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On the computation of proper, feedforward internally stabilizing and pole-placing compensators

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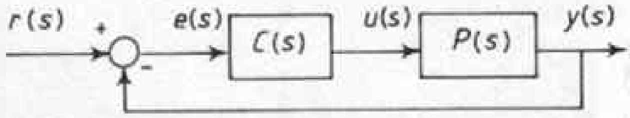
By combining two important results from the 'polynomial' and 'fractional representation' approaches we derive a state-space formula for the direct computation of a proper, feedforward, internally stabilizing, and pole-placing compensator $C(s)$.

1. Introduction

Let Σ be a closed-loop stabilizable (Wolovich 1978, Wolovich and Ferreira 1979) linear multivariable system whose input–output behaviour is described by a strictly proper transfer function matrix $P(s)$. The problem of designing a proper compensator $C(s)$ such that, when employed in the unity feedback loop shown in the Figure it will internally stabilize the resulting closed-loop feedback system. Σ_{c1} has been thoroughly investigated over the past few years using various approaches (see for example Youla *et al.* 1976, Desoer *et al.* 1980, Vidyasagar *et al.* 1982, Saeks and Murray 1981, Callier and Desoer 1981, Iver and Saeks 1984).



Although the above problem has been completely resolved theoretically, the main issue we are still faced with, when we attempt to implement in a computational algorithm the theoretical results of either the 'polynomial' or the 'fractional representation' approach, is that of developing reliable procedures for the computation of $C(s)$.

The polynomial matrix approach for the computation of $C(s)$ (Wolovich 1974, 1978, Kučera 1979) requires various manipulations of polynomial matrices which in effect are the steps required for the solution of polynomial matrix Diophantine equations (Wolovich and Antsaklis 1984), and these are not particularly suitable for computer implementation. The fractional representation approach presents at least equal computational difficulties. The primary purpose of this paper is to show how some theoretical results of the above approaches can be combined in order to develop a state-space oriented procedure (which is thus suitable for implementation in computer-aided control system design algorithms) for the direct computation of a proper feedforward, internally stabilizing and pole-placing compensator $C(s)$.

2. Notation and preliminaries

Let $\mathbb{R}$ be the field of reals, $\mathbb{R}[s]$ the ring of polynomials, $\mathbb{R}(s)$ the field of rational functions and $\mathbb{R}_{pr}(s)$ the ring or proper rational functions. Let Ω be a region of the finite complex plane $\mathbb{C}$, symmetrically located with respect to the real axis and which

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excludes at least one point $-\alpha \in \mathbb{R}$, and let Ω^c be the complement of Ω with respect to $\mathbb{C}$, i.e. $\mathbb{C} = \Omega \cup \Omega^c$. Let $t \in \mathbb{R}(s)$, and factorize it as $t = (n_\Omega \cdot \hat{n})/(d_\Omega \cdot \hat{d})$ where $n_\Omega, d_\Omega \in \mathbb{R}[s]$ coprime with not all their zeros outside Ω and $\hat{\eta}, \hat{d} \in \mathbb{R}[s]$ coprime with all their zeros outside Ω. Let $\mathscr{P} := \Omega \cup \{\infty\}$ and denote by $\mathbb{R}_{\mathscr{P}}(s)$ the subring of $\mathbb{R}(s)$ consisting of all $t \in \mathbb{R}(s)$ with no poles in $\mathscr{P}$, i.e. of 'proper and Ω-stable' rational functions. It is known (Morse 1975, Hung and Anderson 1979) that with 'degree' function $\delta_{\mathscr{P}}(\cdot): \mathbb{R}_{\mathscr{P}}(s) \to \mathbb{Z} \cup \{\infty\}$ defined by $\delta_{\mathscr{P}}(t) :=$ deg $\hat{d}$-deg $\hat{n} \geqq 0$, $\delta_{\mathscr{P}}(0) := \infty$, $\mathbb{R}_{\mathscr{P}}(s)$ is a Euclidean domain. It follows easily that for a $t \in \mathbb{R}_{\mathscr{P}}(s)$, $\delta_{\mathscr{P}}(t)$ gives the total (i.e. with multiplicities accounted for) number of zeros of t in $\mathscr{P} = \Omega \cup \{\infty\}$ and that $t \in \mathbb{R}_{\mathscr{P}}(s)$ is a unit in $\mathbb{R}_{\mathscr{P}}(s)$ iff $\delta_{\mathscr{P}}(t) = 0$ or equivalently iff t has also no zeros in $\mathscr{P}$. If Ω coincides with the closed right half-plane (RHP) $\mathbb{C}_+ := \{s \in \mathbb{C},\ \mathrm{Re}(s) \geqq 0\}$ then $\mathscr{P} \equiv \mathbb{C}_+ \cup \{\infty\} =: \bar{\mathbb{C}}_+$ is the 'extended' closed RHP and $\mathbb{R}_{\bar{\mathbb{C}}_+}(s)$ is the Euclidean ring of 'proper and stable' rational functions. The units in $\mathbb{R}_{\bar{\mathbb{C}}_+}(s)$ are biproper, stable and 'minimum-phase' rational functions. A matrix $T(s) \in \mathbb{R}_{\mathscr{P}}^{p \times p}(s)$ is called $\mathbb{R}_{\mathscr{P}}(s)$-unimodular if there exists a $\hat{T}(s) \in \mathbb{R}_{\mathscr{P}}^{p \times p}(s)$ such that $\hat{T}(s)T(s) = I_p$ or equivalently iff det $T(s)$ is a unit in $\mathbb{R}_{\mathscr{P}}(s)$. Two matrices $(T_1(s), T_2(s))\mathbb{R}^{p \times m}(s) \times \mathbb{R}^{p \times m}(s)$ are called 'equivalent in $\mathscr{P}$' if there exists $\mathbb{R}_{\mathscr{P}}(s)$-unimodular matrices $T_L(s) \in \mathbb{R}_{\mathscr{P}}^{p \times p}(s)$, $T_R(s) \in \mathbb{R}_{\mathscr{P}}^{m \times m}(s)$ such that $T_L(s)T_1(s)T_R(s) = T_2(s)$. Consequently, every $T(s) \in \mathbb{R}^{p \times m}(s)$ is 'equivalent in $\mathscr{P}$' to its 'Smith–MacMillan form in $\mathscr{P}$' $S_T^{\mathscr{P}}(s) =$ block diag $(\varepsilon_i/\psi_i, \ldots, \varepsilon_r/\psi_r, O_{p-r,m-r})$, $(r = \mathrm{rank}_{\mathbb{R}(s)} T(s))$, whose invariant factors ε_i/ψ_i give the zero-pole structure of $T(s)$ in $\mathscr{P}$ and have the form

$$\varepsilon_i = \frac{\varepsilon_{i\Omega}}{(s+\alpha)^{p_i}} \in \mathbb{R}_{\mathscr{P}}(s), \quad \psi_i = \frac{\psi_{i\Omega}}{(s+\alpha)^{q_i}} \in \mathbb{R}_{\mathscr{P}}(s)$$

where $\varepsilon_{i\Omega} \in \mathbb{R}[s]$, $\psi_{i\Omega} \in \mathbb{R}[s]$ are coprime with their zeros not outside Ω and $-\alpha \in \mathbb{R}$ is outside Ω, $p_i, q_i \in \mathbb{Z}$. Thus the finite zeros of ε_i and ψ_i, $i \in \mathbf{r}$, i.e. the zeros of $\varepsilon_{i\Omega}$ and $\psi_{i\Omega}$, give respectively the zeros and poles of $T(s)$ inside Ω and $q^i_{z\infty} := p_i - \deg \varepsilon_{i\Omega} \geqslant 0$, $q^i_{p\infty} := q_i - \deg \psi_{i\Omega} \geqslant 0$ give respectively the orders of the zeros and the poles of $T(s)$ at $s = \infty$. (see MacDuffee 1946, Vardulakis and Karcanias 1983, Hammer and Heymann 1983).

The following is a direct consequence of the above.

Proposition 1 (Vardulakis and Karcanias 1983)

Let $A(s) \in \mathbb{R}_{\mathscr{P}}^{l \times m}(s)$, $B(s) \in \mathbb{R}_{\mathscr{P}}^{t \times m}(s)$ with $k := l + t \geqq m$. Then the following statements are equivalent.

(*a*) $A(s)$ and $B(s)$ are right coprime in $\mathscr{P}$.

(*b*) $T(s) := \begin{bmatrix} A(s) \\ \cdots\cdots \\ B(s) \end{bmatrix} \in \mathbb{R}_{\mathscr{P}}^{k \times m}(s)$ has no zeros in $\mathscr{P}$.

(*c*) There exists in $\mathbb{R}_{\mathscr{P}}(s)$-unimodular matrix $T_L(s) \in \mathbb{R}_{\mathscr{P}}^{k \times k}(s)$ such that

$$T_L(s)T(s) = \begin{bmatrix} I_m \\ 0 \end{bmatrix} \equiv S_T^{\mathscr{P}}(s)$$

(*d*) There exist $X(s) \in \mathbb{R}_{\mathscr{P}}^{m \times l}(s)$, $Y(s) \in \mathbb{R}_{\mathscr{P}}^{m \times t}(s)$ such that

$$[X(s),\ Y(s)] \begin{bmatrix} A(s) \\ \cdots\cdots \\ B(s) \end{bmatrix} = I_m$$

(*e*) There exist $N(s) \in \mathbb{R}_{\mathscr{P}}^{l \times (k-m)}(s)$, $D(s) \in \mathbb{R}_{\mathscr{P}}^{t \times (k-m)}(s)$ such that

$$\begin{bmatrix} A(s) & N(s) \\ B(s) & D(s) \end{bmatrix} \in \mathbb{R}_{\mathscr{P}}^{k \times k}(s) \text{ is } \mathbb{R}_{\mathscr{P}}(s)\text{-unimodular}$$

(*f*) $\operatorname{rank}_{\mathbb{C}} \begin{bmatrix} A(s_0) \\ B(s_0) \end{bmatrix} = m \;\forall\; s_0 \in \Omega$ and

$$\lim_{s \to \infty} \begin{bmatrix} A(s) \\ B(s) \end{bmatrix} = E \in \mathbb{R}^{k \times m} \text{ with } \operatorname{rank}_{\mathbb{R}} E = m.$$

Definition 1

A matrix $T(s) \in \mathbb{R}_{\mathscr{P}}^{k \times m}(s)$ satisfying the equivalent conditions of Proposition 1 is called $\mathbb{R}_{\mathscr{P}}(s)$-left unimodular (or $\mathbb{R}_{\mathscr{P}}(s)$-left invertible) ($\mathbb{R}_{\mathscr{P}}(s)$-right unimodular matrices can be defined in an analogous manner).

Notice that a $\mathbb{R}_{\mathscr{P}}(s)$-left unimodular matrix might have finite poles and finite zeros in Ω^c only. In the particular case when $\Omega \equiv \mathbb{C}_+$ an $\mathbb{R}_{\mathscr{P}}(s)$-left unimodular matrix represents a proper stable and minimum phase transfer function matrix which has also no zeros at $s = \infty$. In such a case Proposition 1 (and its dual) can be viewed as expressing a number of equivalent necessary and sufficient conditions for the existence of proper, stable and 'minimum-phase' left (right) inverses of a $T(s) \in \mathbb{R}_{\text{pr}}^{k \times m}(s)$ with $k \geqq m$ ($k \leqq m$) and $\operatorname{rank}_{\mathbb{R}(s)} T(s) = m$ ($= k$).

3. Internal stabilization and pole placement

Let $P(s) \in \mathbb{R}_{\text{pr}}^{p \times m}(s)$ be the strictly proper transfer function matrix (the 'plant') of a *closed-loop stabilizable* system Σ. We derive below a simple state-space formula for the direct computation of a proper compensator $C(s) \in \mathbb{R}_{\text{pr}}^{m \times p}(s)$ such that the unity feedback system Σ_{c1} described by the Figure is internally stable and the closed-loop transfer function matrix $H_{yr} \in \mathbb{R}_{\text{pr}}^{p \times p}(s) r \mapsto y$ has arbitrary desired poles. The expression for the computation of $C(s)$ will be in terms of a minimal state-space realization of $P(s)$. We assume that we have a right coprime polynomial MFD of $P(s)$

$$P(s) = N(s)D(s)^{-1}$$

with $N(s) \in \mathbb{R}^{p \times m}[s]$, $D(s) \in \mathbb{R}^{m \times m}[s]$ and column proper (Wolovich 1970). An important result that we will use is the following.

Theorem 1 (Wolovich–Falb 'structure theorem' 1969, Wolovich 1970)

Let $D_j \in \mathbb{R}^{m \times 1}[s]$, $j \in \mathbf{m}$ be the jth column of $D(s)$, $v_j := \deg D_j \geqq 0$, $j \in \mathbf{m}$ and $n := \sum_{j=1}^{m} v_j$. Let $[D(s)]_c^h =: \hat{B}_m^{-1} \in \mathbb{R}^{m \times m}$ (where the symbol $[\,]_c^h$ denotes the highest column degree matrix of the matrix inside the brackets. Since $D(s)$ is column proper, $[D(S)]_c^h$ is non-singular). Write

$$D(s) = [D(s)]_c^h \operatorname{diag}[s^{v_1}, \ldots, s^{v_m}] + D_{bc} S(s) \tag{1}$$

where $D_{bc} \in \mathbb{R}^{m \times n}$, $S(s) = \text{block diag}[\hat{S}_1(s), \ldots, \hat{S}_m(s)]$, $\hat{S}_j(s) = [1, s, \ldots, s^{v_j - 1}]^T \in$

$\mathbb{R}^{v_j \times 1}[s]$, $j \in \mathbf{m}$. Finally let

$$\hat{A}_m := -\hat{B}_m D_{bc} \in \mathbb{R}^{m \times n} \tag{2}$$

Then a minimal realization of $P(s)$ (in 'multi-companion' form) is given by $(\hat{A}, \hat{B}, \hat{C})$ where

$$\hat{A} := A_0 + \tilde{B}\tilde{A}_m \in \mathbb{R}^{n \times n} \tag{3}$$

$$\hat{B} := \tilde{B}\hat{B}_m \in \mathbb{R}^{n \times m} \tag{4}$$

$$\hat{C} \in \mathbb{R}^{p \times m} \text{ is such that } \hat{C}S(s) = N(s) \tag{5}$$

with $\tilde{B} = \text{block diag}\,[e_1, e_2, \ldots, e_m] \in \mathbb{R}^{n \times m}$, $e_j = [0, 0, \ldots, 0, 1]^T \in \mathbb{R}^{v_j \times 1}$, $j \in \mathbf{m}$ and $A_0 = \text{block diag}\,[A_{01}, \ldots, A_{0m}] \in \mathbb{R}^{n \times n}$

$$A_{0j} := \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix} \in \mathbb{R}^{v_j \times v_j}, j \in \mathbf{m}$$ □

Let now $\Omega \equiv \mathbb{C}_+$ and

$$P(s) = A_1^{-1}(s)B_1(s) = B_2(s)A_2^{-1}(s) \tag{6}$$

with

$$B_1(s) \in \mathbb{R}_{\mathscr{P}}^{p \times m}(s),\ A_1(s) \in \mathbb{R}_{\mathscr{P}}^{p \times p}(s);\ B_2(s) \in \mathbb{R}_{\mathscr{P}}^{m \times m}(s),\ A_2(s) \in \mathbb{R}_{\mathscr{P}}^{m \times m}(s)$$

being respectively left and right coprime (in $\mathscr{P}$) *'fractional representations'* of $P(s)$, i.e. let the matrices $[A_1(s), B_1(s)] \in \mathbb{R}_{\mathscr{P}}^{p \times (p+m)}(s)$ and $\begin{bmatrix} A_2(s) \\ B_2(s) \end{bmatrix} \in \mathbb{R}_{\mathscr{P}}^{(m+p) \times m}(s)$ be respectively $\mathbb{R}_{\mathscr{P}}(s)$-right and left unimodular. Also let

$$C(s) = D_1^{-1}(s)N_1(s) = N_2(s)D_2^{-1}(s) \tag{7}$$

with $N_1(s) \in \mathbb{R}_{\mathscr{P}}^{m \times p}(s)$, $D_1(s) \in \mathbb{R}_{\mathscr{P}}^{m \times m}(s)$; $N_2(s) \in \mathbb{R}_{\mathscr{P}}^{m \times p}(s)$, $D_2(s) \in \mathbb{R}_{\mathscr{P}}^{p \times p}(s)$ be respectively left and right coprime (in $\mathscr{P}$) fractional representations of $C(s)$. Then it is known that $C(s)$ is a proper stabilizing compensator iff the following Bezout identity is satisfied (e.g. Desoer *et al.* 1980, Vidyasagar *et al.* 1982):

$$\begin{bmatrix} D_1(s) & N_1(s) \\ -B_1(s) & A_1(s) \end{bmatrix} \begin{bmatrix} A_2(s) & -N_2(s) \\ B_2(s) & D_2(s) \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_p \end{bmatrix} \tag{8}$$

In order to determine $C(s)$ we will consider below the identity (matrix Diophantine equation over the ring $\mathbb{R}_{\mathscr{P}}(s)$)

$$[D_1(s), N_1(s)] \begin{bmatrix} A_2(s) \\ B_2(s) \end{bmatrix} = I_m \tag{8 a}$$

which we will solve with respect to $D_1(s)$ and $N_1(s)$ in state-space terms. Firstly we will need a (right) coprime in $\mathscr{P}$ fractional representation $B_2(s)A_2^{-1}(s)$ of $P(s)$. We have the following result.

Proposition 2

Let $d_i(s) = s^{v_i} + a_{i,v_i-1}s^{v_i-1} + \dots + a_{i,1}s + a_{i0} \in \mathbb{R}[s]$ $i \in \mathbf{m}$ arbitrary polynomials with no zeros inside Ω and

$$D_d(s) := \text{diag}\,[d_1(s), \dots, d_m(s)] \in \mathbb{R}^{m\times m}[s].$$

Define

$$A_2(s) := D(s)D_d(s)^{-1} \in \mathbb{R}_{\mathscr{P}}^{m\times m}(s) \tag{9}$$

$$B_2(s) := N(s)D_d(s)^{-1} \in \mathbb{R}_{\mathscr{P}}^{p\times m}(s) \tag{10}$$

Then $B_2(s)A_2(s)^{-1}$ is a (right) coprime in $\mathscr{P} = \mathbb{C}_+ \cup \{\infty\}$ fractional representation of $P(s)$.

Proof

$$\text{Let } T(s) := \begin{bmatrix} A_2(s) \\ \cdots\cdots \\ B_2(s) \end{bmatrix} \in \mathbb{R}_{\mathscr{P}}^{(m+p)\times m}(s)$$

and write

$$\begin{bmatrix} A_2(s) \\ \cdots\cdots \\ B_2(s) \end{bmatrix} = \begin{bmatrix} D(s) \\ \cdots\cdots \\ N(s) \end{bmatrix} D_d(s)^{-1} \tag{11}$$

then since $D(s)$, $N(s)$ are right coprime the 'numerator'

$$\begin{bmatrix} D(s) \\ \cdots\cdots \\ N(s) \end{bmatrix} \in \mathbb{R}^{(m+p)\times m}[s]$$

of $T(s)$ has no zeros in $\mathbb{C}$. We now show that $T(s)$ has also no zeros at $s = \infty$. Write

$$A_2(s) = D(s)D_d(s)^{-1} = R(s)D_d(s)^{-1} + E_1 \tag{12}$$

$R(s) \in \mathbb{R}^{m\times m}[s]$ and such that $R(s)D_d(s)^{-1} \in \mathbb{R}_{\text{pr}}^{m\times m}(s)$ is strictly proper and $E_1 \in \mathbb{R}^{m\times m}$, or

$$D(s) = E_1 D_d(s) + RS(s) \tag{13}$$

$R \in \mathbb{R}^{m\times m}$. Also write

$$D_d(s) = \text{diag}\,[s^{v_1}, \dots, s^{v_m}] - \hat{A}_{dm}S(s) \tag{14}$$

$\hat{A}_{dm} :=$ block diag $[-a_{i0}, -a_{i1}, \dots, -a_{i,v_j-1}] \in \mathbb{R}^{m\times n}$. Then substituting (14) into (13) and making use of (1) and (2) we have

$$\hat{B}_m^{-1}[s^v] - \hat{B}_m^{-1}\hat{A}_m S(s) = E_1[[s^v] - \hat{A}_{dm}S(s)] + RS(s) \tag{15}$$

where $[s^v] := \text{diag}\,[s^{v_1}, \dots, s^{v_m}]$. Equating coefficient matrices of $[s^v]$ and $S(s)$ in (15) we obtain

$$E_1 = \hat{B}_m^{-1} \tag{16}$$

$$R = \hat{B}_m^{-1}[\hat{A}_{dm} - \hat{A}_m] \tag{17}$$

Therefore from (12): $A_2(\infty) = E_1 \equiv \hat{B}_m^{-1}$ is non-singular and $T(\infty) = \begin{bmatrix} \hat{B}_m^{-1} \\ \cdots\cdots \\ 0_{p,m} \end{bmatrix}$ (since $B_2(s)$ is by construction strictly proper), i.e. $T(s)$ has also no zeros at $s = \infty$. Thus from

Part (f) of Proposition 1 $T(s) \in \mathbb{R}_{\mathscr{P}}^{(m+p)\times m}(s)$ and is $\mathbb{R}_{\mathscr{P}}(s)$-left unimodular and so $A_2(s)$, $B_2(s)$ are right coprime in $\mathscr{P}$. □

From the above result and Theorem 1 we have directly the following result.

Proposition 3

A minimal realization (A_T, B_T, C_T, E_T) of $T(s) = \begin{bmatrix} A_2(s) \\ \cdots\cdots \\ B_2(s) \end{bmatrix} \in \mathbb{R}_{\mathscr{P}}^{(m+p)\times m}(s)$ is given by

$$A_T := A_0 + \tilde{B}\hat{A}_{dm} \in \mathbb{R}^{n\times n} \tag{18}$$

$$B_T := \tilde{B} \in \mathbb{R}^{n\times m} \tag{19}$$

$$C_T := \begin{bmatrix} \hat{B}_m^{-1}(\hat{A}_{dm} - \hat{A}_m) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \hat{C} \end{bmatrix} \in \mathbb{R}^{(m+p)\times n} \tag{20}$$

$$E_T := \begin{bmatrix} \hat{B}_m^{-1} \\ \cdots\cdots \\ 0_{p,m} \end{bmatrix} \in \mathbb{R}^{(m+p)\times m} \tag{21}$$

Proof

Since $\begin{bmatrix} D(s) \\ N(s) \end{bmatrix}$ has no zeros inside $\mathbb{C}$ the pair $\left(\begin{bmatrix} D(s) \\ N(s) \end{bmatrix}, D_d(s)\right)$ is right coprime for every choice of $D_d(s)$ and from (9)–(14) we have

$$T(s) = \begin{bmatrix} A_2(s) \\ B_2(s) \end{bmatrix} = \begin{bmatrix} D(s) \\ N(s) \end{bmatrix} D_d(s)^{-1} = \begin{bmatrix} R(s) \\ N(s) \end{bmatrix} D_d(s)^{-1} + \begin{bmatrix} E_1 \\ 0_{p,m} \end{bmatrix} \tag{22}$$

Thus applying Theorem 1 to $T(s)$ and making use of (16) and (17) we have the result. □

Consider now the identity

$$T_{\text{post}}(s)T(s) = T_{L,Q}(s) \tag{23}$$

where

$$T(s) = C_T(sI_n - A_T)^{-1}B_T + E_T \in \mathbb{R}_{\mathscr{P}}^{(m+p)\times m}(s) \tag{24}$$

$$T_{\text{post}}(s) = QC_T(sI_n - A_T - LC_T)^{-1}L + Q \in \mathbb{R}_{\text{pr}}^{m\times(m+p)}(s) \tag{25}$$

$$T_{L,Q}(s) = QC_T(sI_n - A_T - LC_T)^{-1}(B_T + LE_T) + QE_T \in \mathbb{R}_{\text{pr}}^{m\times m}(s) \tag{26}$$

which can be easily verified and it is valid for every state-space quadruple (A_T, B_T, C_T, E_T) and arbitrary matrices $Q \in \mathbb{R}^{m\times(m+p)}$ with $\text{rank}_{\mathbb{R}}Q = m$ and $L \in \mathbb{R}^{n\times(m+p)}$. Partitioning L as $L = [L_1 \quad L_2]$, $L_1 \in \mathbb{R}^{n\times m}$, $L_2 \in \mathbb{R}^{n\times p}$ and letting

$$Q := [\hat{B}_m, 0_{m,p}] \tag{27}$$

$$L_1 := -\hat{B} = -\tilde{B}\hat{B}_m \tag{28}$$

we have that for every $L_2 \in \mathbb{R}^{n\times m}$

$$B_T + LE_T = \tilde{B} + [-\tilde{B}\hat{B}_m \,\vdots\, L_2]\begin{bmatrix} \hat{B}_m^{-1} \\ \cdots\cdots \\ 0_{p,m} \end{bmatrix} = \tilde{B} - \tilde{B} = 0_{n,m} \tag{29}$$

and thus it simply follows from (26), (27) and (21) that for this particular choice of Q and L_1

$$T_{L,Q} = I_m \tag{30}$$

Considering the matrix $A_T + LC_T$ in (25) we also see that for Q and L_1 given by (27) and (28) we have

$$\begin{aligned} A_T + LC_T &= A_0 + \tilde{B}\hat{A}_{dm} + [-\tilde{B}\hat{B}_m \,\vdots\, L_2]\begin{bmatrix} \hat{B}_m^{-1}(\hat{A}_{dm} - \hat{A}_m) \\ \cdots\cdots \\ \hat{C} \end{bmatrix} \\ &= A_0 + \tilde{B}\hat{A}_m + L_2\hat{C} = \hat{A} + L_2\hat{C} \end{aligned} \tag{31}$$

Thus since the pair $(\hat{A}, \hat{C})$ is observable we can choose $L_2 \in \mathbb{R}^{n\times p}$ such that $\hat{A} + L_2\hat{C}$ has arbitrary eigenvalues (outside $\mathbb{C}^+$). It readily follows from (25) that with such a choice for L and Q given by (27) $T_{post}(s) \in \mathbb{R}_{\mathscr{P}}^{m\times(m+p)}(s)$ and it is $\mathbb{R}_{\mathscr{P}}(s)$-right unimodular.

Partitioning now T_{post} as $T_{post} = [\overset{\longleftarrow m \longrightarrow}{T_{post\,1}} \,\vdots\, \overset{\longleftarrow p \longrightarrow}{T_{post\,2}}]$ from (25) we have that

$$T_{post\,1}(s) = QC_T(sI_n - A_T - LC_T)^{-1}L_1 + \hat{B}_m \in \mathbb{R}_{\mathscr{P}}^{m\times m}(s) \tag{32}$$

and is biproper, while

$$T_{post\,2}(s) = QC_T(sI_n - A_T - LC_T)^{-1}L_2 \in \mathbb{R}_{\mathscr{P}}^{m\times p}(s) \tag{33}$$

is strictly proper. So by comparing (8.1) with (23) we may conclude that a (left) coprime in $\mathscr{P}$ fractional representation of $C(s)$ is given by the left formula in (7) with

$$D_1(s) \equiv T_{post\,1}(s) \tag{34}$$

$$N_1(s) \equiv T_{post\,2}(s) \tag{35}$$

The above analysis gives rise to the following result.

Proposition 4

Let the 'plant' $P(s)$ have minimal realization (in multi-companion form) $\hat{A}$, $\hat{B}$, $\hat{C}$.

Then a proper stabilizing and pole-placing compensator $C(s)$ can be calculated from the formula

$$C(s) = \hat{B}_m^{-1}(\hat{A}_{dm} - \hat{A}_m)(sI_n - A_0 - \tilde{B}A_{dm} - L_2\hat{C})^{-1}L_2 \tag{36}$$

where $L_2 \in \mathbb{R}^{n\times p}$ must be chosen so that $\hat{A} + L_2\hat{C}$ has desired eigenvalues (outside $\mathbb{C}^+$). The poles of the resulting closed-loop transfer function matrix $H_{yr}(s) \in \mathbb{R}_{\mathscr{P}}^{p\times p}(s): r(s) \mapsto y(s)$ are in general a subset of the eigenvalues of $A_T := A_0 + \tilde{B}A_{dm}$ and $\hat{A} + L_2\hat{C}$.

Proof

From (7) a proper stabilizing compensator $C(s)$ can be calculated from $D_1(s)^{-1}N_1(s)$. From eqns. (31), (32) and (34) it simply follows that a realization (A_D, B_D, C_D, E_D) of $D_1(s)$ is given by

$$A_D = A_T + LC_T = \hat{A} + L_2\hat{C} \tag{37}$$

$$B_D = L_1 = -\hat{B} \tag{38}$$

$$C_D = QC_T = [\hat{B}_m, 0_{m,p}]\begin{bmatrix}\hat{B}_m^{-1}(\hat{A}_{dm} - \hat{A}_m) \\ \hdashline \hat{C}\end{bmatrix} = \hat{A}_{dm} - \hat{A}_m \tag{39}$$

$$E_D = \hat{B}_m \tag{40}$$

Hence, since a realization of $D_1(s)^{-1}$ is given by $(A_D - B_D E_D^{-1} C_D, B_D E_D^{-1}, -E_D^{-1} C_D, E_D^{-1})$, by invoking (37)–(40) we have

$$\begin{aligned} A_D - B_D E_D^{-1} C_D &= \hat{A} + L_2\hat{C} + \hat{B}\hat{B}_m^{-1}(\hat{A}_{dm} - \hat{A}_m) \\ &= \hat{A}_0 + \tilde{B}\hat{A}_m + L_2\hat{C} + \tilde{B}A_{dm} - \tilde{B}\hat{A}_m = \hat{A}_0 + \tilde{B}\hat{A}_{dm} + L_2\hat{C} = A_T + L_2\hat{C} \end{aligned} \tag{41}$$

$$B_D E_D^{-1} = -\hat{B}\hat{B}_m^{-1} = -\tilde{B} \tag{42}$$

$$-E_D^{-1} C_D = -\hat{B}_m^{-1}(\hat{A}_{dm} - \hat{A}_m) \tag{43}$$

$$E_D^{-1} = \hat{B}_m^{-1}$$

Thus

$$D_1(s)^{-1} = \hat{B}_m^{-1}(\hat{A}_{dm} - \hat{A}_m)(sI_n - A_T - L_2\hat{C})^{-1}\tilde{B} + \hat{B}_m^{-1} \tag{44}$$

Similarly from eqns. (33) and (35) a realization (A_N, B_N, C_N) of $N_1(s)$ is given by

$$A_N = A_T + LC_T = \hat{A} + L_2\hat{C} \tag{45}$$

$$B_N = L_2 \tag{46}$$

$$C_N = QC_T = \hat{A}_{dm} + \hat{A}_m \tag{47}$$

and thus

$$N_1(s) = (\hat{A}_{dm} - \hat{A}_m)(sI_n - \hat{A} - L_2\hat{C})^{-1}L_2 \tag{48}$$

Finally

$$\begin{aligned} D_1(s)^{-1}N_1(s) &= [\hat{B}_m^{-1}(\hat{A}_{dm} - \hat{A}_m)(sI_n - A_T - L_2\hat{C})^{-1}\tilde{B} + \hat{B}_m^{-1}] \\ &\quad \times (\hat{A}_{dm} - \hat{A}_m)(sI_n - \hat{A} - L_2\hat{C})^{-1}L_2 \\ &= \hat{B}_m^{-1}(\hat{A}_{dm} - \hat{A}_m)(sI - A_T - L_2\hat{C})^{-1}[\tilde{B}(\hat{A}_{dm} - \hat{A}_m) + sI_n - \hat{A}_T - L_2\hat{C}] \\ &\quad \times (sI_n - \hat{A} - L_2\hat{C})^{-1}L_2 \\ &= \hat{B}_m^{-1}(\hat{A}_{dm} - \hat{A}_m)(sI - A_0 - \tilde{B}\hat{A}_{dm} - L_2\hat{C})^{-1}L_2 \end{aligned} \tag{49}$$

From $H_{yr}(s) = P(I_m + CP)^{-1}C$ and invoking (6), (7) and (8) it simply follows that

$$H_{yr}(s) = B_2(s)N_1(s) \in \mathbb{R}_{\mathcal{P}}^{p \times p}(s) \tag{50}$$

From (10) a realization (A_B, B_B, C_B) of $B_2(s) \in \mathbb{R}^{p \times m}(s)$ is given by $A_B = A_T$, $B_B = \tilde{B}$, $C_B = \hat{C}$.

The last part of Proposition 4 follows now from the facts that (i) the poles of $B_2(s)$ are in general a subset of the eigenvalues of A_T depending on whether the pair $(A_T, \hat{C})$ is observable (since $(A_T, \tilde{B})$ is controllable for every choice of $d_i(s)$, $i \in \mathbf{m}$) (ii) the poles of $N_1(s)$ are in general a subset of the eigenvalues of $A_N = \hat{A} + L_2\hat{C}$, depending on whenever the triple (A_N, B_N, C_N) is controllable and observable and (iii) that there may be (stable) pole-zero cancellations between $B_2(s)$ and $N_1(s)$ in (50).

4. Conclusions

This paper has analysed an explicit state-space formula for the direct computation of a proper, feedforward, internally stabilizing and pole-placing compensator in a unity feedback loop.

Results regarding connections between state-space and fractional representations appeared originally in the work of Vidyasagar (1984) and Net *et al.* (1984). Our analysis relies on the special multi-companion form of the minimal realization $(\hat{A}, \hat{B}, \hat{C})$ of $P(s)$ and is in effect a state-space method for analysing and solving the 'proper and Ω-stable rational matrix Diophantine equation' (8.1). The use of this particular realization of $P(s)$ has recently been highlighted by Wolovich and Antsaklis (1984) in the resolution of a number of standard problems arising in linear multivariable control which involve solutions of polynomial matrix Diophantine equations.

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