha}\frac{s+\alpha}{s+1}.$$

Hence the g.c.d. (in $\mathscr{P}$) of minors of single order is $1/(s+\alpha) =: m_1(s)$. Writing

$$\frac{1-s}{(s+1)^3} = \frac{1-s}{(s+\alpha)^3}\frac{(s+\alpha)^3}{(s+1)^3}$$

the g.c.d. (in $\mathscr{P}$) of minors of order two is $(1-s)/(s+\alpha)^3 = m_2(s)$, hence with $m_0(s) := 1$, from (3.13)

$$\epsilon_1(s) = m_1(s)m_0(s)^{-1} = m_1(s) = \frac{1}{s+\alpha}$$

$$\epsilon_2(s) = m_2(s)m_1(s)^{-1} = \frac{1-s}{(s+\alpha)^3}\cdot\frac{s+\alpha}{1} = \frac{1-s}{(s+\alpha)^2} \qquad \square$$

The *pole rational function in* $\mathscr{P}$ *of a* $T(s) \in \mathbb{R}^{p\times m}(s)$ is defined as

$$p_{\mathscr{P}}(s) := \prod_{i=1}^{r} \psi_i(s) = \prod_{i=1}^{r} \frac{\psi_{i\Omega}(s)}{(s+\alpha)^{l_i}} \in \mathbb{R}_{\mathscr{P}}(s) \tag{3.14}$$

Obviously

$$\delta_{\mathscr{P}}(p_{\mathscr{P}}(s)) = \sum_{i=1}^{r} l_i$$

and $\delta_M^{\mathscr{P}}(T) := \delta_{\mathscr{P}}(p_{\mathscr{P}}(s))$ is defined as the *MacMillan degree of* $T(s)$ *in* $\mathscr{P}$ and it gives the total number of poles of $T(s)$ in $\mathscr{P}$ (i.e. finite ones in Ω and infinite ones, where multiplicities are accounted for).

The *zero rational function in* $\mathscr{P}$ *of a* $T(s) \in \mathbb{R}^{p\times m}(s)$ is defined as

$$z_{\mathscr{P}}(s) = \prod_{i=1}^{r} \epsilon_i(s) = \prod_{i=1}^{r} \frac{\epsilon_{i\Omega}(s)}{(s+\alpha)^{p_i}} \in \mathbb{R}_{\mathscr{P}}(s) \tag{3.15}$$

where

$$\delta_{\mathscr{P}}(z) = \sum_{i=1}^{r} p_i$$

gives the total number of zeros of $T(s)$ in $\mathscr{P}$ (i.e. finite ones in Ω and infinite ones, where multiplicities are accounted for).

The computation of the pole and zero rational functions in $\mathscr{P}$ of a $T(s) \in \mathbb{R}^{p\times m}(s)$ can be achieved without the prior determination of $S_{T(s)}^{\mathscr{P}}$. The basis of this computation is contained in the following adaptation of Lemma 1 of Kalman (1965) for the ring $\mathbb{R}_{\mathscr{P}}(s)$, which can be stated as follows.

Lemma 3.1

Let $d_\alpha^{\,j}(s) \in \mathbb{R}_{\mathscr{P}}(s)$ be the l.c.m. of the denominators of the jth order minors of $T(s)$ when these minors have been represented as quotients (mod α) of elements in $\mathbb{R}_{\mathscr{P}}(s)$. Then

$$d_\alpha^{\,j}(s) = \prod_{i=1}^{j} \psi_i(s), \quad j \in \mathbf{r}$$

The proof of Lemma 3.1 is identical to that given in Kalman (1965). This result implies the following algorithm for the computation of $p_{\mathscr{P}}(s)$ and $z_{\mathscr{P}}(s)$.

(i) *Pole rational function* $p_{\mathscr{P}}(s)$ *of* $T(s)$ *in* $\mathscr{P}$. Compute all non-zero minors of $T(s)$ of all possible orders and write them as quotients (mod α) of elements in $\mathbb{R}_{\mathscr{P}}(s)$. $p_{\mathscr{P}}(s)$ is then given as the l.c.m. of the denominators (in $\mathbb{R}_{\mathscr{P}}(s)$) of these minors.

(ii) *Zero rational function* $z_{\mathscr{P}}(s)$ *of* $T(s)$ *in* $\mathscr{P}$. Write all maximal order non-zero minors of $T(s)$ as quotients (mod α) of elements from $\mathbb{R}_{\mathscr{P}}(s)$. Adjust all these minors to have as common denominator the $p_{\mathscr{P}}(s)$. The g.c.d. in $\mathbb{R}_{\mathscr{P}}(s)$ of the numerators of these adjusted minors gives $z_{\mathscr{P}}(s)$.

4. Coprimeness in $\mathscr{P}$ of proper and Ω-stable rational matrices

We introduce now the notions right or left coprimeness of rational matrices in the region $\mathscr{P} = \Omega \cup \{\infty\}$. From the definition of the zeros of a $T(s) \in \mathbb{R}^{p\times m}(s)$ in $\mathscr{P} = \Omega \cup \{\infty\}$ via its Smith–MacMillan form : $S_{T(s)}^{\mathscr{P}}$ in $\mathscr{P}$ we have the following proposition.

Proposition 4.1

Let $T(s)\in\mathbb{R}^{p\times m}(s)$, $\operatorname{rank}_{\mathbb{R}(s)} T(s)=r$. Then the following statements are equivalent.

(i) $T(s)$ has no zeros in $\mathscr{P}=\Omega\cup\{\infty\}$.

(ii) $\epsilon_i(s)=1$, $i\in r\Leftrightarrow\epsilon_{i\Omega}(s)=1$ *and* $p_i=0$, $i\in r$.

(iii) $S_{T(s)}{}^{\mathscr{P}}=\operatorname{diag}\left\{\dfrac{1}{\psi_1(s)},\ldots,\dfrac{1}{\psi_r(s)},0,\ldots,0\right\}$. (i.e. $T(s)$ has possibly only *poles* in $\mathscr{P}$.)

Definition 4.1

Given two rational matrices $A(s)\in\mathbb{R}^{l\times m}(s)$, $B(s)\in\mathbb{R}^{t\times m}(s)$ with $p:=l+t\geqslant m$ and $\operatorname{rank}_{\mathbb{R}(s)}\left[\dfrac{A(s)}{B(s)}\right]=m$, then we say that (the rows of) $A(s)$ and $B(s)$ are *right coprime in* $\mathscr{P}=\Omega\cup\{\infty\}$ if $T(s):=\left[\dfrac{A(s)}{B(s)}\right]\in\mathbb{R}^{(l+t)\times m}(s)$ has no zeros in $\mathscr{P}$.

If we restrict ourselves to matrices that are proper and Ω-stable we then have the next proposition.

Proposition 4.2

Let $A(s)\in\mathbb{R}_{\mathscr{P}}{}^{l\times m}(s)$, $B(s)\in\mathbb{R}_{\mathscr{P}}{}^{t\times m}(s)$ (i.e. proper and Ω-stable) with $p:=l+t\geqslant m$. Then the following statements are equivalent.

(i) $A(s)$ and $B(s)$ are right coprime in $\mathscr{P}:=\Omega\cup\{\infty\}$.

(ii) The proper and Ω-stable matrix $T(s)=\left[\dfrac{A(s)}{B(s)}\right]\in\mathbb{R}_{\mathscr{P}}{}^{p\times m}(s)$ has no zeros in $\mathscr{P}$.

(iii) There exists a $\mathbb{R}_{\mathscr{P}}(s)$-unimodular matrix $T_L(s)\in\mathbb{R}_{\mathscr{P}}{}^{p\times p}(s)$ such that

$$T_L(s)T(s)=\begin{bmatrix} I_m \\ \\ 0_{p-m,m}\end{bmatrix}\equiv S_{T(s)}{}^{\mathscr{P}}$$

(iv) There exist proper, Ω-stable rational matrices $X(s)\in\mathbb{R}_{\mathscr{P}}{}^{m\times l}(s)$, $Y(s)\in\mathbb{R}_{\mathscr{P}}{}^{m\times t}(s)$ such that

$$X(s)A(s)+Y(s)B(s)=I_m$$

(v) There exist proper, Ω-stable rational matrices $C(s)\in\mathbb{R}_{\mathscr{P}}{}^{l\times(p-m)}(s)$, $D(s)\in\mathbb{R}_{\mathscr{P}}{}^{t\times(p-m)}(s)$ such that the proper and Ω-stable rational matrix

$$\begin{bmatrix} A(s) & C(s) \\ \\ B(s) & D(s)\end{bmatrix}\in\mathbb{R}_{\mathscr{P}}{}^{p\times p}(s)$$

is $\mathbb{R}_{\mathscr{P}}(s)$-unimodular.

(vi)
$$\operatorname{rank}_{\mathbb{C}}\left[\frac{A(s_0)}{B(s_0)}\right]=m,\quad \forall s_0\in\Omega$$

and

$$\lim_{s\to\infty}\left[\frac{A(s)}{B(s)}\right]=E\in\mathbb{R}^{p\times m}$$

with $\operatorname{rank}_{\mathbb{R}} E=m$.

Proof

Follow the steps given for a similar proposition given in Vardulakis *et al.* (1982).

Definition 4.2

A proper and Ω-stable rational matrix $T(s)\in\mathbb{R}_{\mathscr{P}}^{p\times m}(s)$, $(p\geqslant m)$ satisfying the equivalent conditions of Proposition 4.2, is defined as a $\mathbb{R}_{\mathscr{P}}(s)$-*left unimodular* rational matrix. ($\mathbb{R}_{\mathscr{P}}(s)$-right unimodular matrices can be defined in an analogous manner.)

4.1. *Coprime in $\mathscr{P}$ $\mathbb{R}_{\mathscr{P}}(s)$-matrix function descriptions of a rational matrix*

We now generalize the results concerning the representation of a rational function $t(s)$ as quotient of proper and Ω-stable rational functions for the matrix case. This representation was first introduced by Vidyasagar (1975), and later used by Francis and Vidyasagar (1980), Desoer *et al.* (1980), Saeks and Murray (1981) (see also Vidyasagar *et al.* 1982, Vidyasagar and Viswanadham 1982, Saeks and Murray 1982, Francis and Vidyasagar 1983). In this section, algorithmic procedures for constructing the representations of a general (not necessarily proper) rational matrix $T(s)$ based on the factorizations of the Smith–MacMillan form of $T(s)$ in $\mathscr{P}$ are presented.

We start by introducing the notions of right (common) divisors in $\mathscr{P}$ and of greatest (common) right divisors in $\mathscr{P}$ of (two or more) rational matrices (having the same number of columns). Left (common) and greatest left (common) divisors in $\mathscr{P}$ can be defined analogously. The following result describes an important factorization of a rational matrix which is the matrix analogue of Proposition 2.2.

Proposition 4.3

Any rational matrix $T(s)\in\mathbb{R}^{p\times m}(s)$ with $p\geqslant m$ and $\operatorname{rank}_{\mathbb{R}(s)} T(s)=m$ can be factorized (in a non-unique way) as

$$T(s)=T_1(s)T_{GR}(s) \tag{4.1}$$

where $T_1(s)\in\mathbb{R}_{\mathscr{P}}^{p\times m}(s)$ is $\mathbb{R}_{\mathscr{P}}(s)$-left unimodular and $T_{GR}(s)\in\mathbb{R}^{m\times m}(s)$ has pole-zero structure in $\mathscr{P}=\Omega\cup\{\infty\}$ the same with that of $T(s)$.

Proof

Let $T_L(s)\in\mathbb{R}_{\mathscr{P}}^{p\times p}(s)$ a $\mathbb{R}_{\mathscr{P}}(s)$-unimodular rational matrix such that

$$T_L(s)T(s)=\left[\frac{T_{GR}(s)}{0}\right] \tag{4.2}$$

where $T_{GR}(s)\in\mathbb{R}^{m\times m}(s)$ ($T_L(s)$ can be chosen as the $\mathbb{R}_{\mathscr{P}}(s)$-unimodular matrix which reduces $T(s)$ to a right upper triangular form by elementary row operation over $\mathbb{R}_{\mathscr{P}}(s)$). Let now $T_L(s)^{-1} =: \hat{T}(s)\in\mathbb{R}_{\mathscr{P}}^{p\times p}(s)$ (and $\mathbb{R}_{\mathscr{P}}(s)$-unimodular) and partition $\hat{T}(s)$ as

$$\hat{T}(s)=[T_1(s)\ \ T_2(s)]$$

where $T_1(s)\in\mathbb{R}_{\mathscr{P}}{}^{p\times m}(s)$, $T_2(s)\in\mathbb{R}_{\mathscr{P}}{}^{p\times(p-m)}(s)$ and by Proposition 4.2 (v) both are $\mathbb{R}_{\mathscr{P}}(s)$-left unimodular. Then (4.2) gives

$$T(s)=[T_1(s)\ \ T_2(s)]\begin{bmatrix} I_m \\ 0 \end{bmatrix} T_{GR}(s)=T_1(s)T_{GR}(s)$$

From (4.2) $\left(T(s), \begin{bmatrix} T_{GR}(s) \\ \hline 0 \end{bmatrix}\right)\in\mathscr{E}^{\mathscr{P}}$. Hence the pole-zero structure in $\mathscr{P}$ of $T(s)$ and $\begin{bmatrix} T_{GR}(s) \\ \hline 0 \end{bmatrix}$ and hence of $T_{GR}(s)$ coincide. □

Corollary 4.1

If $T(s)\in\mathbb{R}_{\mathscr{P}}{}^{p\times m}(s)$ then it can be factorized as in (4.1) where $T_{GR}(s)\in\mathbb{R}_{\mathscr{P}}{}^{m\times m}(s)$ has the same ' zero structure ' in $\mathscr{P}$ as that of $T(s)$ (see Remark 4.1 to follow).

Definition 4.3

Let the proper and Ω-stable rational matrices $T(s)\in\mathbb{R}_{\mathscr{P}}{}^{p\times m}(s)$, $T_1(s)\in\mathbb{R}_{\mathscr{P}}{}^{p\times m}(s)$, $T_R(s)\in\mathbb{R}_{\mathscr{P}}{}^{m\times m}(s)$ be related via

$$T(s)=T_1(s)T_R(s) \tag{4.3}$$

Then $T_R(s)$ is called a (common) *right divisor in* $\mathscr{P}$ of (the rows of) $T(s)$.

Definition 4.4

Let $T(s)\in\mathbb{R}_{\mathscr{P}}{}^{p\times m}(s)$ with $p\geqslant m$ and $\operatorname{rank}_{\mathbb{R}(s)} T(s)=m$. Then any proper and Ω-stable rational matrix $T_G(s)\in\mathbb{R}_{\mathscr{P}}{}^{m\times m}(s)$ that satisfies (4.1) for some $\mathbb{R}_{\mathscr{P}}(s)$-left unimodular rational matrix $T_1(s)\in\mathbb{R}_{\mathscr{P}}{}^{p\times m}(s)$ is called a *greatest (common) right divisor* (*g.c.r.d.*) in $\mathscr{P}$ of (the rows of) $T(s)$.

Remark 4.1

If $T_{GR}(s)\in\mathbb{R}_{\mathscr{P}}{}^{m\times m}(s)$ is a g.c.r.d. in $\mathscr{P}$ of (the rows of) $T(s)\in\mathbb{R}_{\mathscr{P}}{}^{p\times m}(s)$ then from Proposition 3.3 it follows that $T_{GR}(s)$ ' contains ' *all* the zeros of $T(s)$ in $\mathscr{P}=\Omega\cup\{\infty\}$ (i.e. the finite ones in Ω and infinite ones, if any). If $T_{GR}(s)\in\mathbb{R}_{\mathscr{P}}{}^{m\times m}(s)$ happens to be $\mathbb{R}_{\mathscr{P}}(s)$-unimodular then $T(s)$ has also no zeros in $\mathscr{P}$ and its rows are said to be right coprime in $\mathscr{P}$. Notice that in such a case $T(s)$ might have *only finite zeros outside* Ω, i.e. in Ω^c.

Proposition 4.4

Let $T(s)\in\mathbb{R}^{p\times m}(s)$, $\operatorname{rank}_{\mathbb{R}(s)} T(s)=r$. Then $T(s)$ can be expressed always (in a non-unique way) as

$$T(s)=B_2(s)A_2(s)^{-1}=A_1(s)^{-1}B_1(s) \tag{4.4}$$

where $B_2(s)\in\mathbb{R}_{\mathscr{P}}{}^{p\times m}(s)$, $A_2(s)\in\mathbb{R}_{\mathscr{P}}{}^{m\times m}(s)$ are *right coprime in* $\mathscr{P}$ and $A_1(s)\in\mathbb{R}_{\mathscr{P}}{}^{p\times p}(s)$, $B_1(s)\in\mathbb{R}_{\mathscr{P}}{}^{p\times m}(s)$ are *left coprime in* $\mathscr{P}$.

Proof

Let $U_L(s)\in\mathbb{R}_{\mathscr{P}}^{p\times p}(s)$, $U_R(s)\in\mathbb{R}_{\mathscr{P}}^{m\times m}(s)$ $\mathbb{R}_{\mathscr{P}}(s)$-unimodular matrices such that

$$U_L(s)T(s)U_R(s)=S_T^{\mathscr{P}}=E^{\mathscr{P}}(s)\Psi_R^{\mathscr{P}}(s)^{-1}=\Psi_L^{\mathscr{P}}(s)^{-1}E^{\mathscr{P}}(s)$$

Then let $B_2(s):=U_L^{-1}(s)E^{\mathscr{P}}(s)$, $A_2(s):=U_R(s)\Psi_R^{\mathscr{P}}(s)$, $A_1(s):=\Psi_L^{\mathscr{P}}(s)U_L(s)$, $B_1(s):=E^{\mathscr{P}}(s)U_R(s)^{-1}$, from which we have

$$\begin{bmatrix} B_2(s) \\ A_2(s) \end{bmatrix}=\begin{bmatrix} U_L^{-1}(s) & 0 \\ 0 & U_R(s) \end{bmatrix}\begin{bmatrix} E^{\mathscr{P}}(s) \\ \Psi_R^{\mathscr{P}}(s) \end{bmatrix}$$

$$[B_1(s),\ A_1(s)]=[E^{\mathscr{P}}(s)\ \ \Psi_L^{\mathscr{P}}(s)]\begin{bmatrix} U_R^{-1}(s) & 0 \\ 0 & U_L(s) \end{bmatrix}$$

and thus it follows that $B_2(s)$, $A_2(s)$ ($B_1(s)$, $A_1(s)$) are right (left) coprime in $\mathscr{P}$. □

Definition 4.5

A pair $(B_2(s), A_2(s))$ $((B_1(s), A_1(s)))$ satisfying Proposition 4.4 is called a right (left) coprime in $\mathscr{P}$ $\mathbb{R}_{\mathscr{P}}(s)$-matrix fraction description of $T(s)\in\mathbb{R}^{p\times m}(s)$ ($\mathbb{R}_{\mathscr{P}}(s)$-MFD of $T(s)$).

Proposition 4.5

Let $T(s)\in\mathbb{R}^{p\times m}(s)$ and let

$$T(s)=B_2(s)A_2(s)^{-1}=\bar{B}_2(s)\bar{A}_2(s)^{-1}$$

be two right coprime in $\mathscr{P}$ $\mathbb{R}_{\mathscr{P}}(s)$-MFDs of $T(s)$. Then there exist $U(s)\in\mathbb{R}_{\mathscr{P}}^{m\times m}(s)$ and $\mathbb{R}_{\mathscr{P}}(s)$-unimodular such that

$$B_2(s)=\bar{B}_2(s)U(s),\quad A_2(s)=\bar{A}_2(s)U(s)$$

i.e. all right ' numerators ' and all right ' denominators ' appearing in right coprime in $\mathscr{P}$ $\mathbb{R}_{\mathscr{P}}(s)$-MFDs of $T(s)$ are column equivalent in $\mathscr{P}$. (A similar result holds for any two left coprime in $\mathscr{P}$ $\mathbb{R}_{\mathscr{P}}(s)$-MFDs of $T(s)$.

Proof

See the steps given in Rosenbrock (1970, p. 139) for an analogous result concerning polynomial MFDs of a $T(s)\in\mathbb{R}^{p\times m}(s)$.

Proposition 4.6

Let $T(s)\in\mathbb{R}^{p\times m}(s)$. Then $T(s)\in\mathbb{R}_{\mathrm{pr}}{}^{p\times m}(s)$ if and only if in every right or left coprime in $\mathscr{P}$ $\mathbb{R}_{\mathscr{P}}(s)$-MFD of $T(s)=B_2(s)A_2(s)^{-1}=A_1^{-1}(s)B_1(s)$, the ' denominator ' matrices $A_2(s)\in\mathbb{R}_{\mathscr{P}}^{m\times m}(s)$, $A_1(s)\in\mathbb{R}_{\mathscr{P}}^{p\times p}(s)$ are *biproper* rational matrices.

Proof

From Propositions 4.4 and 4.5, if

$$S_{T(s)}{}^{\mathscr{P}}=E^{\mathscr{P}}(s)\Psi_R^{\mathscr{P}}(s)^{-1}=\Psi_L^{\mathscr{P}}(s)^{-1}E^{\mathscr{P}}(s)$$

then $A_2(s)=U_R(s)\Psi_R^{\mathscr{P}}(s)$ and $A_1(s)=\Psi_{LR}(s)U_L(s)$ for some $\mathbb{R}_{\mathscr{P}}(s)$-unimodular and therefore biproper rational matrices $U_R(s)\in\mathbb{R}_{\mathscr{P}}^{m\times m}(s)$ and $U_L(s)\in\mathbb{R}_{\mathscr{P}}^{p\times p}(s)$. Now from Corollary 2.2 $T(s)\in\mathbb{R}_{\mathrm{pr}}^{p\times m}(s)$ implies both $\Psi_{R\mathscr{P}}(s)\in\mathbb{R}_{\mathscr{P}}^{m\times m}(s)$ and $\Psi_L^{\mathscr{P}}(s)\in\mathbb{R}_{\mathscr{P}}^{p\times p}(s)$ are biproper, hence $A_2(s)$, $A_1(s)$ are biproper. □

Proposition 4.7

Let $T(s)\in\mathbb{R}_{\mathrm{pr}}^{p\times m}(s)$ with $p\geqslant m$ $(p\leqslant m)$. Then $T(s)$ is left (right) biproper (Vardulakis and Karcanias 1982) if and only if in every right (left) coprime in $\mathscr{P}$ $\mathbb{R}_{\mathscr{P}}(s)$-MFD of $T(s)=B_2(s)A_2(s)^{-1}=A_1(s)^{-1}B_1(s)$ the 'numerator' matrix $B_2(s)\in\mathbb{R}_{\mathscr{P}}^{p\times m}(s)$ $(B_1(s)\in\mathbb{R}_{\mathscr{P}}^{p\times m}(s))$ is left biproper.

Proof

From Propositions 4.4 and 4.5, if

$$S_{T(s)}^{\mathscr{P}}=E^{\mathscr{P}}(s)\Psi_R^{\mathscr{P}}(s)^{-1}=\Psi_L^{\mathscr{P}}(s)^{-1}E^{\mathscr{P}}(s)$$

then $B_2(s)=U_L^{-1}(s)E^{\mathscr{P}}(s)$, $B_1(s)=E^{\mathscr{P}}(s)U_R^{-1}(s)$ for some $\mathbb{R}_{\mathscr{P}}(s)$-unimodular and therefore biproper rational matrices $U_L(s)\in\mathbb{R}_{\mathscr{P}}^{p\times p}(s)$ and $U_R(s)\in\mathbb{R}_{\mathscr{P}}^{m\times m}(s)$. Now from Corollary 3.3 : $T(s)\in\mathbb{R}_{\mathrm{pr}}^{p\times m}(s)$ is left (right) biproper$\Leftrightarrow E^{\mathscr{P}}(s)$ is left (right) biproper, hence $B_2(s)$ $(B_1(s))$ left (right) biproper. □

Proposition 4.8

Let $T(s)\in\mathbb{R}^{p\times m}(s)$. Then $T(s)\in\mathbb{R}_{\mathscr{P}}^{p\times m}(s)$, i.e. $T(s)$ is proper and Ω-stable if and only if in every right (left) coprime in $\mathscr{P}$ $\mathbb{R}_{\mathscr{P}}(s)$-MFD of $T(s)$ the 'denominator' matrix $A_2(s)\in\mathbb{R}_{\mathscr{P}}^{m\times m}(s)$ $(A_1(s)\in\mathbb{R}_{\mathscr{P}}^{p\times p}(s))$ is $\mathbb{R}_{\mathscr{P}}(s)$-unimodular.

Proof

From Propositions 4.4 and 4.5 we have that : $A_2(s)=U_R(s)\Psi_R^{\mathscr{P}}(s)$ and $A_1(s)=\Psi_L^{\mathscr{P}}(s)U_L(s)$ for some $\mathbb{R}_{\mathscr{P}}(s)$-unimodular rational matrices $U_R(s)\in\mathbb{R}_{\mathscr{P}}^{m\times m}(s)$ and $U_L(s)\in\mathbb{R}_{\mathscr{P}}^{p\times p}(s)$. Now from Corollary 3.3 : $T(s)\in\mathbb{R}_{\mathscr{P}}^{p\times m}(s)\Leftrightarrow \Psi_R^{\mathscr{P}}(s)=I_m$, $\Psi_L^{\mathscr{P}}(s)=I_p$ and hence $A_2(s)\in\mathbb{R}_{\mathscr{P}}^{m\times m}(s)$, $A_1(s)\in\mathbb{R}_{\mathscr{P}}^{p\times p}(s)$ are $\mathbb{R}_{\mathscr{P}}(s)$-unimodular. □

Remark 4.2

This corollary in fact says that if $T(s)\in\mathbb{R}^{p\times m}(s)$ is proper and 'stable' and $T(s)=N_R(s)D_R(s)^{-1}$ is a polynomial right coprime MFD of $T(s)$ then $N_R(s)D_R(s)^{-1}=B_2(s)$ is a 'numerator' in $\mathbb{R}_{\mathscr{P}}^{p\times m}(s)$ and $A_2(s)=I_m$ is a $\mathbb{R}_{\mathscr{P}}^{m\times m}(s)$-unimodular 'denominator' of $T(s)$.

5. The function Ω δ (·) for rational matrices

We now generalize the definition of the map $\delta_\Omega(\cdot)$ introduced in § 2.0 for the case of rational matrices. Let $T(s)\in\mathbb{R}^{p\times m}(s)$, $\mathrm{rank}_{\mathbb{R}(s)}\,T(s)=r$. We define the map δ_Ω : $\mathbb{R}^{p\times m}(s)\rightarrow\mathbb{Z}\cup\{+\infty\}$ via

$$\delta_\Omega(T)=\begin{cases}\min\left\{\begin{array}{l}\delta_\Omega(\cdot)\text{ among the }\delta_\Omega(\cdot)\text{ of all}\\ r\text{th (non-zero) minors of }T(s)\end{array}\right. & \text{if } r>0\\ +\infty & \text{if } r=0\end{cases}\tag{5.1}$$

As in the scalar case the restriction of the map $\delta_\Omega : \mathbb{R}^{p\times m}(s)\to\mathbb{Z}\cup\{+\infty\}$ to its subdomain $\mathbb{R}_{\mathscr{P}}{}^{p\times m}(s)\subset\mathbb{R}^{p\times m}(s)$ we will denote by the symbol $\delta_{\mathscr{P}}(\cdot)$. We now give a number of properties regarding the $\delta_\Omega(\cdot)$ of rational matrices. These results are stated for the case of rational matrices $T(s)\in\mathbb{R}^{p\times m}(s)$ with $p\geqslant m$ and similar ones can be stated for the case $p\leqslant m$. The following proposition describes the $\delta_\Omega(\cdot)$ of a product of two rational matrices which have special sizes.

Proposition 5.1

Let $T_1(s)\in\mathbb{R}^{p\times m}(s)$, $p\geqslant m$, $\text{rank}_{\mathbb{R}(s)}\, T_1(s)=r$ and $T_2(s)\in\mathbb{R}^{m\times m}(s)$, $\text{rank}_{\mathbb{R}(s)}\, T_2(s)=m$ and let

$$T(s) := T_1(s)T_2(s) \tag{5.2}$$

Then

$$\delta_\Omega(T)=\delta_\Omega(T_1)+\delta_\Omega(T_2) \tag{5.3}$$

Proof

Let $m_i(T)$, $m_i(T_1)\in\mathbb{R}(s)$, $i\in\mathbf{g}$, $g=\binom{p}{r}$ be the rth order (non-zero) minors of $T(s)$ and $T_1(s)$, respectively. From (5.2) $m_i(T)=m_i(T_1)\det T_2(s)$, $i\in\mathbf{g}$, and hence $\delta_\Omega(m_i(T))=\delta_\Omega(m_i(T_1))+\delta_\Omega(\det T_2(s))$. Thus

$$\begin{aligned}\delta_\Omega(T) &:= \min_{i\in\mathbf{g}}\{\delta_\Omega(m_i(T))\}=\min_{i\in\mathbf{g}}\{\delta_\Omega(m_i(T_1))\}+\delta_\Omega(\det T_2(s))\\ &=\delta_\Omega(T_1)+\delta_\Omega(T_2)\end{aligned}$$ □

Proposition 5.2

Let $T(s)\in\mathbb{R}_{\mathscr{P}}{}^{p\times m}(s)$. Then $\delta_{\mathscr{P}}(T)\geqslant 0$. Moreover, if $p=m=\text{rank}_{\mathbb{R}(s)}\, T(s)$ then $T(s)$ is $\mathbb{R}_{\mathscr{P}}(s)$-unimodular if and only if $\delta_{\mathscr{P}}(T)=0$.

Proof

If $T(s)\in\mathbb{R}_{\mathscr{P}}{}^{p\times m}(s)$ then all its rth order minors $m_i(T)$ are elements of $\mathbb{R}_{\mathscr{P}}(s)$ and the non-zero ones have $\delta_{\mathscr{P}}(m_i(T))\geqslant 0$. Hence $\delta_{\mathscr{P}}(T)\geqslant 0$. The second part follows from the definition of an $\mathbb{R}_{\mathscr{P}}(s)$-unimodular matrix and that of $\delta_{\mathscr{P}}(T)$. □

The following proposition describes the fact that $\delta_\Omega(\cdot)$ is an *invariant* of the column (row) $\mathbb{R}_{\mathscr{P}}(s)$-equivalence class of a rational matrix $T(s)\in\mathbb{R}^{p\times m}(s)$ with $p\geqslant m$ $(p\leqslant m)$.

Proposition 5.3

Let $T_1(s)\in\mathbb{R}^{p\times m}(s)$, $T_2(s)\in\mathbb{R}^{p\times m}(s)$ with $p\geqslant m$ $(p\leqslant m)$ be column (row) equivalent in $\mathscr{P}$, i.e. let $T_1(s)=T_2(s)T_R(s)$ $(T_1(s)=T_L(s)T_2(s))$ for some $\mathbb{R}_{\mathscr{P}}(s)$-unimodular matrix $T_R(s)\in\mathbb{R}_{\mathscr{P}}{}^{m\times m}(s)$ $(T_L(s)\in\mathbb{R}_{\mathscr{P}}{}^{p\times p}(s))$. Then $\delta_\Omega(T_1)=\delta_\Omega(T_2)$.

Proof

This follows directly from Propositions 5.1 and 5.2.

From the above and Definitions 4.3 and 4.4 of a (common) right divisor in $\mathscr{P}$ and of a greatest (common) right divisor in $\mathscr{P}$ of a proper and Ω-stable rational matrix $T(s)\in\mathbb{R}_{\mathscr{P}}{}^{p\times m}(s)$, we easily obtain a corollary.

Corollary 5.1

Let $T_1(s)\in\mathbb{R}_{\mathscr{P}}^{p\times m}(s)$, $p\geqslant m$, $\mathrm{rank}_{\mathbb{R}(s)}\, T_1(s)=m$ and let $T_R(s)\in\mathbb{R}_{\mathscr{P}}^{m\times m}(s)$ be a (common) right divisor in $\mathscr{P}$ (greatest (common) right divisor in $\mathscr{P}$) of (the rows of) $T_1(s)$, i.e. let $T_1(s)=T_2(s)T_R(s)$ for some $T_2(s)\in\mathbb{R}_{\mathscr{P}}^{p\times m}(s)$ (for some $\mathbb{R}_{\mathscr{P}}(s)$-left unimodular $T_2(s)\in\mathbb{R}_{\mathscr{P}}^{p\times m}(s)$). Then from (5.3)

$$\delta_{\mathscr{P}}(T_1)=\delta_{\mathscr{P}}(T_2)+\delta_{\mathscr{P}}(T_R) \tag{5.4}$$

and since from Proposition 4.2 $\delta_{\mathscr{P}}(T_i)\geqslant 0$, $i=1, 2$, we have

$$\delta_{\mathscr{P}}(T_1)\geqslant\delta_{\mathscr{P}}(T_2) \tag{5.5}$$

with equality holding if and only if $T_2(s)\in\mathbb{R}_{\mathscr{P}}^{m\times m}(s)$ is $\mathbb{R}_{\mathscr{P}}(s)$-unimodular.

In order to proceed in the examination of properties of $\delta_{\mathscr{P}}(\cdot)$ for a rational matrix $T(s)\in\mathbb{R}_{\mathscr{P}}^{p\times m}(s)$, we will need an expression for the $\delta_{\Omega}(\cdot)$ of a polynomial matrix. To this end we need another concept, that of the *degree* of a polynomial matrix (Vardulakis and Karcanias 1983).

Definition 5.1

The degree of a polynomial matrix $T(s)\in\mathbb{R}^{p\times m}[s]$, denoted by $\deg T$, is defined as the *maximum* degree among the degrees of all its maximum order (non-zero) minors.

If $T_1(s)\in\mathbb{R}^{p\times m}[s]$, $p\geqslant m$, and $T_2(s)\in\mathbb{R}^{m\times m}[s]$ with $\mathrm{rank}_{\mathbb{R}(s)}\, T_2(s)=m$ then it follows from Definition 5.1 and the Binet–Cauchy theorem that $\deg(T_1T_2)=\deg T_1+\deg T_2$. Now, let $T(s)\in\mathbb{R}^{m\times m}[s]$, $\mathrm{rank}_{\mathbb{R}(s)}\, T(s)=m$, and let $S_{T(s)}{}^{\mathbf{C}}=\mathrm{diag}\,(\epsilon_1(s), \ldots, \epsilon_m(s))$ the (finite) Smith form of $T(s)$. By factorizing each invariant polynomial $\epsilon_i(s)$, $i\in\mathbf{m}$, of $T(s)$ as $\epsilon_i(s)=\epsilon_{i\Omega}(s)\hat{\epsilon}_i(s)$ where $\epsilon_{i\Omega}(s)$ has no zeros outside Ω and $\hat{\epsilon}_i(s)$ has zeros outside Ω it simply follows that every $T(s)$ can always be factorized (non-uniquely) as

$$T(s)=T_{\Omega}(s)\hat{T}(s) \tag{5.6}$$

where $T_{\Omega}(s)$ has no (finite) zeros outside Ω and $\hat{T}(s)$ has (finite) zeros outside Ω. Thus we obtain the next proposition.

Proposition 5.4

Let $T(s)\in\mathbb{R}^{m\times m}[s]$, $\mathrm{rank}_{\mathbb{R}(s)}\, T(s)=m$ and factorize it as in (5.6). Then

$$\delta_{\Omega}(T)=-\deg\det\hat{T}=-\deg\hat{T} \tag{5.7}$$

Proof

From (4.1)

$$\begin{aligned}\delta_{\Omega}(T):=&\ \delta_{\Omega}\,(\det T(s))=\delta_{\Omega}\,(\det T_{\Omega}(s)\det\hat{T}(s))=-\deg\det\hat{T}(s)\\ =&-\deg\hat{T}(s)\end{aligned}$$

□

We now investigate the $\delta_{\Omega}(\cdot)$ of a *non-square polynomial matrix*. Let $N(s)\in\mathbb{R}^{p\times m}[s]$, $p\geqslant m$, $\mathrm{rank}_{\mathbb{R}(s)}\, N(s)=m$. It is well known (Forney 1975) that $N(s)$ can be factorized as $N(s)=N_L(s)N_R(s)$ with $N_R(s)\in\mathbb{R}^{m\times m}[s]$ a greatest (common) right divisor of (the rows of) $N(s)$ and $N_L(s)\in\mathbb{R}^{p\times m}[s]$ a (polynomial) *minimal* basis for the rational vector space $\mathscr{N}(s)$ spanned by the columns of

$N(s)$ (Forney 1975), i.e. $N_L(s) \in \mathbb{R}^{p\times m}[s]$ is column proper (Wolovich 1974) and has relatively right coprime rows. Also factorizing $N_R(s) \in \mathbb{R}^{m\times m}[s]$ as in (5.6), i.e. as $N_R(s) = N_{R\Omega}(s)\hat{N}_R(s)$, we have finally that $N(s)$ can be factorized as

$$N(s) = N_L(s)N_{R\Omega}(s)\hat{N}_R(s) \tag{5.8}$$

Denoting now by $m_i(N)$ and $m_i(N_L)$, $i \in \mathbf{g}$, the $g = \binom{p}{m}$ mth order minors of $N(s)$ and $N_L(s)$, respectively, we have $m_i(N) = m_i(N_L)$ det $N_{R\Omega}(s)$ det $\hat{N}_R(s)$ and hence

$$\begin{aligned}\delta_\Omega[m_i(N)] &= \delta_\Omega[m_i(N_L) \det N_{R\Omega}(s) \det \hat{N}_R(s)] \\ &= \delta_\Omega[m_i(N_L)] + \delta_\Omega[\hat{N}_R(s)] = \delta_\Omega[m_i(N_L)] - \deg \hat{N}_R \\ &= \delta_\Omega[m_i(N_L)] - z_f^{\Omega c}(N)\end{aligned}$$

where $z_f^{\Omega c}(N)$ denotes the number of (finite) zeros of $N(s)$ outside Ω (i.e. in Ω^c). Thus from (5.1) we develop the next proposition.

Proposition 5.5

Let $N(s) \in \mathbb{R}^{p\times m}[s]$ a polynomial matrix with $p \geq m$ and $\text{rank}_{\mathbb{R}(s)} N(s) = m$. Then

$$\delta_\Omega(N) := \min_{i\in\mathbf{g}} \{\delta_\Omega[m_i(N)]\} = \min_{i\in\mathbf{g}} \{\delta_\Omega[m_i(N_L)]\} - z_f^{\Omega c}(N) \tag{5.9}$$

where $N_L(s) \in \mathbb{R}^{p\times m}[s]$ is a *minimal basis* for the rational vector space $\mathcal{N}(s)$ spanned by the columns of $N(s)$ and $z_f^{\Omega c}(N)$ denotes the number of (finite) zeros of $N(s)$ outside Ω.

Proposition 5.6

The quantity

$$k(\mathcal{N}(s)) := \min_{i\in\mathbf{g}} \{\delta_\Omega[m_i(N_L)]\} \tag{5.10}$$

appearing in (5.9) is an invariant of the rational vector space $\mathcal{N}(s)$ spanned by the columns of $N(s)$, i.e. if $\bar{N}_L(s) \in \mathbb{R}^{p\times m}[s]$ is any other *minimal basis* of $\mathcal{N}(s)$ then $\min_{i\in\mathbf{g}} \{\delta_\Omega[m_i(N_L)]\} = \min_{i\in\mathbf{g}} \{\delta_\Omega[m_i(\bar{N}_L)]\}$.

Proof

If $N_L(s)$ and $\bar{N}_L(s)$ are both polynomial minimal bases of $\mathcal{N}(s)$ then $N_L(s) = \bar{N}_L(s)U_R(s)$ for some unimodular matrix $U_R(s) \in \mathbb{R}^{m\times m}[s]$. Hence, $m_i(N_L) = c_i \cdot m_i(\bar{N}_L)$, $c_i \in \mathbb{R}$ $(\neq 0)$, $i \in \mathbf{g}$, and the result follows. □

Remark 5.1

If we denote by $\text{ord}_F(\mathcal{N}(s))$ the *Forney invariant dynamical order of* $\mathcal{N}(s)$ (which by definition is given by the sum of column degrees (known as the 'invariant dynamical indices of $\mathcal{N}(s)$') of any minimal basis $N_L(s)$ of $\mathcal{N}(s)$ (Forney 1975), then we have the following properties.

(i) If $m_i(N_L)\in\mathbb{R}[s]$ have zeros outside Ω *for every* $i\in\mathbf{g}$ then $\delta_\Omega(m_i(N_L)) = -\deg\cdot m_i(N_L)$ and hence

$$k(\mathcal{N}(s) := \min_{i\in\mathbf{g}}\{-\deg\cdot m_i(N_L)\} = -\max_{i\in\mathbf{g}}\{\deg\cdot m_i(N_L)\}$$
$$= -\deg\cdot N_L = -\mathrm{ord}_F(\mathcal{N}(s)) \tag{5.11}$$

(ii) If $m_i(N_L)$ has no zeros outside Ω, *for every* $i\in\mathbf{g}$, then $\delta_\Omega(m_i(N_L))=0$, $i\in\mathbf{g}$, and hence $k(\mathcal{N}(s))=0$.

(iii) In general, $-\mathrm{ord}_F(\mathcal{N}(s))\leqslant k(\mathcal{N}(s))\leqslant 0$.

The next proposition generalizes to the matrix case, the definition in (1.2) of the function $\delta_\Omega(\cdot)$ of a scalar rational function.

Proposition 5.7

Let $T(s)\in\mathbb{R}^{p\times m}(s)$, $p\geqslant m$, $\mathrm{rank}_{\mathbb{R}(s)}\,T(s)=m$ and $T(s)=N(s)D(s)^{-1}$ be a right coprime polynomial MFD, i.e. $N(s)\in\mathbb{R}^{p\times m}[s]$, $D(s)\in\mathbb{R}^{m\times m}[s]$. Then

$$\delta_\Omega(T) = p_f^{\Omega c}(T) + k(\mathcal{T}(s)) - z_f^{\Omega c}(T) \tag{5.12}$$

where $p_f^{\Omega c}(T)$ is the number of (finite) poles of $T(s)$ outside Ω (i.e. in Ω^c).

Proof

From $D(s)D(s)^{-1}=I_m$ and Proposition 5.1 we have $\delta_\Omega(D^{-1})=-\delta_\Omega(D)$. Factorizing $N(s)$ as in (5.8) and $D(s)$ as in (5.6) and making use of Propositions 5.1, 5.4, 5.5, we have

$$\delta_\Omega(T) = \delta_\Omega(N)-\delta_\Omega(D) = k(\mathcal{T}(s)) - z_f^{\Omega c}(T) + \deg\,\hat{D}(s)$$
$$= k(\mathcal{T}(s)) - z_f^{\Omega c}(T) + p_f^{\Omega c}(T)$$

The above result gives rise to a relationship between the function $\delta_{\mathscr{P}}(T(s))$ of a proper and Ω-stable rational matrix $T(s)\in\mathbb{R}_{\mathscr{P}}^{p\times m}(s)$ and its MacMillan degree $\delta_M(T)$, which we may state as a corollary.

Corollary 5.2

Let $T(s)\in\mathbb{R}_{\mathscr{P}}^{p\times m}(s)$, $p\geqslant m$, $\mathrm{rank}_{\mathbb{R}(s)}\,T(s)=m$. Then

$$\delta_{\mathscr{P}}(T) = \delta_M(T) + k(\mathcal{T}(s)) - z_f^{\Omega c}(T) \tag{5.13}$$

Proof

Let $T(s)=N(s)D(s)^{-1}$ be a right coprime polynomial MFD of $T(s)$, and factorize $D(s)$ as in (5.6), i.e. let $D(s)=D_\Omega(s)\hat{D}(s)$. If $T(s)\in\mathbb{R}_{\mathscr{P}}^{p\times m}(s)$ then $D_\Omega(s)\in\mathbb{R}^{m\times m}[s]$ is unimodular and hence $p_f^{\Omega c}(T) := \deg\,\hat{D}(s) = \deg\,D(s) = \delta_M(T)$ and (5.13) follows from (5.12).

6. Conclusions

The fine algebraic aspects of rational matrices with respect to the ring of proper rational functions with no poles inside a region $\Omega\subset\mathbb{C}$ has been examined. Emphasis has been given to those aspects related to computational problems regarding coprime (in $\mathscr{P}$) rational representations of general rational matrices. A number of tests for coprimeness (in $\mathscr{P}$) of rational matrices have been given.

The key tool in this analysis is the Smith–MacMillan form in $\mathscr{P}$ of a general matrix $T(s)$.

This work provides the algorithmic tools necessary for computations needed in the recently developed algebraic design techniques which are based on *proper and stable* matrix fraction representations of linear systems.

References

Desoer, C. A., Liu, R. W., Murray, J., and Saeks, R., 1980, *I.E.E.E. Trans. autom. Control*, **25,** 399.

Forney, G. D., 1975, *SIAM Jl Control*, **13,** 493.

Francis, B. A., and Vidyasagar, M., 1980, S & IS Report No. 8003 ; 1983, *Automatica*, **19,** 87.

Gantmacher, M., 1960, *The Theory of Matrices* (New York : Chelsea).

Hung, N. T., and Anderson, B. D. O., 1979, *I.E.E.E. Trans. autom. Control*, **24,** 455.

Kalman, R. E., 1965, *SIAM Jl Control*, **13,** 520 ; 1966, *Ibid.*, **14,** 152.

MacDuffee, C. C., 1946, *The Theory of Matrices* (New York : Chelsea), reprinted.

Morse, A. S., 1975, *Lecture Notes in Economics and Mathematical Systems Theory*, Vol. 131 (New York : Springer), pp. 61–74.

Newman, M., 1972, *Integral Matrices* (Academic Press).

Rosenbrock, H. H., 1970, *State Space and Multivariable Theory* (London : Nelson).

Saeks, R., and Murray, J., 1981, *I.E.E.E. Trans. autom. Control*, **26,** 203 ; 1982, *Ibid.*, **27,** 895.

Vardulakis, A. I. G., and Karcanias, N., 1983, *Int. J. Control* **38,** 779.

Vardulakis, A. I. G., Limebeer, D. J. N., and Karcanias, N., 1982, *Int. J. Control*, **35,** 701.

Vidyasagar, M., 1975, *SIAM Jl Control*, **13,** 1144 ; 1978, *I.E.E.E. Trans. Circuits Systems*, **25,** 916.

Vidyasagar, M., Schneider, H., and Francis, B. A., 1982, *I.E.E.E. Trans. autom. Control*, **27,** 880.

Vidyasagar, M., and Viswanadham, N., 1982, *I.E.E.E. Trans. autom. Control*, **27,** 1085.

Wolovich, W. A., 1974, *Linear Multivariable Systems* (Berlin : Springer-Verlag).