Proper Rational Matrix Diophantine Equations and the Exact Model Matching Problem

A. I. G. VARDULAKIS

Abstract—**Relying on the theory behind the Smith–MacMillan form of a rational matrix at $s=\infty$, a necessary and sufficient condition is derived from the solvability of matrix Diophantine equations of the form $A(s)X(s)+B(s)Y(s)=M(s)$, where $A(s)$, $B(s)$, and $M(s)$ are given proper rational matrices and $X(s)$ and $Y(s)$ are unknown proper rational matrices. It is shown that the above result can be used in order to resolve in a new illuminating way the exact model matching problem (EMMP). If a solution to EMMP exists, then the family of all solutions is parametrized.**

INTRODUCTION

The primary purpose of this note is to investigate the exact model matching problem (EMMP) in the light of some concepts and results that concern the structure and the Smith–MacMillan form of a rational matrix at infinity [1]. It is shown that the resolution of EMMP is equivalent to the investigation of existence of solutions of proper rational matrix Diophantine equations, i.e., equations having the form $A(s)X(s)+B(s)Y(s)=M(s)$, where $A(s)$, $B(s)$, $M(s)$ are known proper rational matrices and $X(s)$ and $Y(s)$ unknown proper rational matrices. Thus, by analogy to other forms of Diophantine equations [2] and relying on the theory behind the Smith–MacMillan form of a rational matrix at $s=\infty$, a necessary and sufficient condition is derived for the solvability of Diophantine equations of the above form. If a solution to EMMP exists, then the family of all solutions is parametrized.

I. BACKGROUND

Let $\mathbb{R}$ be the field of reals, $\mathbb{R}[s]$ the ring of polynomials with coefficients in $\mathbb{R}$, and $\mathbb{R}(s)$ the field of rational functions $t(s)=n(s)/d(s)$, $n(s),d(s)\in\mathbb{R}[s]$, $d(s)\not\equiv 0$. Define the map $\delta_\infty:\mathbb{R}(s)\to\mathbb{Z}\cup\{\infty\}$ [1], ($\mathbb{Z}$: the ring of integers), via $\delta_\infty(t(s))=\deg\cdot d(s)-\deg\cdot n(s)$, $\delta_\infty(0)=\infty$. The map $\delta_\infty(\cdot)$ is a discrete valuation on $\mathbb{R}(s)$ [3] and every $t(s)=n(s)/d(s)\in\mathbb{R}(s)$ can be factored as $t(s)=1/s)^{q_\infty}n_1(s)/d_1(s)$ where $q_\infty:=\delta_\infty(t(s))$ and $\deg\cdot n_1(s)=\deg\cdot d_1(s)$. If $q_\infty>0$ we say that $t(s)$ has a zero at $s=\infty$ of order q_∞ and if $q_\infty<0$, then we say that $t(s)$ has a pole at $s=\infty$ of order $|q_\infty|$. If $t(s)\in\mathbb{R}(s)$ has $\delta_\infty(t(s))\geqslant 0$, then $t(s)$ is called a proper rational function and if the inequality is strict, then $t(s)$ is called strictly proper. The set of all proper rational functions, which we denote by $\mathbb{R}_{pr}(s)$, is a Euclidean ring with "degree" given by the map $\delta_\infty(\cdot)$ [1]. The units in $\mathbb{R}_{pr}(s)$ are proper rational functions with $\deg\cdot n(s)=\deg\cdot d(s)$ (i.e., having also no zeros at $s=\infty$). These units we call *biproper* rational functions. We denote by $\mathbb{R}^{p\times m}(s)$ the set of $p\times m$ matrices with elements in $\mathbb{R}(s)$ and by $\mathbb{R}_{pr}^{p\times m}(s)$ the subset of $\mathbb{R}^{p\times m}(s)$ consisting of all $p\times m$ matrices with elements in $\mathbb{R}_{pr}(s)$. A $T(s)\in\mathbb{R}_{pr}^{p\times p}(s)$ is called $\mathbb{R}_{pr}(s)$-unimodular or *biproper* if there exists a $\bar{T}(s)\in\mathbb{R}_{pr}^{p\times p}(s)$ such that $T(s)\bar{T}(s)=I_p$. "Elementary row and column operations" and a $T(s)\in\mathbb{R}^{p\times m}(s)$ are defined in the following usual way [1], [4]: 1) interchanging any two rows (columns) of $T(s)$, 2) multiply row (column) i of $T(s)$ by a biproper rational function $u(s)\in\mathbb{R}_{pr}(s)$, and 3) add to row (column) i of $T(s)$ a multiple by a $t(s)\in\mathbb{R}_{pr}(s)$ of row (column) j. These elementary operations can be accomplished by multiplying the given $T(s)$ on the left (right) by "elementary" biproper rational matrices, obtained by performing the above elementary operations on the identity matrix $I_{p(m)}$. $T_1(s)\in\mathbb{R}^{p\times m}(s)$, $T_2(s)\in\mathbb{R}^{p\times m}(s)$ are called *equivalent at* $s=\infty$ if there exist biproper rational matrices $T_L(s)\in\mathbb{R}_{pr}^{p\times p}(s)$, $T_R(s)\in\mathbb{R}_{pr}^{m\times m}(s)$, such that

$$T_L(s)T_1(s)T_R(s)=T_2(s). \tag{1}$$

Equation (1) defines an equivalence relation on $\mathbb{R}^{p\times m}(s)$, which we denote by $\mathscr{E}^\infty$ and if $T_1(s),T_2(s)$ are equivalent at $s=\infty$ we denote this fact by writing $(T_1(s),T_2(s))\in\mathscr{E}^\infty$. We have the following.

Theorem 1 [1]: (Smith form of a proper rational matrix at $s=\infty$.) Let $T(s)\in\mathbb{R}_{pr}^{p\times m}(s)$, $\operatorname{rank}_{\mathbb{R}(s)}T(s)=r$. Then $T(s)$ is equivalent at $s=\infty$ to a diagonal matrix $S_{T(s)}^{\infty}=\text{block diag}(1/s^{q_1},1/s^{q_2},\cdots,1/s^{q_r},0_{p-r,m-r})$ where $q_i\in\mathbb{Z}$, $i\in\mathbf{r}$ and

$$q_1\geqslant q_2\geqslant\cdots\geqslant q_r\geqslant 0. \tag{2}$$

Proof: See [1], [4].

Remark: If z_∞ is the number of q_i's in (2) satisfying $q_i>0$, $i\in z_\infty$, then we say that $T(s)$ has z_∞ zeros at $s=\infty$, each one of order q_i. The ordered set of indexes $(q_1,q_2,\cdots,q_r)$ we call the "zero structure at $s=\infty$" of $T(s)$.

II. DIVISORS AND GREATEST (COMMON) LEFT OR RIGHT DIVISORS AT $s=\infty$ OF PROPER RATIONAL MATRICES. COPRIMENESS AT $s=\infty$

We introduce now the notions of (common) left divisors and of greatest (common) left divisors at $s=\infty$ of proper rational matrices. (Greatest) common right divisors at $s=\infty$ can be defined analogously. We start with the following.

Proposition 1 [1]: Let $T_1(s)\in\mathbb{R}_{pr}^{p\times l}(s)$, $T_2(s)\in\mathbb{R}_{pr}^{p\times t}(s)$ with $m:=l+t\geqslant p$. Then the following statements are equivalent. 1) $T_1(s),T_2(s)$ are left coprime at $s=\infty$. 2) The proper rational matrix $T(s):=[T_1(s),T_2(s)]\in\mathbb{R}_{pr}^{p\times m}(s)$ has no zeros at $s=\infty$. 3) $S_{T(s)}^{\infty}=[I_p,0p,m-p]$. 4) There exists a *biproper* rational matrix $T_R(s)\in\mathbb{R}_{pr}^{m\times m}(s)$ such that $T(s)T_R(s)=[I_p,0_{p,m-p}]=S_{T(s)}^{\infty}$. 5) There exist proper rational matrices $X(s)\in\mathbb{R}_{pr}^{l\times p}(s)$, $Y(s)\in\mathbb{R}_{pr}^{t\times p}(s)$ such that $T_1(s)X(s)+T_2(s)Y(s)=I_p$. 6) There exist proper rational matrices $T_3(s)\in\mathbb{R}_{pr}^{(m-p)\times l}(s)$, $T_4(s)\in\mathbb{R}_{pr}^{(m-p)\times t}(s)$ such that the proper rational matrix

$$\begin{bmatrix} T_1(s) & T_2(s)\\ T_3(s) & T_4(s)\end{bmatrix}\in\mathbb{R}_{pr}^{m\times m}(s)$$

is biproper. 7) $\lim_{s\to\infty}T(s)=E\in\mathbb{R}^{p\times m}$ and $\operatorname{rank}_{\mathbb{R}}E=p$.

Proof: See [1].

Definition 1: A proper rational matrix $T(s)\in\mathbb{R}_{pr}^{p\times m}(s)$ with $p\leqslant m$ satisfying the equivalent conditions of Proposition 1 is called *right biproper.*

Proposition 2: Any proper rational matrix $T(s)\in\mathbb{R}_{pr}^{p\times m}(s)$ with $p\leqslant m$ can be factorized (in a nonunique way) as

$$T(s)=T_{GL}(s)\bar{T}(s) \tag{3}$$

where $T_{GL}(s)\in\mathbb{R}_{pr}^{p\times p}(s)$ has zero structure at $s=\infty$, same with that of $T(s)$ and $\bar{T}(s)\in\mathbb{R}_{pr}^{p\times m}(s)$ is right biproper.

Proof: Let $T_R(s)\in\mathbb{R}_{pr}^{m\times m}(s)$ be a biproper rational matrix such that

$$T(s)T_R(s)=[T_{GL}(s),0_{p,m-p}] \tag{4}$$

where $T_{GL}(s)\in\mathbb{R}_{pr}^{p\times p}(s)$. ($T_R(s)$ can be chosen as the biproper rational matrix which reduces $T(s)$ to a left lower triangular form (see [1, Theorem 1].) Now let $T_R(s)^{-1}=:\hat{T}(s)\in\mathbb{R}_{pr}^{m\times m}(s)$ (and biproper) and partition $\hat{T}(s)$ as $\hat{T}(s)=\begin{bmatrix}\bar{T}(s)\\ \tilde{T}(s)\end{bmatrix}$, where $\bar{T}(s)\in\mathbb{R}_{pr}^{p\times m}(s)$, $\tilde{T}(s)\in\mathbb{R}_{pr}^{(m-p)\times m}(s)$, and by 4) of Proposition 1, both are right biproper. Then (4) gives

$$T(s)=[T_{GL}(s),0]\hat{T}(s)=T_{GL}(s)[I_p,0]\begin{bmatrix}\bar{T}(s)\\ \tilde{T}(s)\end{bmatrix}=T_{GL}(s)\bar{T}(s).$$

From (4) $(T(s),[T_{GL}(s),0])\in\mathscr{E}^\infty$, hence $S_{T(s)}^{\infty}\equiv S_{[T_{GL}(s),0]}^{\infty}=[S_{T_{GL}(s)}^{\infty},0]]$, and hence the zero structure at $s=\infty$ of $T(s)$ is given by that of $T_{GL}(s)$. ∎

Definition 2: Let the proper matrices $T(s)\in\mathbb{R}_{pr}^{p\times m}(s)$, $T_L(s)\in\mathbb{R}_{pr}^{p\times p}(s)$, $\bar{T}(s)\in\mathbb{R}_{pr}^{p\times m}(s)$ be related via $T(s)=T_L(s)\bar{T}(s)$. Then $T_L(s)$ is called a (common) left divisor at $s=\infty$ of (the columns of) $T(s)$.

Manuscript received February 21, 1983; revised March 8, 1983 and July 7, 1983.
The author was with the Control and Management Systems Division, Department of Engineering, Cambridge University, Cambridge, England. He is now with the Department of Mathematics, Aristotelian University of Thessaloniki, Thessaloniki, Greece.

0018-9286/84/0500-0475$01.00

Definition 3: Let $T(s) \in \mathbb{R}_{pr}^{p\times m}(s)$ with $p \leq m$ and $\mathrm{rank}_{\mathbb{R}(s)} T(s) = p$. Then any proper rational matrix $T_{GL}(s) \in \mathbb{R}_{pr}^{p\times p}(s)$ that satisfies (3) for some right biproper rational matrix $\overline{T}(s) \in \mathbb{R}_{pr}^{p\times m}(s)$, is called a greatest (common) left divisor at $s = \infty$ of (the columns of) $T(s)$.

III. Precompensator Design for Exact Model Matching

In light of the above concepts and results we are now able to address the main question of this note which is that of the resolution of the "exact model matching problem" (EMMP) [5]. As it has been shown by Wolovich [5] the solution of EMMP via combined linear state variable feedback (l.s.v.f.) and input dynamics reduces to that of solving a matrix equation over proper rational matrices. This result [5, Theorem 8.5.2] will be the starting point of our analysis for the investigation of the EMMP, and we restate here as the following.

Lemma 1 [5]: Let Σ_p be a given system (the "plant") fully described by its transfer function matrix $T(s) \in \mathbb{R}_{pr}^{p\times m}(s)$ and let Σ_m be a "model" system again fully described by its transfer function matrix $M(s) \in \mathbb{R}_{pr}^{p\times q}(s)$. Then there exists a compensation scheme which employs input dynamics in combination with l.s.v.f (as defined in [5, sect. 7.4]), such that the compensated transfer matrix of the given system is equal to $M(s)$ if and only if

$$T(s)K(s) = M(s) \tag{5}$$

for some $K(s) \in \mathbb{R}_{pr}^{m\times q}(s)$.

As shown in [5] the existence of a $K(s) \in \mathbb{R}_{pr}^{m\times q}(s)$ satisfying (5) in the particular case when $p \geq m$ and $\mathrm{rank}_{\mathbb{R}} T(s) = m$, can be easily resolved (see [5, Corollary 8.5.8]). In the following the "much more difficult to resolve" case when $\mathrm{rank}_{\mathbb{R}(s)} T(s) = p < m$ will be considered.

A necessary and sufficient condition for the existence of a proper solution $K(s)$ of (5), when $p \leq m$, has been given in [6] using the notion of the "intractor" $\xi_{T(s)}$ of a proper rational matrix $T(s)$ [6, Theorem 4.5]. In the sequel we derive an equivalent to the above necessary and sufficient condition for the existence of a proper solution $K(s)$ of (5) which relies on the concept of a greatest left divisor at $s = \infty$, $T_{GL}(s)$ of $T(s)$.

We start by first partitioning $T(s)$ as: $T(s) = [T_1(s), T_2(s)]$ where $T_1(s) \in \mathbb{R}_{pr}^{p\times p}(s)$, $T_2(s) \in \mathbb{R}_{pr}^{p\times (m-p)}(s)$. Then (5) can be written as follows:

$$T_1(s)K_1(s) + T_2(s)K_2(s) = M(s) \tag{6}$$

where $K_1(s) \in \mathbb{R}_{pr}^{p\times q}(s)$, $K_2(s) \in \mathbb{R}_{pr}^{(m-p)\times q}(s)$ are unknown matrices and $K(s) = \begin{bmatrix} K_1(s) \\ \hdashline K_2(s) \end{bmatrix}$. Equation (6) is a matrix Diophantine equation where all matrices have elements in the ring $\mathbb{R}_{pr}(s)$. By analogy to the polynomial matrix Diophantine equations [2] we have the following.

Theorem 2: Equation (6) has a solution $K_1(s) \in \mathbb{R}_{pr}^{p\times q}(s)$, $K_2(s) \in \mathbb{R}_{pr}^{(m-p)\times q}(s)$ if and only if every greatest common left divisor at $s = \infty$: $T_{GL}(s)$ of $T_1(s)$ and $T_2(s)$ is a left divisor at $s = \infty$ (not necessarily greatest) of $M(s)$.

Proof:

If: Let $T_{GL}(s) \in \mathbb{R}_{pr}^{p\times p}(s)$ be a g.(c.)l.d. at $s = \infty$ of $T(s) = [T_1(s), T_2(s)]$. Then from Proposition 2 there exists a biproper matrix $T_R(s) \in \mathbb{R}_{pr}^{m\times m}(s)$ such that (4) is satisfied and if $T_R(s)$ is partitioned as

$$T_R(s) = \begin{bmatrix} U_1(s) & U_2(s) \\ U_3(s) & U_4(s) \end{bmatrix} \begin{matrix} p \\ m-p \end{matrix}$$
$$\qquad\qquad\quad p \qquad\quad m-p$$

we have:

$$T_1(s)U_1(s) + T_2(s)U_3(s) = T_{GL}(s) \tag{7}$$

$$T_1(s)U_2(s) + T_2(s)U_4(s) = 0_{p,m-p}. \tag{8}$$

Now by assumption $T_{GL}(s)$ is a left divisor at $s = \infty$ of $M(s)$, i.e., $M(s) = T_{GL}(s)M_0(s)$ for some $M_0(s) \in \mathbb{R}_{pr}^{p\times q}(s)$. Thus, multiplying (7) on the right by $M_0(s)$ we obtain

$$T_1(s)U_1(s)M_0(s) + T_2(s)U_3(s)M_0(s) = T_{GL}(s)M_0(s) = M(s). \tag{9}$$

Hence,

$$K_1(s) := U_1(s)M_0(s), \qquad K_2(s) := U_3(s)M_0(s) \tag{10}$$

is a solution of (6).

Only If: Let $T_{GL}(s) \in \mathbb{R}_{pr}^{p\times p}(s)$ be a g.(c.)r.d. at $s = \infty$ of $T(s) = [T_1(s), T_2(s)]$. Then from Proposition 2:

$$T(s) = T_{GL}(s)\overline{T}(s) \tag{11}$$

where

$$\overline{T}(s) = \Big[\underset{p}{\overline{T}_1(s)}, \underset{m-p}{\overline{T}_2(s)}\Big] \in \mathbb{R}_{pr}^{p\times m}(s)$$

is right biproper. Assuming that $K_1(s), K_2(s)$ is a solution of (6), then from (6) and (11) we have

$$T_{GL}(s)\left[\overline{T}_1(s)K_1(s) + \overline{T}_2(s)K_2(s)\right] = M(s) \tag{12}$$

which implies that $T_{GL}(s)$ is a left divisor at $s = \infty$ of $M(s)$. □

Corollary 1: If the "plant" transfer function matrix $T(s)$ is right biproper, then (6) has always a proper solution for every $M(s) \in \mathbb{R}_{pr}^{p\times q}(s)$.

Corollary 2: If (6) has a solution $K(s) = \begin{bmatrix} K_1(s) \\ \hdashline K_2(s) \end{bmatrix}$, then the general solution is given by:

$$K(s) = \begin{bmatrix} K_1(s) \\ K_2(s) \end{bmatrix} + \begin{bmatrix} U_2(s) \\ U_4(s) \end{bmatrix} Q(s) \tag{13}$$

where $U_2(s) \in \mathbb{R}_{pr}^{p\times (m-p)}(s)$, $U_4(s) \in \mathbb{R}_{pr}^{(m-p)\times (m-p)}(s)$ are the blocks in $T_R(s)$ (see Theorem 2) and $Q(s) \in \mathbb{R}_{pr}^{(m-p)\times q}(s)$ is arbitrary.

Proof: Let $K_1(s) \in \mathbb{R}_{pr}^{p\times q}(s)$, $K_2(s) \in \mathbb{R}_{pr}^{(m-p)\times q}(s)$ be a solution of (6). Combining (6) and (8) in matrix form we have:

$$[T_1(s), T_2(s)] \begin{bmatrix} K_1(s) & U_2(s) \\ K_2(s) & U_4(s) \end{bmatrix} = \left[M(s), 0_{p,m-p}\right]. \tag{14}$$

Multiplying the above equation on the right by the proper rational matrix

$$\begin{bmatrix} I_q & 0_{q,m-p} \\ Q(s) & I_{m-p} \end{bmatrix},$$

where $Q(s) \in \mathbb{R}_{pr}^{(m-p)\times q}(s)$ and otherwise arbitrary we obtain the general solution of (6) given by (13). □

IV. EMMP Algorithm

Thus given $T(s) \in \mathbb{R}_{pr}^{p\times m}(s)$ with $\mathrm{rank}_{\mathbb{R}(s)} T(s) = p < m$ and $M(s) \in \mathbb{R}_{pr}^{p\times q}(s)$, the algorithm to see whether the EMMP: $T(s)K(s) = M(s)$ has a proper solution $K(s)$ can be summarized as follows.

1) Using elementary column operations over $\mathbb{R}_{pr}(s)$ determine a biproper matrix $T_R(s) \in \mathbb{R}_{pr}^{m\times m}(s)$ that reduces $T(s)$ to $[T_{GL}(s), 0_{p,m-p}]$ where $T_{GL}(s) \in \mathbb{R}_{pr}^{p\times p}(s)$ [see (4)].

2) Examine if $T_{GL}(s)$ is a left divisor at $s = \infty$ of $M(s)$, i.e., examine if $M(s) = T_{GL}(s)M_0(s)$ for some $M_0(s) \in \mathbb{R}_{pr}^{p\times q}(s)$, thus see if $M_0(s) := T_{GL}(s)^{-1}M(s)$ is proper.

3) If $M_0(s)$ is proper an infinite number of solutions exist, parametrized by $Q(s)$ and given by (13). If $M_0(s)$ is nonproper, then no solution exists.

Conclusions

In this note, proper rational matrix Diophantine equations have been considered and their use to the resolution of the exact model matching problem has been elucidated. The equivalence between our condition for the solvability of EMMP and that given in [6] can be seen as follows. According to [6] if $\xi_T(s)$ is the "intractor" of $T(s)$, then $\xi_T(s)T(s) =: \overline{T}(s) \in \mathbb{R}_{pr}^{p\times m}(s)$ and $\lim_{s\to\infty} \overline{T}(s) = E \in \mathbb{R}^{p\times m}$ with $\mathrm{rank}_{\mathbb{R}} E = p$. From the above, Definition 3 and condition 4) in Proposition 1 it simply follows that $\xi_T(s)^{-1}$ is a greatest left divisor at $s = \infty$ of $T(s)$ which has a special structure (see [6]). Thus, if $\xi_M(s)$ is the intractor of $M(s)$

$(\xi_M(s)M(s) =: \overline{M}(s)$, $\lim_{s\to\infty}\overline{M}(s) = E_{\overline{M}}$, $\text{rank}_{\mathbf{R}} E_{\overline{M}} = p)$ our condition for the solvability of EMMP with $T_{GL}(s) \equiv \xi_T(s)^{-1}$ gives that $T_{GL}(s)^{-1}M(s) = \xi_T(s)\xi_M(s)^{-1}\overline{M}(s)$ must be proper. In view of the fact that $\overline{M}(s)$ is proper the above is equivalent to $\xi_T(s)\xi_M(s)^{-1}$ being proper, which is the necessary and sufficient condition given in [6].

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Convergence Properties of LMS Adaptive Estimators with Unbounded Dependent Inputs

ROBERT R. BITMEAD

***Abstract*—This note presents limit theorems for the behavior of adaptive estimators using the LMS algorithm when the driving or input sequence is a member of a broad class of random processes which are not necessarily almost surely bounded and are dependent over time. Convergence in distribution of the estimates is established in the stationary case while general nonstationary tracking is characterized in the nonstationary case. These results follow from the exponential convergence of the homogeneous algorithm which in turn follows from a strong limit theorem for infinite products of ergodic and mixing sequences of matrices.**

I. INTRODUCTION

The estimators which will be considered here are based on the LMS algorithm

$$w_{k+1} = w_k + \mu u_k\left(y_k - u_k' w_k\right) \tag{1.1}$$

which attempts to identify the best fitting parameter vector w^* as measured by the performance index $E(y_k - u_k'w)^2$ with $\{y_k\}$ scalars and $\{u_k\}$ N-vectors [1]. This class of model and estimator falls into the area of equation-error identification methods [2] and the results to be presented have application to many adaptive areas such as control, filtering, and identification [16].

The LMS algorithm has been analyzed by many people, most of whom have worked under the assumption of independent random sequences $\{u_k\}$ or deterministic $\{u_k\}$. With this assumption plus stationarity it is straightforward to prove convergence of the mean squared parameter error and the mean squared output error, as well as to tie an exponential rate to this convergence. This has been the approach of Widrow *et al.* [1], Gersho [3], Weiss and Mitra [4], and others. Once the convergence rate is established, an adaptation time constant may be defined (depending on μ) which characterizes nonstationary parameter tracking.

The algorithm has been studied with stationary dependent $\{u_k\}$ by Davisson [5], Daniell [6], Kim and Davisson [7], Jones [8], Farden *et al.* [9], Reed *et al.* [10], and Bitmead and Anderson [11]. In general these approaches assume a restricted class of dependent processes such as: M-dependence with M known or Markov structure, almost-surely bounded $\{u_k\}$ or its conditional moments, or w^* belonging to a known compact set. Results based on these assumptions have been extended by some of the above authors to provide nonstationary results.

The convergence analyses of (1.1) are usually based on Lyapunov stability techniques, specialized moment inequalities, or modified algorithms, hence the need for explicit and sometimes abstruse assumptions. The approach here will be to appeal to strong limit theorems for infinite products of random matrices which describe the exponentially convergent behavior of the homogeneous LMS algorithm (i.e., when there exists a w^* such that $y_k = u_k'w^*$ precisely). Having established almost sure exponential convergence in the sense of [15], the results of [11] may be appealed to qualitatively to describe nonstationary performance, or the theorems of [12] applied to establish stationary limit results. The major novelties here are the matrix limit theorems themselves and their application to the LMS algorithm to establish exponential convergence for a very broad class of unbounded dependent inputs. The proofs of the theorems are presented as an Appendix to [22].

Manuscript received April 10, 1981; revised May 23, 1983.

The author was with the Department of Electrical and Electronic Engineering, James Cook University of North Queensland, Townsville, Australia. He is now with the Department of Systems Engineering, Research School of Physical Sciences, Australian National University, Canberra, ACT, Australia.

DEFINITIONS OF TERMS

By exponential convergence to zero (almost surely, in mean square, in probability) of a stochastic process $\{z_k\}$, we mean that the related stochastic process $\{(1+\beta)^k z_k\}$ converges to zero as $k \to \infty$ (almost surely, in mean square, in probability) for some $\beta > 0$ independent of the realization [15].

Mean square exponential convergence implies almost sure exponential convergence, although not necessarily with the same rate. Consequently, we shall refer simply to exponential convergence of the LMS parameter estimates $\{w_k\}$ to mean almost sure.

II. EXPONENTIAL CONVERGENCE OF THE HOMOGENEOUS ALGORITHM

The homogeneous LMS algorithm arises when the processes $\{u_k\}$ and $\{y_k\}$ are precisely related by $y_k = u_k'w^*$ for some w^* [11]. In this case (1.1) takes the simple (homogeneous) form

$$v_{k+1} = \left(I - \mu u_k u_k'\right)V_K \tag{2.1}$$

where $v_k = w_k - w^*$, the parameter error. Putting A_k for $u_k u_k'$ the solution to (2.1) may be written as

$$v_k = \prod_{i=1}^{k-1}\left(I - \mu A_i\right)v_0 \tag{2.2}$$

where the product entails successive left multiplication. We now turn to the characterization of the matrix products appearing in (2.2)

We shall appeal to the following theorem of [22] which generalizes the familiar analytical result $(1+n^{-1}x)^n \to \exp(x)$ to stationary dependent sequences of random square matrices.

Theorem 1: Suppose that $\{A_i;\ i = 1,2,\cdots\}$ is an ergodic sequence of square matrices and that all moments of A_i exist with $E(A_i) = \hat{A}$. Then

$$\lim_{n\to\infty}\prod_{i=1}^{n}\left(I + n^{-1}A_i\right) = \exp(\hat{A}) \qquad \text{a.s.}$$

Hölder's inequality is used in the proof of Theorem 1 and is the source of the requirement of finiteness of all moments. If we suppose that the A_i satisfy the following mixing condition due to Phillip [13] then we may remove this finiteness condition.

Denote by $M_{a,b}$ the sigma-algebra generated by the $\{A_i;\ 1 \leqslant a \leqslant i \leqslant b\}$. We consider the following mixing condition.

For any events $A \in M_{1,t}$ and $B \in M_{t+n,\infty}$ we have

$$|P(AB) - P(A)P(B)| \leqslant \psi(n)P(A)P(B) \tag{2.3}$$

with $\psi(n) \downarrow 0$.

0018-9286/84/0500-0477$01.00