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Internal Stabilization and Decoupling in Linear Multivariable Systems by Unity Output Feedback Compensation

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Abstract—**We consider the problem of internally stabilizing and simultaneously diagonally decoupling a linear multivariable system by unity output feedback compensation. A sufficient condition is derived for the existence of a cascade proper compensator $C(s)$ such that when employed in a unity feedback loop involving the proper transfer function matrix P_o of a free of unstable hidden modes system $\Sigma(P_o)$, will not only internally stabilize the feedback closed-loop system $\Sigma(P_o, C)$ but will also give rise to a closed-loop transfer function matrix H_{yr}^{diag}, which is nonsingular, diagonal, and has desired poles. Based on this analysis, an algorithmic procedure for the computation of such a compensator is presented.**

Introduction

It is known, [1]–[8] that given a linear, time-invariant multivariable system $\Sigma(P_o)$ which is free of unstable hidden modes and whose input–output behavior is described by a $p \times m$ proper (strictly proper) rational transfer function matrix P_o (the "plant") then there always exists a nonempty family $\phi(P_o)$ of (proper) *stabilizing compensators,* i.e., of (proper) rational matrices C_W such that for each compensator C_W in $\phi(P_o)$ employed as in the unity feedback loop of Fig. 1, the resulting closed-loop system $\Sigma(P_o, C_W)$ is internally asymptotically stable, i.e., all its modes, including the unobservable and the uncontrollable ones, are stable and the closed-loop transfer function matrix $H_{yr}: r \to y$ has poles inside any arbitrary (symmetric with respect to the real axis) subset Ω^c of the open left-half complex plane $\mathbb{C}_o^- := \{s \in \mathbb{C}, \text{Re}(s) < 0\}$ (see Fig. 2).

The problem of diagonally decoupling $\Sigma(P_o)$ by unity output feedback compensation is to choose a $C_W \in \phi(P_o)$ such that in addition to the above requirements the transfer function matrix H_{yr} is also nonsingular and *diagonal* with desired poles in Ω^c.

The primary purpose of this note is to examine conditions under which such a compensator C_W exists and if it does exist how it can be computed. In such a case we say that P_o [or $\Sigma(P_o)$] is *decouplable* (by unity output feedback compensation). If P_o is decouplable we examine also the structure of the possible diagonal closed-loop transfer function matrices H_{yr}^{diag} which are obtainable by some $C_W \in \phi(P_o)$. The paper is organized as follows. The necessary mathematical background is given in Section II. The problem of simultaneous internal stabilization and decoupling with arbitrary pole assignment inside Ω^c for the resulting closed-loop transfer function matrix and by unity output feedback compensation is analyzed in Section III. This analysis gives rise to a sufficient condition for P_o to be decouplable which is our main result and is stated as Theorem 1. This condition turns out to be generically satisfied depending on the noncoincidence of unstable poles and unstable zeros of P_o and roughly can be described as follows.

It is well known that in general the transfer function matrix P_o of a linear multivariable system might have coinciding poles and zeros that do not "cancel out" each other due to the fact that they correspond denominators and numerators of different entries in the McMillan form of P_o.

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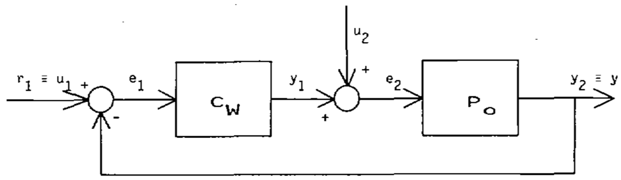

Fig. 1.

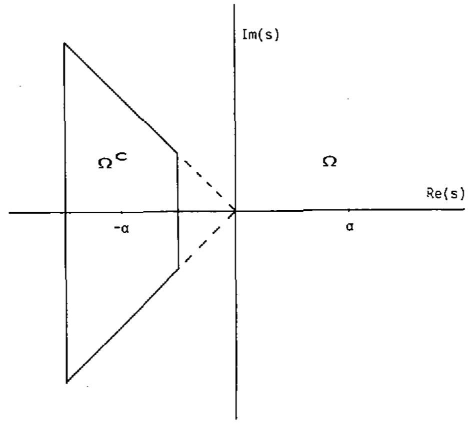

Fig. 2.

A sufficient condition for P_o to be decouplable is that P_o is free of such uncancelable coincidences of *unstable* poles and zeros. More formally this condition can be stated as follows. Let $S^{\mathbb{C}}{}_{P_o}$ = block diag $[\tilde{\epsilon}_1/\tilde{\psi}_1, \tilde{\epsilon}_2/\tilde{\psi}_2, \cdots, \tilde{\epsilon}_p/\tilde{\psi}_p, 0_{p,m-p}]$ be the McMillan form of P_o and factorize the (minimal) "pole" and "zero" polynomials $\tilde{\psi}_1$ and $\tilde{\epsilon}_p$ as $\tilde{\psi}_1 = \psi_{1\Omega}\hat{\psi}_1$, $\tilde{\epsilon}_p = \epsilon_{p\Omega}\hat{\epsilon}_p$ where $\psi_{1\Omega}$, $\epsilon_{p\Omega}$ have no zeros inside $\Omega^c \equiv \mathbb{C}_o^-$ and $\hat{\psi}_1$, $\hat{\epsilon}_p$ have no zeros outside Ω^c. Then P_o is decouplable if the polynomials $\psi_{1\Omega}$ and $\epsilon_{p\Omega}$ are coprime. Since generically any two polynomials are coprime the above result implies that generically any free of unstable hidden modes $\Sigma(P_o)$, with full row rank p, will be decouplable. Finally, a simple algorithm for the computation of an internally stabilizing, decoupling, and pole assigning compensator is presented and the whole procedure is illustrated by an example.

Some of the related work in this area can be summarized as follows. Decoupling (by dynamic unity feedback compensation) as one objective among others was considered in [9] where a similar condition to ours was given for square plants only but pole assignment and the dependence of the decoupling compensator and its construction on the unstable pole and zero minimal polynomials $\psi_{1\Omega}$ and $\epsilon_{p\Omega}$ was not examined. Necessary and sufficient conditions for a plant to be decoupled by a compensator in the feedback loop were established in [10] and recently an algebraic theory for the design of a two-parameter decoupling compensator was presented in [11].

II. Notation and Background

Let $\mathbb{R}$ be the field of reals, $\mathbb{R}[s]$ the ring of polynomials, $\mathbb{R}(s)$ the field of rational functions, and $\mathbb{R}_{pr}(s)$ the ring of proper rational functions, all with coefficients in $\mathbb{R}$. Let Ω be a region of the finite complex plane $\mathbb{C}$ symmetrically located with respect to $\mathbb{R}$ and which excludes at least one point $-\alpha \in \mathbb{R}$ $(\alpha > 0)$ and let Ω^c be the complement of Ω with respect to $\mathbb{C}$, i.e., let $\mathbb{C} = \Omega \cup \Omega^c$. Let $\bar{\Omega} := \Omega \cup \{\infty\}$ and denote by S the subring of $\mathbb{R}(s)$ consisting of $t \in \mathbb{R}(s)$ with no poles in $\bar{\Omega}$, i.e., of all "proper and Ω-stable" rational functions. It is known [12], [13] that with "degree" function $\delta(\cdot): S \to \mathbb{Z} \cup \{\infty\}$ ($\mathbb{Z}$: nonnegative integers) defined by $\delta(t) := \deg \cdot \hat{d} - \deg \cdot \hat{n} \geq 0$, $\delta(0) := +\infty$, where $t = (n_\Omega \cdot \hat{n})/\hat{d} \in S$ and $n_\Omega \in \mathbb{R}[s]$ with no zeros outside Ω and $\hat{n}, \hat{d} \in \mathbb{R}[s]$ coprime with zeros outside Ω, S is a Euclidean domain. Two matrices $T_1, T_2 \in \mathbb{R}(s)^{p \times m}$ are called

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"equivalent in $\bar{\Omega}$" if there exist S-unimodular matrices $T_L \in S^{p\times p}$, $T_R \in S^{m\times m}$ such that $T_L T_1 T_R = T_2$. Consequently, every matrix $T \in \mathbb{R}(s)^{p\times m}$ is "equivalent in $\bar{\Omega}$" to its "Smith–McMillan form in $\bar{\Omega}$": $S_T^{\bar{\Omega}}$ = block diag $(\epsilon_1/\psi_1, \epsilon_2/\psi_2, \cdots, \epsilon_r/\psi_r, 0_{p-r,m-r}) \in \mathbb{R}(s)^{p\times m}$, [14], [15] $r = \text{rank}_{\mathbb{R}(s)} T$, whose invariant factors ϵ_i/ψ_i give the zero-pole structure of T in $\bar{\Omega}$ and have the form $\epsilon_i = \epsilon_{i\Omega}/(s+\alpha)^{q_i}$, $\psi_i = \psi_{i\Omega}/(s+\alpha)^{l_i} \in S$, $i \in r$, where $\epsilon_{i\Omega}, \psi_{i\Omega} \in \mathbb{R}[s]$ are coprime with their zeros not outside Ω, $q_i, l_i \in \mathbb{Z}$. Thus, the finite zeros of ϵ_i and ψ_i, $i \in r$, i.e., the zeros of $\epsilon_{i\Omega}$ and $\psi_{i\Omega}$, give, respectively, the zeros and poles of T in Ω and $q_i^{z\infty} := q_i - \deg \cdot \epsilon_{i\Omega} \geqq 0$, $q_i^{p\infty} := l_i - \deg \cdot \psi_{i\Omega} \geqq 0$, $i \in r$, respectively, the orders of the zeros and poles of T at $s = \infty$ (see [14]–[18]).

Proposition 1 [19]: Any rational matrix $T \in \mathbb{R}(s)^{p\times m}$, $\text{rank}_{\mathbb{R}(s)} T = r \leqq p$, can be factorized (in a nonunique way) as

$$T = T_{\bar{\Omega}} T_1 \tag{1}$$

where $T_1 \in S^{r\times m}$ is S-right invertible and $T_{\bar{\Omega}} \in S^{p\times r}$, $\text{rank}_{\mathbb{R}(s)} T_{\bar{\Omega}} = r$ and has pole zero structure in $\bar{\Omega}$ identical with that of T, i.e., if $S_{T_{\bar{\Omega}}}^{\bar{\Omega}}$ = block diag $(\epsilon_1/\psi_1, \cdots, \epsilon_r/\psi_r, 0_{p-r,m-r})$ then

$$S^{\bar{\Omega}}_{T_{\bar{\Omega}}} = \begin{bmatrix} \epsilon_1/\psi_1 & & \\ & \ddots & \\ & & \epsilon_r/\psi_r \\ \hline & 0_{p-r,r} & \end{bmatrix}.$$

Definition: The matrix $T_{\bar{\Omega}}$ in (1) is called a "structure matrix of T in $\bar{\Omega}$."

III. Internal Stabilization and Decoupling

Let $\Sigma(P_o)$ be a linear, time-invariant, and free of unstable hidden modes multivariable system whose transfer function matrix is $P_o \in \mathbb{R}_{pr}^{p\times m}(s)$ and let

$$P_o = A_1^{-1} B_1 = B_2 A_2^{-1} \tag{2}$$

with $A_1 \in S^{p\times p}$, $A_2 \in S^{m\times m}$, $B_1, B_2 \in S^{p\times m}$, be, respectively, any left and right coprime in $\bar{\Omega}$ fractional representations of P_o. It is then known [3]–[8] that if $C_o \in \mathbb{R}_{pr}(s)^{m\times p}$ is a stabilizing compensator for $\Sigma(P_o)$, then C_o has unique left and right coprime in $\bar{\Omega}$ fractional representations

$$C_o = D_1^{-1} N_1 = N_2 D_2^{-1} \tag{3}$$

with $D_1 \in S^{m\times m}$, $D_2 \in S^{p\times p}$, $N_1, N_2 \in S^{m\times p}$, satisfying the Bezout identity

$$\begin{bmatrix} D_1 & N_1 \\ -B_1 & A_1 \end{bmatrix} \begin{bmatrix} A_2 & -N_2 \\ B_2 & D_2 \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_p \end{bmatrix}. \tag{4}$$

Multiplying (4) on the left and right by the S-unimodular matrices $\begin{bmatrix} I_m & W \\ 0 & I_p \end{bmatrix}$ and $\begin{bmatrix} I_m & -W \\ 0 & I_p \end{bmatrix}$, respectively, where $W \in S^{m\times p}$ and such that $|(D_1 - WB_1)(\infty)| \neq 0$, $|(D_2 - B_2 W)(\infty)| \neq 0$, we obtain the Bezout identity:

$$\begin{bmatrix} D_1 - WB_1 & N_1 + WA_1 \\ -B_1 & A_1 \end{bmatrix} \begin{bmatrix} A_2 & -(N_2 + A_2 W) \\ B_2 & D_2 - B_2 W \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_p \end{bmatrix} \tag{5}$$

which due to the results mentioned above clearly shows that the set $\phi(P_o)$ of all stabilizing compensators C_W of $\Sigma(P_o)$ is parametrized by C_o and $W \in S^{m\times p}$ and it is given by [3]–[8]

$$\begin{aligned} \phi(P_o) = \{ C_W &= (D_1 - WB_1)^{-1}(N_1 + WA_1) \\ &= (N_2 + A_2 W)(D_2 - B_2 W)^{-1} \mid W \in S^{m\times p} \} \end{aligned} \tag{6}$$

$$|(D_1 - WB_1)(\infty)| \neq 0, \quad |(D_2 - WB_2)(\infty)| \neq 0. \tag{7}$$

A. Model Matching by Dynamic Compensation and Unity Output Feedback

Consider now the closed-loop transfer function matrix from $r := u_1$ to $y := y_2$ (Fig. 1)

$$H_{yr} = H_{yr}(P_o, C_W) = (I_p + P_o C_W)^{-1} P_o C_W = P_o (I_m + C_W P_o)^{-1} C_W \tag{8}$$

with $C_W \in \phi(P_o)$ so that $H_{yr} \in S^{p\times p}$. Then we have the following.

Proposition 2: Let $P_o \in \mathbb{R}_{pr}^{p\times m}(s)$ with $P_o = A_1^{-1} B_1 = B_2 A_2^{-1}$ and $C_W \in \phi(P_o)$ with $C_o = D_1^{-1} N_1 = N_2 D_2^{-1}$ so that (4) holds true. Then H_{yr} satisfies the relations

$$H_{yr} = B_2 (N_1 + WA_1) \tag{9}$$

$$I_p - H_{yr} = (D_2 - B_2 W) A_1. \tag{10}$$

Proof: A similar proposition regarding the sensitivity matrix $(I_p + P_o C_W)^{-1}$ has been given in [9]. We give a simple proof for ease of reference. Equation (9) can be obtained from (8) by substituting P_o from (2), C_W from (6) and making use of (5). Now consider (5) which by reversing the order of the matrices involved gives

$$\begin{bmatrix} A_2 & -(N_2 + A_2 W) \\ B_2 & D_2 - B_2 W \end{bmatrix} \begin{bmatrix} D_1 - WB_1 & N_1 + WA_1 \\ -B_1 & A_1 \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_p \end{bmatrix} \tag{11}$$

from which

$$B_2 (N_1 + WA_1) + (D_2 - B_2 W) A_1 = I_p. \tag{12}$$

The above identity, due to (9), gives rise to (10). □

From Proposition 2 it follows that the matrices $X := N_1 + WA_1 \in S^{m\times p}$ and $Y := D_2 - B_2 W \in S^{p\times p}$ represent a pair of solutions to the matrix equations

$$H_{yr} = B_2 X \tag{13}$$

$$I_p - H_{yr} = Y A_1. \tag{14}$$

Assume now that the matrices $B_2 \in S^{p\times m}$, $A_1 \in S^{p\times p}$, and $H_{yr} \in S^{p\times p}$ are all given and consider for a moment the problem of determining conditions under which the matrix equations (13), (14) have solutions $X \in S^{m\times p}$, $Y \in S^{p\times p}$. The above represents a particular case of a more general problem, studied in detail in [19] and known as the "stable exact model matching problem." Regarding (13) we have the following.

Proposition 3 [19]: Let $B_2 \in S^{p\times m}$ and $H_{yr} \in S^{p\times p}$ with $\text{rank}_{\mathbb{R}(s)} B_2 = \text{rank}_{\mathbb{R}(s)} H_{yr} = p$. Let $T_R \in S^{m\times m}$ be an S-unimodular matrix reducing B_2 to $[B_{2\bar{\Omega}}, 0_{p,m-p}]$, i.e., let

$$B_2 T_R = [B_{2\bar{\Omega}}, 0_{p,m-p}]. \tag{15}$$

Then

i) $B_{2\bar{\Omega}} \in S^{p\times p}$ is a "structure matrix in $\bar{\Omega}$" of B_2.

ii) Equation (13) has a solution $X \in S^{m\times p}$ iff

$$H_1 := B_{2\bar{\Omega}}^{-1} H_{yr} \in S^{p\times p}. \tag{16}$$

iii) If condition (16) is satisfied, then the general solution $X \in S^{m\times p}$ of (13) is given by

$$X = T_R \begin{bmatrix} H_1 \\ \hline Z \end{bmatrix} \in S^{m\times p} \tag{17}$$

where $Z \in S^{(m-p)\times p}$ and is otherwise arbitrary.

Remark 1: If $p = m = \text{rank}_{\mathbb{R}(s)} P_o$, then $B_{2\bar{\Omega}} \equiv B_2$ and $T_R \equiv I_p$. In such a case from (16) (17) we see that (13) has a solution $X \in S^{p\times p}$ iff

$$H_1 = X := B_2^{-1} H_{yr} \in S^{p\times p} \tag{18}$$

which for fixed H_{yr}, is also the *unique* solution of (13). From the above it also follows that (14) has a solution $Y \in S^{p\times p}$ iff

$$H_2 = Y := (I_p - H_{yr}) A^{-1} \in S^{p\times p} \tag{19}$$

which is also the *unique* solution of (14).

As it is shown in [19]–[21] the above results characterize in effect the family of all "model" closed-loop transfer function matrices $H_{yr} \in S^{p\times p}$ which are obtainable from (13) and (14) by some $X \in S^{m\times p}$, $Y \in S^{p\times p}$ or equivalently by some compensator $C_W \in \phi(P_o)$ (see Proposition 2). Thus, from (16) and (19) the family Ψ of all such H_{yr} is given by

$$\Psi = \{H_{yr} \in S^{p\times p} \mid H_{yr} = B_{2\bar{\Omega}}H_1 \text{ and } I_p - H_{yr} = H_2A_1 \text{ where } H_i \in S^{p\times p},\ \text{rank}_{\mathbb{R}(s)}H_i = p,\ i = 1, 2 \text{ and are related through a } W \in S^{m\times p} \text{ via the formulas:}$$

$$T_R\begin{bmatrix} H_1 \\ Z \end{bmatrix} = N_1 + WA_1,\ H_2 = D_2 - B_2W\}. \tag{20}$$

Remark 2: Notice that by Propositions 1 and 3 i) the zero structure in $\bar{\Omega}$ of $B_{2\bar{\Omega}}$ is identical to the zero structure in $\bar{\Omega}$ of B_2 which in turn is identical to the zero structure in $\bar{\Omega}$ of the plant P_o. On the other hand the zero structure in Ω of A_1 is identical to the pole structure in Ω of P_o (see [15, Proposition 4.4]). Thus,

$$H_{yr} = B_{2\bar{\Omega}}H_1,\ H_1 \in S^{p\times p} \tag{21}$$

$$I_p - H_{yr} = H_2A_1,\ H_2 \in S^{p\times p} \tag{22}$$

imply that a matrix $H_{yr} \in S^{p\times p}$ can be obtained as a closed-loop transfer function matrix for some $C_W \in \phi(P_o)$ iff it is chosen so that: i) H_{yr} has, at least, zero structure in $\bar{\Omega}$ which is identical to the zero structure in $\bar{\Omega}$ of P_o, and ii) $I_p - H_{yr}$ has, at least, zero structure in Ω which is identical to the pole structure in Ω of P_o. (See also [9], [22].)

B. Decoupling

In order now to state our main theorem we introduce some more notation and three lemmas. Thus, let $\text{rank}_{\mathbb{R}(s)}P_o = p$ and consider the Smith–McMillan form in $\bar{\Omega}$ of P_o:

$$S^{\bar{\Omega}}_{P_o} = \text{block diag } (\epsilon_1/\psi_1, \cdots, \epsilon_p/\psi_p, 0_{p,m-p}) \in \mathbb{R}^{p\times m}_{pr}(s) \tag{23}$$

where

$$\epsilon_i = \frac{\epsilon_{i\Omega}}{(s+\alpha)^{q_i}},\ \psi_i = \frac{\psi_{i\Omega}}{(s+\alpha)^{l_i}} \in S, \qquad i \in \boldsymbol{p} \tag{24}$$

with $q_i := \delta(\epsilon_i) \geqq 0$, $l_i := \delta(\psi_i) \geqq 0$ and $l_i = \deg\cdot\psi_{i\Omega}$, $i \in \boldsymbol{p}$, (so that ψ_i are biproper)

$$\epsilon_i \mid \epsilon_{i+1},\ i \in \boldsymbol{p}-1 \text{ or equivalently } \epsilon_{i\Omega} \mid \epsilon_{i+1,\Omega},\quad i \in \boldsymbol{p}-1 \text{ and } 0 \leqq q_1 \leqq q_2 \leqq \cdots \leqq q_p \tag{25}$$

$$\psi_{i+1} \mid \psi_i,\ i \in \boldsymbol{p}-1 \text{ or equivalently } \psi_{i+1,\Omega} \mid \psi_{i\Omega},\quad i \in \boldsymbol{p}-1 \text{ and } l_1 \geqq l_2 \geqq \cdots \geqq l_p \geqq 0. \tag{26}$$

Lemma 1: Let $S^{\bar{\Omega}}_{B_{2\bar{\Omega}}}$ and $S^{\bar{\Omega}}_{A_1}$ be the Smith forms in $\bar{\Omega}$ of $B_{2\bar{\Omega}}$ and A_1, respectively. Then

$$S^{\bar{\Omega}}_{B_{2\bar{\Omega}}} = \text{diag } [\epsilon_1, \epsilon_2, \cdots, \epsilon_p] \in S^{p\times p} \tag{27}$$

$$S^{\bar{\Omega}}_{A_1} = \text{diag } [\psi_p, \psi_{p-1}, \cdots, \psi_1] \in S^{p\times p}. \tag{28}$$

Proof: See [15].

The following two lemmas will be used in the proof of our main theorem below. As it will be seen there, this part of the proof is constructive and proceeds along slightly different lines depending on whether the given plant P_o is an element of $S^{p\times m}$ or not (or equivalently whether $l_1 = \deg\cdot\psi_{1\Omega} > 0$ or $l_1 = 0$). We start with the following.

Lemma 2: Let $\psi_{1\Omega}$, $\epsilon_{p\Omega}$ be coprime with $l_1 = \deg\cdot\psi_{1\Omega} > 0$ and let d_i, $i \in \boldsymbol{p}$ be polynomials having no zeros in Ω with

$$\deg\cdot d_i \geqq l_1 + q_p, \qquad i \in \boldsymbol{p} \tag{29}$$

and otherwise arbitrary. Let n_i, $m_i \in \mathbb{R}[s]$, $i \in \boldsymbol{p}$ such that

$$n_i\epsilon_{p\Omega} + m_i\psi_{1\Omega} = d_i \quad i \in \boldsymbol{p} \tag{30}$$

$$\deg\cdot n_i < \deg\cdot\psi_{1\Omega} = l_1 \quad i \in \boldsymbol{p}. \tag{31}$$

Then, $e_i := n_i(s+\alpha)^{q_p}/d_i \in S$, $f := m_i(s+\alpha)^{l_1}/d_i \in S$, $i \in \boldsymbol{p}$.

Proof: From (29) and (31) it follows that

$$\deg\cdot d_i > \deg\cdot n_i + q_p \tag{32}$$

so that $e_i \in S$. Also $\epsilon_p = \epsilon_{p\Omega}/(s+\alpha)^{q_p} \in S$ implies that

$$q_p \geqq \deg\cdot\epsilon_{p\Omega}. \tag{33}$$

In view of (33), (32) gives $\deg\cdot d_i > \deg\cdot n_i + \deg\cdot\epsilon_{p\Omega}$ which implies that $(n_i\epsilon_{p\Omega})/d_i \in S$ and are strictly proper, so from (30): $(m_i\psi_{i\Omega})/d_i = 1 - (n_i\epsilon_{p\Omega})/d_i \in S$, $i \in \boldsymbol{p}$ which in turn implies that $f_i \in S$, $i \in \boldsymbol{p}$, and are biproper. □

Lemma 3: Let $l_1 = 0$ and choose n_i, $d_i \in \mathbb{R}[s]$, $i \in \boldsymbol{p}$ with d_i having no zeros in Ω and such that

$$\deg\cdot d_i > \deg\cdot n_i + q_p \qquad i \in \boldsymbol{p}. \tag{34}$$

Define polynomials m_i via

$$m_i := d_i - n_i\epsilon_{p\Omega} \qquad i \in \boldsymbol{p}. \tag{35}$$

Then

$$h_i := (\epsilon_{p\Omega}n_i)/d_i \in S,\ e_i := n_i(s+\alpha)^{q_p}/d_i \in S,\ f_i := m_i/d_i \in S, \quad i \in \boldsymbol{p}. \tag{36}$$

Proof: Since $\epsilon_p \in S$, from (24) we have again (33). Now (33) combined with (34) gives that $\deg\cdot d_i > \deg\cdot n_i + \deg\cdot\epsilon_{p\Omega}$ which implies that $h_i \in S$ and the fact that h_i are strictly proper. The fact that $e_i \in S$ follows from (34). Finally from (35) $f_i = m_i/d_i = 1 - h_i \in S$, $i \in \boldsymbol{p}$. □

Having stated the above we are now able to state our main result as follows.

Theorem 1: Let $P_o \in \mathbb{R}^{p\times m}_{pr}(s)$, $\text{rank}_{\mathbb{R}(s)}P_o = p$. Let $S^{\mathbb{C}}_{P_o}$ = block diag $[\tilde{\epsilon}_1/\tilde{\psi}_1, \cdots, \tilde{\epsilon}_p/\tilde{\psi}_p, 0_{p,m-p}] \in \mathbb{R}^{p\times m}(s)$, $\tilde{\epsilon}_i$, $\tilde{\psi}_i \in \mathbb{R}[s]$, $i \in \boldsymbol{p}$ be the (ordinary) McMillan form of P_o and factorize $\tilde{\psi}_1$ and $\tilde{\epsilon}_p$ as $\tilde{\psi}_1 = \psi_{1\Omega}\hat{\psi}_1$, $\tilde{\epsilon}_p = \epsilon_{p\Omega}\hat{\epsilon}_p$ where the polynomials $\psi_{1\Omega}$ and $\epsilon_{p\Omega}$ have no zeros outside Ω and the polynomials $\hat{\psi}_1$ and $\hat{\epsilon}_p$ have zeros outside Ω. Then there exists a compensator $C \in \phi(P_o)$ such that $H_{yr}(P_o, C) =: H^{\text{diag}}_{yr} \in S^{p\times p}$ is nonsingular, diagonal, and has arbitrary desired poles outside Ω if the polynomials $\psi_{1\Omega}$ and $\epsilon_{p\Omega}$ are coprime.

Proof: Let $\psi_{1\Omega}$, $\epsilon_{p\Omega}$ be coprime. Then depending on whether $l_1 > 0$ or $l_1 = 0$ we can choose d_i, n_i, m_i, $i \in \boldsymbol{p}$ as in Lemmas 2 or 3, respectively. Let $\hat{T}_L$, $\hat{T}_R$, $\tilde{T}_L$, $\tilde{T}_R \in S^{p\times p}$ be S-unimodular matrices such that $S^{\bar{\Omega}}_{B_{2\bar{\Omega}}} = \hat{T}_LB_{2\bar{\Omega}}\hat{T}_P$, $S^{\bar{\Omega}}_{A_1} = \tilde{T}_LA_1\tilde{T}_R$. Define [21] $\bar{\epsilon}_i := \epsilon_p\epsilon_i^{-1}$, $\bar{\psi}_i := \psi_1\psi_i^{-1}$, $i \in \boldsymbol{p}$ where due to the "division properties" in (25), (26) we have that $\bar{\epsilon}_i$, $\bar{\psi}_i \in S$, $i \in \boldsymbol{p}$ and let $\Delta_\epsilon := \text{diag } [\bar{\epsilon}_1, \bar{\epsilon}_2, \cdots, \bar{\epsilon}_p] \in S^{p\times p}$ so that $S^{\bar{\Omega}}_{B_{2\bar{\Omega}}}\Delta_\epsilon = \epsilon_pI_p$ and $\Delta_\psi = \text{diag } [\bar{\psi}_p, \bar{\psi}_{p-1}, \cdots, \bar{\psi}_1] \in S^{p\times p}$ so that $\Delta_\psi S^{\bar{\Omega}}_{A_1} = \psi_1I_p$. Finally, let

$$H_1 := \hat{T}_R\Delta_\epsilon\hat{T}_L\Delta_1 \in S^{p\times p},\ H_2 := \Delta_2\tilde{T}_{\tilde{T}_R}\Delta_\psi\tilde{T}_L \in S^{p\times p} \tag{37}$$

with $\Delta_1 := \text{diag } [e_1, \cdots, e_p] \in S^{p\times p}$, $\Delta_2 := \text{diag } [f_1, \cdots, f_p] \in S^{p\times p}$ and the $e_i, f_i \in S$, $i \in \boldsymbol{p}$ have been defined in Lemmas 2 or 3. With H_1 given by (37) we see that all H_{yr} obtainable as closed-loop transfer function matrices are given by

$$H_{yr} = B_{2\bar{\Omega}}H_1 = \hat{T}_L^{-1}S^{\bar{\Omega}}_{B_{2\bar{\Omega}}}\hat{T}_R^{-1}T_R\Delta_\epsilon T_L\Delta_1 = \epsilon_p\Delta_1$$

$$= \text{diag}\begin{bmatrix}\frac{\epsilon_{p\Omega}n_1}{d_1}, \cdots, \frac{\epsilon_{p\Omega}n_p}{d_p}\end{bmatrix} =: H^{\text{diag}}_{yr} \in S^{p\times p}, \tag{38}$$

i.e., they are nonsingular and diagonal with desired poles outside Ω and satisfy requirement i) of Remark 2. Also with H_2 given by (37) condition (22) gives:

$$I_p - H^{\text{diag}}_{yr} = \text{diag}\begin{bmatrix}\frac{d_1 - \epsilon_{p\Omega}n_1}{d_1}, \cdots, \frac{d_p - \epsilon_{p\Omega}n_p}{d_p}\end{bmatrix}$$

$$= H_2A_1 = \Delta_2\tilde{T}_R\Delta_\psi\tilde{T}_L\tilde{T}_L^{-1}S^{\bar{\Omega}}_{A_1}\tilde{T}_R^{-1}$$

$$= \Delta_\psi\psi_1 = \text{diag}\begin{bmatrix}\frac{\psi_{1\Omega}m_1}{d_1}, \cdots, \frac{\psi_{1\Omega}m_p}{d_p}\end{bmatrix} \in S^{p\times p} \tag{39}$$

which holds true because of (30) or (35) and shows that requirement ii) of Remark 2 is also satisfied. □

Corollary 1: If the condition of Theorem 1 is satisfied then the decoupling compensator is proper and is given by

$$C=A_2T_R\begin{bmatrix}B_{2\Omega}^{-1}H_{yr}^{\text{diag}}\\ \hline Z\end{bmatrix}(I_p-H_{yr}^{\text{diag}})^{-1}\in\phi(P_o). \quad (40)$$

Proof: If $\psi_{1\Omega}$, $\epsilon_{p\Omega}$ are coprime then depending on whether $l_1>0$ or $l_1=0$ we can choose d_i, m_i, n_i, $i\in p$ as in Lemmas 2 or 3, respectively, and H_{yr}^{diag} as in (38). Then from the identity $A_2(N_1+WA_1)=(N_2+A_2W)A_1$ in (11) we obtain $N_2+A_2W=A_2XA_1^{-1}$ and from (6)

$$C_W=(N_2+A_2W)(D_2-B_2W)^{-1}=A_2XA_1^{-1}Y^{-1}=:C\in\phi(P_o) \quad (41)$$

which due to (16), (17), and (19) gives rise to (40). The fact that C is proper follows from (40) since $I_p-H_{yr}^{\text{diag}}$ is biproper and all the rest of the matrices involved in (40) are proper (and Ω-stable).

Corollary 2: If $p=m$, then

$$C=P_o^{-1}\frac{\epsilon_{p\Omega}}{\psi_{1\Omega}}\text{diag}(n_1/m_1,\cdots,n_p/m_p). \quad (42)$$

Proof: If $p=m$, then $T_R\equiv I_p$, $B_{2\tilde\Omega}\equiv B_2$ and (42) is obtained from (40) by making use of (38) and (39). □

Corollary 3: If $P_o\in S^{p\times m}$, i.e., if $\psi_{1\Omega}=1$ and $\text{rank}_{\mathbb{R}(s)}P_o=p$, then the condition of Theorem 1 is satisfied and the decoupling compensator is given by

$$C=T_R\begin{bmatrix}P_{\tilde\Omega}^{-1}H_{yr}^{\text{diag}}\\ \hline Z\end{bmatrix}\text{diag}(d_1/m_1,\cdots,d_p/m_p) \quad (43)$$

where $T_R\in S^{m\times m}$ a S-unimodular matrix such that $P_oT_R=[P_{\tilde\Omega},0_{p,m-p}]$ and $P_{\tilde\Omega}\in S^{p\times p}$.

Proof: If $P_o\in S^{p\times m}$, then we can take $A_2=I_m$, $B_2=P_o$. Then according to the constriction of Lemma 3 $I_p-H_{yr}^{\text{diag}}=\text{diag}(m_1/d_1,\cdots,m_p/d_p)$ so that (43) follows from (40). □

Corollary 4: If $P_o\in S^{p\times p}$ then the decoupling compensator is given by

$$C=P_o^{-1}\epsilon_{p\Omega}\text{diag}(n_1/m_1,\cdots,n_p/m_p). \quad (44)$$

Proof: Obvious from (42). Of course n_i, m_i must be chosen as in Lemma 3. □

The following obvious corollary gives a number of sufficient conditions for decouplability.

Corollary 5: Let $\Sigma(P_o)$ be free of unstable hidden modes with $\text{rank}_{\mathbb{R}(s)}P_o=p$. Then $\Sigma(P_o)$ is decouplable if any of the following conditions is satisfied i) the polynomials $\tilde\psi_1$ and $\tilde\epsilon_p$ in the (ordinary) McMillan form $S_{P_o}^{\mathbb{C}}$ of P_o are coprime ii) P_o is asymptotically stable, i.e., $P_o\in S^{p\times m}$ or equivalently $\psi_{1\Omega}=1$ with $\Omega\equiv\{s\in\mathbb{C},\text{Re}(s)\geqq0\}$ iii) P_o is "minimum phase" or equivalently $\epsilon_{p\Omega}=1$ with $\Omega\equiv\{s\in\mathbb{C},\text{Re}(s)\geqq0\}$.

The above analysis gives rise to the following algorithm for the computation of decoupling and pole assigning compensator $C\in\phi(P_o)$.

Algorithm: Data: $P_o\in\mathbb{R}_{pr}^{p\times m}(s)$, $\text{rank}_{\mathbb{R}(s)}P_o=p$.

1) *Choose* Ω (Fig. 2) and compute the Smith–McMillan form in $\tilde\Omega=\Omega\cup\{\infty\}$, $S_{P_o}^{\tilde\Omega}$ of P_o [15].
2) Examine if $\psi_{1\Omega}$, $\epsilon_{p\Omega}$ are coprime. If they are go to (3), otherwise write: "existence of decoupling compensator cannot be determined by present method." Go to 1).
3) Store l_1 and q_p. If $l_1>0$ choose $d_i\in\mathbb{R}[s]$ with $\deg\cdot d_i\geqq l_1+q_p$, $i\in p$ and with no zeros inside Ω but otherwise arbitrary, and go to (4). If $l_1=0$ choose d_i, $n_i\in\mathbb{R}[s]$ such that $\deg\cdot d_i>\deg\cdot n_i+q_p$, $i\in p$, the d_i having no zeros inside Ω and otherwise arbitrary, and go to 5).
4) Compute polynomials n_i, m_i, $i\in p$ such that (30), (31) are satisfied and go to 6).
5) Define polynomials $m_i:=d_i-\epsilon_{p\Omega}n_i$, $i\in p$ and go to 6).
6) Form H_{yr}^{diag} as in (38). If $l_1>0$ and $p<m$ compute C via (40). If $l_1>0$ and $p=m$ compute C via (42). If $l_1=0$ and $p<m$ compute C via (43). If $l_1=0$ and $p=m$ compute C via (44).

Example: Let $\Omega\equiv\{s\in\mathbb{C},\text{Re}(s)\geqq0\}=:\mathbb{C}^+$, $\tilde\Omega=\mathbb{C}^+\cup\{\infty\}$ and

$$P_o=\begin{bmatrix}\frac{s+1}{s^2} & 0\\ \frac{1}{s(s-1)} & \frac{1}{1-s}\end{bmatrix}\in\mathbb{R}_{pr}^{2\times2}(s).$$

Computing the Smith–McMillan form in $\tilde\Omega$ of P_o we get: $\epsilon_1=\epsilon_2=1/(s+\alpha)$, $\psi_1=s^2(s-1)/(s+\alpha)^3)$ $\psi_2=1$, $\alpha>0$, from which $\epsilon_{1\Omega}=\epsilon_{2\Omega}=1$, $q_1=q_2=1$ and P_o has two zeros at $s=\infty$ each one of order $q_1^{z\infty}=q_2^{z\infty}=1$. Also $\psi_{1\Omega}=s^2(s-1)$, $\psi_{2\Omega}=1$, $\deg\cdot\psi_{1\Omega}=l_1=3$. So $\psi_{1\Omega}$, $\epsilon_{2\Omega}$ are coprime and P_o is decouplable. From $\deg\cdot d_i=l_1+q_2=3+1=4$, $i=1,2$ we choose $d_1=d_2=(s+1)^4$. Then $m_i=s+5$, $n_i=11s^2+4s+1$, $i=1,2$. Since $p=m=2$ we compute C from (42):

$$C=P_o^{-1}\epsilon_{2\Omega}/\psi_{1\Omega}\text{diag}(n_1/m_1,n_2/m_2)$$

$$=\begin{bmatrix}\frac{11s^2+4s+1}{(s+1)(s-1)(s+5)} & \\ \frac{11s^2+4s+1}{s(s+1)(s-1)(s+5)} & \frac{-(11s^2+4s+1)}{s^2(s+5)}\end{bmatrix}.$$

With this compensator the resulting closed-loop system is internally stable and $H_{yr}(P_o,C)=(\epsilon_{2\Omega}n_1/d_1)I_2=(11s^2+4s+1)/(s+1)^4)I_2\in S^{2\times2}$. Notice that in general we can choose different dynamics in each loop, i.e., $d_i\neq d_j$, $i\neq j$.

IV. Conclusions

In this note we have shown that coprimeness of the (minimal) Ω-unstable "pole" and "zero" polynomials $\psi_{1\Omega}$ and $\epsilon_{p\Omega}$ of a plant P_o constitutes a sufficient condition for the existence of an internally stabilizing proper cascade compensator C such that under unity feedback the resulting closed-loop transfer matrix will be diagonal with desired poles inside an arbitrary region Ω^c of $\mathbb{C}$. We have also presented a design procedure for C and shown that its structure depends on the integers $l_1=\deg\cdot\psi_{1\Omega}$, and $q_p=\deg\cdot\epsilon_{p\Omega}+q_p^{z\infty}$ where $q_p^{z\infty}$ is the maximum order of the "zero at infinity" or equivalently the order of the pth "zero at infinity" of the plant P_o.

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A Matrix Inequality Associated with Bounds on Solutions of Algebraic Riccati and Lyapunov Equations

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Abstract—**A new proof is presented for the inequality** $\operatorname{tr}(XY) \le \|X\|_2 \cdot \operatorname{tr} Y$. **This argument is valid under the condition that** Y **be real symmetric nonnegative definite;** X **may be any square matrix.**

Introduction

Much work has been done in recent years to establish bounds on the eigenvalues, and, in particular, on the spectral norm, of solutions to the algebraic matrix Riccati equations and the Lyapunov equations of control and estimation theory. The derivations in, e.g., [2]–[5] have either used or implied the fact that, for $X, Y \in R^{n\times n}$ with both $X, Y \ge 0$,[1]

$$\operatorname{tr}(XY) \le \|X\|_2 \cdot \operatorname{tr}(Y) \tag{1}$$

where, following [1], $\|\cdot\|_2$ denotes the spectral norm or largest singular value. In [2]–[6], this inequality was only applied to matrices X, Y that were guaranteed to be symmetric and either nonnegative definite or positive definite. A more recent work on eigenvalue bounds [7] contains the related result for $Y = Y' \ge 0$ and $X = X'$

$$\lambda_{\min}(X) \operatorname{tr}(Y) \le \operatorname{tr}(XY) \le \lambda_{\max}(X) \operatorname{tr}(Y)$$

which implies (1). However, (1) holds whenever *at least* the matrix Y is symmetric and nonnegative definite; the other can be an arbitrary real-valued square matrix. This more general result has not, to the authors' knowledge, previously appeared in the control literature.

Results

Theorem: Let $X, Y \in R^{n\times n}$ with Y symmetric and nonnegative definite. Then

$$\operatorname{tr}(XY) \le |\operatorname{tr}(XY)| \le \|X\|_2 \cdot \operatorname{tr}(Y). \tag{2}$$

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[1] [6] assumes both $X, Y > 0$.

Proof: Obviously we need only prove the second inequality. Since Y is symmetric and nonnegative definite, it can be decomposed as $Y = QQ'$, where $Q \in R^{n\times m}$ has full rank $m \le n$. Write

$$Q = [q_1 | q_2 | q_3 | \cdots | q_m].$$

Then

$$\operatorname{tr} XY = \operatorname{tr} X(QQ') = \operatorname{tr} X[q_1|q_2|\cdots|q_m] \begin{bmatrix} q_1' \\ \cdots \\ q_2' \\ \cdots \\ \vdots \\ \cdots \\ q_m' \end{bmatrix}$$

$$= \operatorname{tr} \sum_{j=1}^{m} X(q_j q_j')$$

and clearly the trace and finite sum can be interchanged to give

$$\operatorname{tr} XY = \sum_{j=1}^{m} \operatorname{tr} X(q_j q_j')$$

$$= \sum_{j=1}^{m} \operatorname{tr}(q_j' X q_j)$$

or

$$\operatorname{tr}(XY) = \sum_{j=1}^{m} q_j' X q_j. \tag{3}$$

Thus,

$$|\operatorname{tr}(XY)| \le \sum_{j=1}^{m} |q_j' X q_j|$$

$$\le \sum_{j=1}^{m} \|q_j\| \cdot \|Xq_j\|$$

$$\le \sum_{j=1}^{m} \|X\|_2 \cdot \|q_j\|^2 = \|X\|_2 \cdot \sum_{j=1}^{m} \|q_j\|^2 \tag{4}$$

by the Cauchy–Schwarz inequality and the definition of the spectral norm. But

$$\sum_{j=1}^{m} \|q_j\|^2 = \sum_{i=1}^{n} \sum_{j=1}^{m} |q_{ij}|^2$$

$$\triangleq \|Q\|_F^2;$$

and if we recall that $\|Q\|_F^2 = \operatorname{tr} Q'Q = \operatorname{tr} QQ'$, (4) becomes

$$|\operatorname{tr}(XY)| \le \|X\|_2 \cdot \operatorname{tr} QQ' = \|X\|_2 \cdot \operatorname{tr} Y.$$

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