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INFINITE ELEMENTARY DIVISORS OF POLYNOMIAL MATRICES AND IMPULSIVE SOLUTIONS OF LINEAR HOMOGENEOUS MATRIX DIFFERENTIAL EQUATIONS*

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Abstract. Impulsive solutions of linear homogeneous matrix differential equations are re-examined in the light of the theory of Jordan chains that correspond to infinite elementary divisors of the associated polynomial matrix. Infinite elementary divisors of general polynomial matrices are defined and their relation to the pole-zero structure of polynomial matrices at infinity is examined. It is shown that impulsive solutions are due to Jordan chains of a "dual" polynomial matrix that correspond to infinite elementary divisors that are associated with the orders of "zeros at infinity" of the original matrix.

1. Introduction

It is well known [1]–[3] that when appropriate initial conditions are permitted then the response of the (free) generalized state-space system

$$E\dot{x}(t) = Ax(t), \tag{1.1}$$

where $E \in \mathbb{R}^{r\times r}$, $1 \le \operatorname{rank}_{\mathbb{R}} E < r$, $A \in \mathbb{R}^{r\times r}$, exhibits an impulsive behavior at $t=0$ which is associated to "zeros at $s=\infty$" of the matrix "pencil" $sE - A \in \mathbb{R}[s]^{r\times r}$, i.e., to the fact that, in this case, the natural modes of (1), defined as the values of s where $sE-A$ loses rank, also include the point at $s=\infty$. The zeros at $s=\infty$ of the matrix pencil $sE-A$ are closely related to the structure of its infinite elementary divisors (IEDs). In general, given a matrix pencil

$$A(s) = A_1 s + A_0 \in \mathbb{R}[s]^{r\times m}, \tag{1.2}$$

$A_i \in \mathbb{R}^{r\times m}$, $i = 0, 1$, the IEDs of $A(s)$ are defined [4] as the finite elementary divisors (FEDs) of the "dual" pencil:

$$\tilde{A}(w) := A_1 + A_0 w \in \mathbb{R}[w]^{r\times m} \tag{1.3}$$

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at $w=0$, i.e., the FEDs having the form w^{μ_i}, $\mu_i>0$. The exact relation between the degrees μ_i of the IEDs of $A(s)$ and the orders $\hat{q}_i$ of its zeros at $s=\infty$ has been examined in [5] where it is shown that if $m=r$ and

$$S^{\infty}_{A(s)}(s)=\operatorname{diag}\left[s^{q_1}, s^{q_2}, \ldots, s^{q_k}, \frac{1}{s^{\hat{q}_{k+1}}}, \ldots, \frac{1}{s^{\hat{q}_r}}\right]\in\mathbb{R}(s)^{r\times r} \tag{1.4}$$

is the Smith–McMillan form at $s=\infty$ of $A(s)=A_1s+A_0$, then

$$q_1=q_2=\cdots=q_k=1 \tag{1.5}$$

and $A(s)$ has $l=$ rank defect of $A_1=r-\operatorname{rank}_{\mathbb{R}} A_1\geq 0$ IEDs w^{μ_i} with

$$\mu_i=\hat{q}_i+1, \qquad i=k+1, k+2, \ldots, r, \tag{1.6}$$

where $k=\operatorname{rank}_{\mathbb{R}} A_1$, so that if $S^{0}_{\tilde{A}(w)}(w)\in\mathbb{R}[w]^{r\times r}$ is the Smith form of $\tilde{A}(w)$ at $w=0$, then

$$S^{0}_{\tilde{A}(w)}(w)=\operatorname{diag}[1, 1, \ldots, 1, w^{\mu_{k+1}}, \ldots, w^{\mu_r}]. \tag{1.7}$$

Generalized state-space systems as in (1.1) represent a particular case of a more general class of systems described by linear homogeneous matrix differential equations having the form

$$A(\rho)\beta(t)=0, \qquad t\geq 0, \tag{1.8}$$

where $\rho := d/dt$ is the differential operator, $A(\rho)$ is an $r\times r$ nonsingular (over the field $\mathbb{R}(\rho)$ of real rational functions in ρ) polynomial matrix, and $\beta(t)\colon (0-, \infty)\to\mathbb{R}^r$ is a vector-valued function to be found which is continuous for $t>0+$, possibly with a step discontinuity at the origin, i.e., such that $\beta(0-)\neq\beta(0+)$. Let

$$A(\rho)=A_\nu\rho^\nu+A_{\nu-1}\rho^{\nu-1}+\cdots+A_1\rho+A_0, \tag{1.9}$$

where $A_i\in\mathbb{R}^{r\times r}$, $i=0, 1, 2, \ldots, \nu>0$, and let $\beta(0-)$, $\beta^{(1)}(0-), \ldots$, $\beta^{(\nu-1)}(0-)$ be the initial values of $\beta(t)$ and its $\nu-1$ derivatives at $t=0-$.

Considering the L_- Laplace transform $\hat{\beta}(s)$ of $\beta(t)$:

$$\hat{\beta}(s) := L_-\beta(t)=\int_{0-}^{\infty}\beta(t)\,e^{-st}\,dt \tag{1.10}$$

and taking the L_- Laplace transform of (1.8) we obtain

$$L_-\{A(\rho)\beta(t)\}=A(s)\hat{\beta}(s)-\alpha(s)=0, \tag{1.11}$$

where

$$A(s)=A_\nu s^\nu+A_{\nu-1}s^{\nu-1}+\cdots+A_1s+A_0\in\mathbb{R}[s]^{r\times r} \tag{1.12}$$

and $\alpha(s)\in\mathbb{R}[s]^{r\times 1}$ is the "initial condition" vector associated with $\beta(0-)$,

$\beta^{(1)}(0-),\ldots,\beta^{(\nu-1)}(0-)$ given by [6]

$$\alpha(s)=[s^{\nu-1}I_r, s^{\nu-2}I_r,\ldots, sI_r, I_r]$$
$$\times\begin{bmatrix} A_\nu & 0 & \cdots & 0 & 0\\ A_{\nu-1} & A_\nu & \cdots & 0 & 0\\ \vdots & & \ddots & & \vdots\\ A_2 & A_3 & \cdots & A_\nu & 0\\ A_1 & A_2 & \cdots & A_{\nu-1} & A_\nu\end{bmatrix}\begin{bmatrix}\beta(0-)\\ \beta^{(1)}(0-)\\ \vdots\\ \beta^{(\nu-2)}(0-)\\ \beta^{(\nu-1)}(0-)\end{bmatrix}. \tag{1.13}$$

Now it is also known [7] that if $\hat{q}_r>0$, i.e., if $A(s)$ has at least one zero at $s=\infty$, then $\operatorname{rank}_{\mathbb{R}} A_r<r$ and, as in the case of the free generalized state-space system in (1.1), when appropriate initial conditions are imposed on $\beta(t)$ and its $(\nu-1)$ derivatives at $t=0-$, $\beta(t)$ may exhibit an impulsive behavior at $t=0$ which is a consequence of the fact that (1.8) forces $\beta(t)$ and $\beta^{(q)}(t)$, $q=1,2,\ldots,\nu-1$, to satisfy certain constraints at $t=0-$.

The exact derivation of these constraints and their relation to the zero structure at $s=\infty$ of the polynomial matrix $A(s)$ (as this structure is demonstrated by its Smith-McMillan form at $s=\infty$: $S^{\infty}_{A(s)}$) and the structure of its IEDs has not been examined in the past. These problems are examined here in some detail. It is shown that the constraints mentioned, that $\beta(t)$ and $\beta^{(q)}(t)$ must satisfy, are in effect conditions satisfied by certain Jordan chains [8] that correspond to IEDs of $A(s)$ or equivalently, and as is shown, by certain Jordan chains that correspond to FEDs at $w=0$ of the "dual" to $A(s)$ polynomial matrix $\tilde{A}(w)$ which is again defined by generalizing the definition of the "dual" pencil $\tilde{A}(w)=A_0w+A_1$ to the general polynomial matrix case.

This paper is organized as follows. In Section 2 we give some background theory regarding the pole-zero structure at infinity of general rational matrices. The IEDs of a polynomial matrix $A(s)$ are defined in Section 3 by generalizing the classical definition regarding matrix pencils [4]. Then the relation of IEDs to the orders of the poles and zeros at infinity of $A(s)$ is examined. Finally, the theory of Jordan chains corresponding to IEDs of $A(s)$ that in turn correspond to zeros at infinity of $A(s)$ is examined in Section 4 and the whole theory is demonstrated by an example.

2. Background

Let $\mathbb{R}$ be the field of reals, let $\mathbb{R}[s]$ be the ring of polynomials, and let $\mathbb{R}(s)$ be the field of rational functions $t(s)=n(s)/d(s)$, $n(s)$, $d(s)\in\mathbb{R}[s]$, both with coefficients in $\mathbb{R}$. Define the map $\delta_\infty:\mathbb{R}(s)\to\mathbb{Z}\cup\{+\infty\}$ via

$$\delta_\infty(t(s)):=\deg d(s)-\deg n(s),\qquad t(s)\not\equiv 0, \tag{2.1}$$

$$\delta_\infty(t(s)):=+\infty,\qquad t(s)\equiv 0.^2 \tag{2.2}$$

[2] $\deg(0):=-\infty$.

The map $\delta_\infty(\cdot)$ is a discrete valuation on $\mathbb{R}(s)$ [9] and every $t(s) \in \mathbb{R}(s)$ can be factored as $t(s) = (1/s)^{q_\infty}(n_1(s)/d_1(s))$ where $q_\infty := \delta_\infty(t(s))$ and $\deg n_1(s) = \deg d_1(s)$. If $q_\infty > 0$ we say that $t(s)$ has a *zero at* $s = \infty$ of order q_∞ and if $q_\infty < 0$, then we say that $t(s)$ has a *pole at* $s = \infty$ of order $|q_\infty|$. If $t(s) \in \mathbb{R}(s)$ and $\delta_\infty(t(s)) \geq 0$, then $t(s)$ is called a *proper* rational function and if the inequality is strict, then $t(s)$ is called *strictly proper*. The set of proper rational functions, which we denote by $\mathbb{R}_{\text{pr}}(s)$, is a Euclidean ring with "degree" given by map $\delta_\infty(\cdot)$. The units in $\mathbb{R}_{\text{pr}}(s)$ are proper rational functions with $\deg n(s) = \deg d(s)$ (i.e., also having no zeros at $s = \infty$). These units we call *biproper* rational functions. Denote by $k^{p\times m}$ the set of $p \times m$ matrices with elements in a set k. If $k \equiv \mathbb{R}_{\text{pr}}(s)$, then such matrices are called proper rational matrices. A $U(s) \in \mathbb{R}_{\text{pr}}(s)^{p\times p}$ is called $\mathbb{R}_{\text{pr}}(s)$-unimodular or *biproper* if $\det U(s) \in \mathbb{R}_{\text{pr}}(s)$ is a unit. $T_1(s)$, $T_2(s) \in \mathbb{R}(s)^{p\times m}$ are called *equivalent at* $s = \infty$ if there exist biproper rational matrices $U_L(s) \in \mathbb{R}_{\text{pr}}(s)^{p\times p}$, $U_R(s) \in \mathbb{R}_{\text{pr}}(s)^{m\times m}$, such that

$$U_L(s)T_1(s)U_R(s) = T_2(s). \tag{2.3}$$

Equation (2.3) defines an equivalence relation on $\mathbb{R}(s)^{p\times m}$, which we denote by E^∞ and if $T_1(s)$, $T_2(s)$ are equivalent at $s = \infty$ we denote this fact by writing $(T_1(s), T_2(s)) \in E^\infty$. We have the following:

Theorem 2.1 (Smith–McMillan form of a rational matrix at $s = \infty$) [10]. *Let* $T(s) \in \mathbb{R}(s)^{p\times m}$, $\text{rank}_{\mathbb{R}(s)} T(s) = r$. *Then* $T(s)$ *is equivalent at* $s = \infty$ *to a diagonal matrix having the form*

$$S^\infty_{T(s)}(s) = \text{block diag}[s^{q_1}, s^{q_2}, \ldots, s^{q_r}, 0_{p-r,m-r}], \tag{2.4}$$

where

$$q_1 \geq q_2 \geq \cdots \geq q_r, \qquad q_i \in \mathbb{Z}, \quad i \in \mathbf{r}. \tag{2.5}$$

Remark 2.1. If $T(s) \in \mathbb{R}_{\text{pr}}(s)^{p\times m}$, then $q_i \leq 0$, $i \in \mathbf{r}$, i.e., $S^\infty_{T(s)}(s) \in \mathbb{R}_{\text{pr}}(s)^{p\times m}$ and in such a case $S^\infty_{T(s)}(s)$ is called the *Smith form of* $T(s)$ *at* $s = \infty$. Otherwise, i.e., if $T(s)$ is nonproper, then some of the q_i satisfy $q_i > 0$ for $i = 1, 2, \ldots, k \leq r$, i.e., $S^\infty_{T(s)}(s)$ is also nonproper and in this case is called the *McMillan form of* $T(s)$ *at* $s = \infty$. Thus in general a rational matrix may have poles *and* zeros at $s = \infty$ and in order to stress this fact we denote the Smith–McMillan form at $s = \infty$ of $T(s)$ as

$$S^\infty_{T(s)}(s) = \text{block diag}\left[s^{q_1}, s^{q_2}, \ldots, s^{q_k}, \frac{1}{s^{\hat{q}_{k+1}}}, \ldots, \frac{1}{s^{\hat{q}_r}}, 0_{p-r,m-r}\right], \tag{2.6}$$

where $1 \leq k \leq r$ and $\hat{q}_i = -q_i$, $i = k+1, \ldots, r$, so that

$$q_1 \geq q_2 \geq \cdots \geq q_k \geq 0, \tag{2.7}$$

$$\hat{q}_r \geq \hat{q}_{r-1} \geq \cdots \geq \hat{q}_{k+1} \geq 0. \tag{2.8}$$

If p_∞ is the number of the q_i in (2.6) satisfying $q_i > 0$, $i \in \mathbf{p}_\infty$, then we say that $T(s)$ has p_∞ *poles at* $s = \infty$ each one of order q_i and if z_∞ is the number of the $\hat{q}_i$ in (2.6) satisfying $\hat{q}_i > 0$, $i \in \mathbf{z}_\infty$, then we say that $T(s)$ has z_∞ *zeros at* $s = \infty$ each one of order $\hat{q}_i$.

It can be proved [10] that if by $\xi_i(T)$ we denote the least $\delta_\infty(\cdot)$ among the $\delta_\infty(\cdot)$ of all minors of $T(s)$ of order i, $i \in \boldsymbol{\tau}$, then

$$\begin{aligned} q_1 &= \xi_0(T) - \xi_1(T), \\ q_2 &= \xi_1(T) - \xi_2(T), \\ &\vdots \\ q_r &= \xi_{r-1}(T) - \xi_r(T), \end{aligned} \tag{2.9}$$

where $\xi_0(T) := 0$.

We now give a number of results that are necessary in our development in the following. We start by examining some relations between the Smith–McMillan form at $s = \infty$ of a general rational matrix $T(s) \in \mathbb{R}(s)^{r \times m}$ and its Laurent expansion at $s = \infty$. The next proposition and its corollary generalize, to the matrix case, some well-known properties of scalar rational functions.

Proposition 2.1. *Let $T(s) \in \mathbb{R}(s)^{p \times m}$, let $\operatorname{rank}_{\mathbb{R}(s)} T(s) = r$, and let $S^\infty_{T(s)}(s)$ be the Smith–McMillan form of $T(s)$ at $s = \infty$ as in (2.6). Let $q_1 > 0$, i.e., assume that $T(s)$ is nonproper and let*

$$T(s) = T_\nu s^\nu + T_{\nu-1} s^{\nu-1} + \cdots + T_1 s + T_0 + T_{-1}\frac{1}{s} + T_{-2}\frac{1}{s^2} + \cdots \tag{2.10}$$

be the Laurent expansion at $s = \infty$ of $T(s)$, where $T_i \in \mathbb{R}^{p \times m}$, $i = \nu - \kappa$, $\kappa = 0, 1, 2, \ldots$, $T_\nu \neq 0$, $\nu > 0$, and $T_{\nu+\kappa} = 0$, $\kappa = 1, 2, \ldots$. Then

$$\nu = q_1. \tag{2.11}$$

Proof. From the construction of $S^\infty_{T(s)}(s)$ [10] we have that $q_1 := -\xi_1(T)$ where $\xi_i(T)$ is the minimum $\delta_\infty(\cdot)$ among the $\delta_\infty(\cdot)$s of all minors of $T(s)$ of order i, $i \in \boldsymbol{\tau}$. Thus q_1 is the minimum of the $\delta_\infty(\cdot)$ of all elements $t_{ij}(s)$ of $T(s)$. As $T(s)$ has the representation (2.10) there is at least one element $t_{ij}(s)$ of $T(s)$ with $\delta_\infty(t_{ij}(s)) = -\nu$. □

Corollary 2.1. *Let $T(s) \in \mathbb{R}[s]^{p \times m}$ and write it as a matrix polynomial, i.e., write*

$$T(s) = T_\nu s^\nu + T_{\nu-1} s^{\nu-1} + \cdots + T_1 s + T_0, \tag{2.12}$$

$T_i \in \mathbb{R}^{p \times m}$, $i = 0, 1, 2, \ldots$, $\nu > 0$. Let $S^\infty_{T(s)}(s)$ be as in (2.6). Then $\nu = q_1$.

Proof. The expression of $T(s)$ as a matrix polynomial is its Laurent expansion at $s=\infty$ and the rest follows from Proposition 2.1. □

3. Infinite elementary divisors of polynomial matrices and their relation to pole-zero structure at infinity

Consider the polynomial matrix

$$A(s)=A_\nu s^\nu + A_{\nu-1}s^{\nu-1}+\cdots+A_1 s + A_0 \in \mathbb{R}[s]^{r\times r}, \tag{3.1}$$

$A_i \in \mathbb{R}^{r\times r}$, $i=0,1,2,\ldots,\nu>0$, $A_\nu \neq 0$, $\operatorname{rank}_{\mathbb{R}(s)} A(s)=r$, and define the "dual" polynomial matrix $\tilde{A}(w)$ of $A(s)$ as

$$\begin{aligned}\tilde{A}(w) &:= A_\nu + A_{\nu-1}w+\cdots+A_1 w^{\nu-1}+A_0 w^\nu \in \mathbb{R}[w]^{r\times r}\\ &= w^\nu A\left(\frac{1}{w}\right).\end{aligned} \tag{3.2}$$

Then we have

Definition 3.1. The IEDs of $A(s)$ are defined as the FEDs of $\tilde{A}(w)$ at $w=0$, i.e., as the FEDs of $\tilde{A}(w)$ that have the form

$$w^{\mu_j},\ \mu_j>0. \tag{3.3}$$

In order to examine the structure of the IEDs of $A(s)$ we thus see that we need the zero structure at $w=0$ of $\tilde{A}(w)$. Let $S^0_{\tilde{A}(w)}(w)$ denote the (local) Smith form of $\tilde{A}(w)$ at $w=0$ [8]. Then we have

Proposition 3.1. *Let $A(s)\in\mathbb{R}[s]^{r\times r}$ as in* (3.1) *and $U_L(s)$, $U_R(s)\in\mathbb{R}_{pr}(s)^{r\times r}$ biproper matrices and such that*

$$U_L(s)A(s)U_R(s)=S^\infty_{A(s)}(s)=\operatorname{diag}\left[s^{q_1},\ldots,s^{q_k},\frac{1}{s^{\hat{q}_{k+1}}},\ldots,\frac{1}{s^{\hat{q}_r}}\right]. \tag{3.4}$$

Let $\tilde{U}_L(w)$, $\tilde{U}_R(w)\in\mathbb{R}(w)^{r\times r}$ rational matrices having no poles and zeros at $w=0$[3] *and such that*

$$\tilde{U}_L(w)\tilde{A}(w)\tilde{U}_R(w)=S^0_{\tilde{A}(w)}(w). \tag{3.5}$$

Then

$$\begin{aligned}S^0_{\tilde{A}(w)}(w) &= w^{q_1}S^\infty_{A(s)}\left(\frac{1}{w}\right)\\ &= \operatorname{diag}[1, w^{q_1-q_2},\ldots,w^{q_1-q_k},w^{q_1+\hat{q}_{k+1}},\ldots,w^{q_1+\hat{q}_r}]\end{aligned} \tag{3.6}$$

and

$$\tilde{U}_L(w)=U_L\left(\frac{1}{w}\right),\qquad \tilde{U}_R(w)=U_R\left(\frac{1}{w}\right). \tag{3.7}$$

[3] That is, "unimodular" over the ring of rational functions $t(w)$ with no poles at $w=0$.

Proof. Notice that from (3.4) and the fact that $\nu = q_1$ in Corollary 2.1 it simply follows that the *proper* rational matrix

$$\frac{1}{s^{q_1}} S^{\infty}_{A(s)}(s) = \operatorname{diag}\left[1, \frac{1}{s^{q_1-q_2}}, \ldots, \frac{1}{s^{q_1-q_k}}, \frac{1}{s^{q_1+\hat{q}_{k+1}}}, \ldots, \frac{1}{s^{q_1+\hat{q}_r}}\right] \quad (3.8)$$

is the "Smith" form at $s = \infty$ of the proper rational matrix

$$\frac{1}{s^{q_1}} A(s) = A_{q_1} + A_{q_1-1}\frac{1}{s} + \cdots + A_1 \frac{1}{s^{q_1-1}} + A_0 \frac{1}{s^{q_1}} \in \mathbb{R}_{pr}(s)^{r\times r}. \quad (3.9)$$

Now using the transformation $s = 1/w$, from (3.8) and (3.9) we have that

$$\begin{aligned} w^{q_1} S^{\infty}_{A(s)}\left(\frac{1}{w}\right) &= \operatorname{diag}[1, w^{q_1-q_2}, \ldots, w^{q_1-q_k}, w^{q_1+\hat{q}_{k+1}} \cdots w^{q_1+\hat{q}_r}] \\ &=: S^{0}_{\tilde{A}(w)}(w) \end{aligned} \quad (3.10)$$

is the (local) Smith form at $w = 0$ of $w^{q_1}A(1/w) =: \tilde{A}(w)$. From (3.4) we have

$$U_L(s)\frac{1}{s^{q_1}} A(s) U_R(s) = \frac{1}{s^{q_1}} S^{\infty}_{A(s)}(s) \quad (3.11)$$

which for $s = 1/w$ gives

$$U_L\left(\frac{1}{w}\right) w^{q_1} A\left(\frac{1}{w}\right) U_R\left(\frac{1}{w}\right) = w^{q_1} S^{\infty}_{A(s)}\left(\frac{1}{w}\right) \quad (3.12)$$

or, due to (3.2) and (3.10), that

$$U_L\left(\frac{1}{w}\right) \tilde{A}(w) U_R\left(\frac{1}{w}\right) = S^{0}_{\tilde{A}(w)}(w) \quad (3.13)$$

and (3.7) follows by comparing (3.13) and (3.5). □

Corollary 3.1. *The IEDs of a polynomial matrix* $A(s)$ *as in* (3.1) *are given by*

$$w^{\mu_j}, \qquad j = 2, 3, \ldots, r, \quad (3.14)$$

$$\mu_j := q_1 - q_j > 0, \qquad j = 2, 3, \ldots, k, \quad (3.15)$$

$$\mu_j := q_1 + \hat{q}_j > 0, \qquad j = k+1, k+2, \ldots, r. \quad (3.16)$$

Remark 3.1. The above result generalizes the results in [5] regarding the structure of IEDs of matrix pencils that we mentioned in the introduction. We see that polynomial matrices in general have two kinds of IEDs. The first kind are IEDs that correspond to *poles* at $s = \infty$ of $A(s)$ with orders $q_j < q_1$, $j = 2, 3, \ldots, k$. The second kind of IEDs correspond to poles *and* zeros at $s = \infty$ [11]. Notice that the first kind of IEDs always exists since $\mu_j = q_1 - q_j > 0$, $j = 2, 3, \ldots, k$, and the second kind exists only when $A(s)$ has zeros at $s = \infty$.

The first kind of IEDs, i.e., the ones with degrees $\mu_j = q_1 - q_j > 0$, $j = 1, 2, \ldots, k$, we call "*infinite pole IEDs.*" The second kind of IEDs, i.e., the ones with degrees $\mu_j = q_1 + \hat{q}_j > 0$ with $\hat{q}_j > 0$, $j = k+1, \ldots, r$, we call "*infinite zero IEDs.*" As it is shown in the following, impulsive solutions to (1.8) are only due to "infinite zero IEDs" and appropriate initial conditions which are determined by vectors $x_{jq} \in \mathbb{R}^{r \times 1}$, $j = k+1, \ldots, r$, $q = 0, 1, 2, \ldots, \hat{q}_j, \hat{q}_j + 1, \ldots, \hat{q}_j + q_1 - 1$ $(= \mu_j - 1)$, that form Jordan chains of lengths $\mu_j = q_1 + \hat{q}_j$ that correspond to zeros at $w = 0$ of "infinite zero IEDs" w^{μ_j} of $A(s)$.

4. Jordan chains and impulsive solutions

Now let $\tilde{U}_R(w) = [\tilde{u}_1(w), \tilde{u}_2(w), \ldots, \tilde{u}_r(w)]$, $\tilde{u}_j(w) \in \mathbb{R}(w)^{r \times 1}$. Then from (3.5) we have

$$\tilde{A}(w)\tilde{u}_j(w) = v_j(w) w^{\mu_j}, \qquad j = k+1, k+2, \ldots, r, \tag{4.1}$$

where $\mu_j = q_1 + \hat{q}_j$, $j = k+1, \ldots, r$, and $v_j(w)$ the jth column of $\tilde{U}_L(w)^{-1}$.

Proposition 4.1. *Let $\tilde{u}_j^{(q)}(w)$, $\tilde{A}^{(q)}(w)$ be the qth derivatives of $\tilde{u}_j(w)$ and $\tilde{A}(w)$ with respect to w for $q = 0, 1, 2, \ldots, \mu_j - 1$, $j = k+1, \ldots, r$, and define*

$$x_{jq} := \frac{1}{q!} \tilde{u}_j^{(q)}(0) \tag{4.2}$$

for $q = 0, 1, \ldots, \mu_j - 1$, $j = k+1, \ldots, r$. Then the vectors

$$x_{j0}, x_{j1}, \ldots, x_{j\mu_j - 1} \in \mathbb{R}^r, \qquad j = k+1, \ldots, r, \tag{4.3}$$

form Jordan chains for $\tilde{A}(w)$ corresponding to $w = 0$ or equivalently they satisfy the conditions [8]

$$\begin{aligned} \tilde{A}(0)x_{j0} &= 0, \\ \tilde{A}^{(1)}(0)x_{j0} + \tilde{A}(0)x_{j1} &= 0, \\ &\vdots \\ \frac{1}{(\mu_j - 1)!}\tilde{A}^{(\mu_j - 1)}(0)x_{j0} + \frac{1}{(\mu_j - 2)!}\tilde{A}^{(\mu_j - 2)}(0)x_{j1} + \cdots + \tilde{A}(0)x_{j\mu_j - 1} &= 0. \end{aligned} \tag{4.4}$$

Proof. Since $\tilde{U}_R(w)$ has no poles or zeros at $w = 0$, $\tilde{u}_j(0) \neq 0$, from (4.1) and for $w = 0$

$$\tilde{A}(0)\tilde{u}_j(0) = 0, \qquad j = k+1, \ldots, r. \tag{4.5}$$

Taking the first derivative of (4.1) with respect to w we have

$$\tilde{A}^{(1)}(w)\tilde{u}_j(w) + \tilde{A}(w)\tilde{u}_j^{(1)}(w) = v_j^{(1)}(w)w^{\mu_j} + v_j(w)\mu_j w^{\mu_j - 1} \tag{4.6}$$

which for $w = 0$ gives

$$\tilde{A}^{(1)}(0)\tilde{u}_j(0) + \tilde{A}(0)\tilde{u}_j^{(1)}(0) = 0, \qquad j = k+1, \ldots, r. \tag{4.7}$$

In a similar manner and taking the derivative of (4.6), i.e., the second derivative of (4.1) and evaluating the resulting expression for $w=0$ we finally obtain for $j=k+1,\ldots,r$

$$\tilde{A}^{(2)}(0)\tilde{u}_j(0)+2\tilde{A}^{(1)}(0)\tilde{u}_j^{(1)}(0)+\tilde{A}(0)\tilde{u}_j^{(2)}(0)=0. \tag{4.8}$$

We continue with this procedure until the (μ_j-1)st derivative of (4.1) since this is the highest-order derivative of (4.1) for which the right-hand side is zero for $w=0$. Now from (4.5) and the definition of $x_{j0}:=\tilde{u}_j(0)$ in (4.2) we obtain the first condition in (4.4). Similarly, from (4.7) and definition (4.2) we obtain the second condition in (4.4). Finally, equation (4.8) gives

$$\tilde{A}^{(2)}(0)x_{j0}+2\tilde{A}^{(1)}(0)x_{j1}+2\tilde{A}(0)x_{j2}=0 \tag{4.9}$$

which can be written as

$$\frac{1}{2!}\tilde{A}^{(2)}(0)x_{j0}+\tilde{A}^{(1)}(0)x_{j1}+\tilde{A}(0)x_{j2}=0.$$

Continuing with this procedure with higher derivatives of (4.1) evaluated at $w=0$ we see that the conditions obtained using (4.2) are indeed conditions (4.4) of the proposition. □

Now from (3.1) we easily obtain

$$\tilde{A}^{(q)}(0)=q!A_{q_1-q}, \qquad q=0,1,2,\ldots,q_1, \tag{4.10}$$

and

$$\tilde{A}^{(q)}(0)=0, \qquad q=q_1+1,q_1+2,\ldots,q_1+\hat{q}_j-1=\mu_j-1, \tag{4.11}$$

so that conditions (4.4) of Proposition 4.1 reduce to the conditions

$$\begin{aligned}
\tilde{A}(0)x_{j0}&=0,\\
\tilde{A}^{(1)}(0)x_{j0}+\tilde{A}(0)x_{j1}&=0,\\
&\vdots\\
\frac{1}{q_1!}\tilde{A}^{(q_1)}(0)x_{j0}+\frac{1}{(q_1-1)!}\tilde{A}^{(q_1-1)}(0)x_{j1}+\cdots+\tilde{A}(0)x_{jq_1}&=0,\\
\frac{1}{q_1!}\tilde{A}^{(q_1)}(0)x_{j1}+\frac{1}{(q_1-1)!}\tilde{A}^{(q_1-1)}(0)x_{j2}+\cdots+\tilde{A}^{(1)}(0)x_{jq_1}+\tilde{A}(0)x_{jq_1+1}&=0,\\
&\vdots\\
\frac{1}{q_1!}\tilde{A}^{(q_1)}(0)x_{j\mu_j-(q_1+1)}+\frac{1}{(q_1-1)!}\tilde{A}^{(q_1-1)}(0)x_{j\mu_j-q_1}&\\
+\cdots+\tilde{A}^{(1)}(0)x_{j\mu_j-2}+\tilde{A}(0)x_{j\mu_j-1}&=0
\end{aligned} \tag{4.12}$$

for $j=k+1,\ldots,r$ or eqiuvalently to the conditions

$$\left.\begin{aligned} A_{q_1}x_{j0}&=0,\\ A_{q_1-1}x_{j0}+A_{q_1}x_{j1}&=0,\\ A_{q_1-2}x_{j0}+A_{q_1-1}x_{j1}+A_{q_1}x_{j2}&=0,\\ &\vdots\\ A_0x_{j0}+A_1x_{j1}+\cdots+A_{q_1}x_{jq_1}&=0,\\ A_0x_{j1}+A_1x_{j2}+\cdots+A_{q_1-1}x_{jq_1}+A_{q_1}x_{jq_1+1}&=0,\\ &\vdots\\ A_0x_{j\hat{q}_j-1}+A_1x_{j\hat{q}_j}+\cdots+A_{q_1-1}x_{j\mu_j-2}+A_{q_1}x_{j\mu_j-1}&=0, \end{aligned}\right\}\quad j=k+1,\ldots,r, \tag{4.13}$$

where we made use of (4.10) and (4.11) and the fact that $\mu_j:=q_1+\hat{q}_j$, $j=k+1,\ldots,r$, and which can be written in more compact matrix form as

$$\begin{array}{c} \\ r q_1 \updownarrow \\ \\ r\mu_j \updownarrow \\ \\ r\hat{q}_j \updownarrow \\ \\ \\ \end{array} \underbrace{\begin{bmatrix} A_{q_1} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\ A_{q_1-1} & A_{q_1} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & & \ddots & & & & & \vdots\\ A_1 & A_2 & A_3 & \cdots & A_{q_1} & 0 & 0 & \cdots & 0\\ A_0 & A_1 & A_2 & \cdots & A_{q_1-1} & A_{q_1} & 0 & \cdots & 0\\ 0 & A_0 & A_1 & \cdots & A_{q_1-2} & A_{q_1-1} & A_{q_1} & \cdots & 0\\ \vdots & & & & & & & & \vdots\\ 0 & 0 & A_0 & \cdots & A_1 & A_2 & A_{q_1-1} & \cdots & A_{q_1} \end{bmatrix}}_{r\mu_j} \begin{bmatrix} x_{j0}\\ x_{j1}\\ \vdots\\ x_{jq_1-1}\\ x_{jq_1}\\ x_{jq_1+1}\\ \vdots\\ x_{j\mu_j-1} \end{bmatrix} = \begin{bmatrix} 0\\ 0\\ 0\\ \vdots\\ 0\\ 0\\ \vdots\\ 0 \end{bmatrix} \tag{4.14}$$

for $j=k+1,\ldots,r$.

For $q=0,1,2,\ldots,\hat{q}_j-1$ and $j=k+1,\ldots,r$ define, now, the polynomial vectors

$$\beta_{jq}^{\infty}(s):=x_{j0}s^{q}+x_{j1}s^{q-1}+\cdots+x_{jq-1}s+x_{jq}\in\mathbb{R}[s]^{r\times 1}, \tag{4.15}$$

i.e., the polynomial vectors

$$\left.\begin{aligned} \beta_{j0}^{\infty}(s)&:=x_{j0},\\ \beta_{j1}^{\infty}(s)&:=x_{j0}s+x_{j1},\\ &\vdots\\ \beta_{j\hat{q}_j-1}^{\infty}(s)&:=x_{j0}s^{\hat{q}_j-1}+x_{j1}s^{\hat{q}_j-2}+\cdots+x_{j\hat{q}_j-2}s+x_{j\hat{q}_j-1}, \end{aligned}\right\}\quad j=k+1,\ldots,r, \tag{4.16}$$

and let

$$\psi_j(s):=[\beta_{j0}^{\infty}(s),\beta_{j1}^{\infty}(s),\ldots,\beta_{j\hat{q}_j-1}^{\infty}]\in\mathbb{R}[s]^{r\times\hat{q}_j},\qquad j=k+1,\ldots,r. \tag{4.17}$$

Then for $q=0,1,2,\ldots,\hat{q}_j-1$ and $j=k+1,\ldots,r$ we have

$$
\begin{aligned}
A(s)\beta_{jq}^{\infty}(s) &= s^{q}A(s)x_{j0}+s^{q-1}A(s)x_{j1}+\cdots+sA(s)x_{jq-1}+A(s)x_{jq}\\
&= [s^{q+q_1}I_r, s^{q+q_1-1}I_r,\ldots,sI_r,I_r]
\begin{array}{c} \\ rq_1 \\ \\ \\ \\ \\ r(q+1) \\ \\ \end{array}
\underbrace{\begin{bmatrix}
A_{q_1} & 0 & \cdots & & 0\\
A_{q_1-1} & A_{q_1} & \cdots & & 0\\
\vdots & & \ddots & & \\
A_1 & A_2 & \cdots & A_{q_1} & 0\\
A_0 & A_1 & & A_2 & A_{q_1}\\
0 & A_0 & & A_1 & A_{q_1-1}\\
\vdots & \vdots & \ddots & & \ddots\\
0 & 0 & 0 & A_0 & A_1
\end{bmatrix}}_{r(q+1)}
\begin{bmatrix} x_{j0}\\ x_{j1}\\ \vdots\\ x_{jq}\end{bmatrix}\\
&= [s^{q+q_1}I_r, s^{q+q_1-1}I_r,\ldots,sI_r,I_r]
\underbrace{\begin{bmatrix}
0 & 0 & \cdots & & 0\\
0 & 0 & \cdots & & 0\\
A_{q_1} & 0 & \cdots & & 0\\
A_{q_1-1} & A_{q_1} & \cdots & & 0\\
\vdots & & \ddots & & \\
A_2 & A_3 & \cdots & A_{q_1} & 0\\
A_1 & A_2 & \cdots & A_3 & A_{q_1}
\end{bmatrix}}_{rq_1}
\begin{bmatrix} -x_{jq+1}\\ -x_{jq+2}\\ \vdots\\ -x_{jq+q_1}\end{bmatrix}
\begin{array}{c} \\ rq_1 \\ \\ \end{array}\\
&= [s^{q_1-1}I_r, s^{q_1-2}I_r,\ldots,sI_r,I_r]
\begin{bmatrix}
A_{q_1} & 0 & \cdots & 0\\
A_{q_1-1} & A_{q_1} & \cdots & 0\\
\vdots & \vdots & & \\
A_1 & A_2 & \cdots & A_{q_1}
\end{bmatrix}
\begin{bmatrix} -x_{jq+1}\\ -x_{jq+2}\\ \vdots\\ -x_{jq+q_1}\end{bmatrix},
\end{aligned}
\tag{4.18}
$$

where we made use of the fact that if we consider $q+1$ at a time (block) columns of the matrix in (4.14) we have that

$$
\begin{array}{c} \\ rq_1 \\ \\ \\ \\ \\ \\ r(q+1) \\ \\ \end{array}
\underbrace{\begin{bmatrix}
A_{q_1} & 0 & \cdots & & & 0\\
A_{q_1-1} & A_{q_1} & \cdots & & & 0\\
\vdots & & \ddots & & & \\
A_1 & A_2 & \cdots & A_{q_1} & \cdots & 0\\
A_0 & A_1 & & A_2 & \cdots & A_{q_1}\\
0 & A_0 & & A_1 & \cdots & A_{q_1-1}\\
0 & 0 & & A_0 & \cdots & A_{q_1-2}\\
\vdots & \vdots & & & & \vdots\\
0 & 0 & & 0 & \cdots & A_0
\end{bmatrix}}_{r(q+1)}
\begin{bmatrix} x_{j0}\\ x_{j1}\\ \vdots\\ x_{jq}\end{bmatrix}
$$

$$
= \begin{array}{c} r(q+1) \\ \\ rq_1 \end{array}
\begin{bmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
A_{q_1} & 0 & \cdots & 0 \\
A_{q_1-1} & A_{q_1} & \cdots & 0 \\
\vdots & & \ddots & \vdots \\
A_1 & A_2 & \cdots & A_{q_1}
\end{bmatrix}
\begin{bmatrix}
-x_{jq+1} \\ -x_{jq+2} \\ \vdots \\ -x_{jq+q_1}
\end{bmatrix}
\tag{4.19}
$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{rq_1}$$

for $q=0,1,2,\dots,\hat{q}_j-1$ and $j=k+1,\dots,r$.

Now comparing (4.18) with (1.13) and taking into account that $v=q_1$ we see that if the initial conditions $\beta(0-)$, $\beta^{(1)}(0-),\dots,\beta^{(q_1-1)}(0-)$ are chosen so that

$$
\begin{gathered}
\beta(0-)=-x_{jq+1}, \\
\beta^{(1)}(0-)=-x_{jq+2},\dots,\beta^{(q_1-1)}(0-)=-x_{jq+q_1}\begin{Bmatrix} q=0,1,\dots,\hat{q}_j-1 \\ j=k+1,\dots,r \end{Bmatrix},
\end{gathered}
\tag{4.20}
$$

then the polynomial vector $\beta_{jq}^{\infty}(s)$ in (4.15) will satisfy equation (1.11) and thus, for $q=0,1,2,\dots,\hat{q}_j-1$ and $j=k+1,k+2,\dots,r$,

$$\beta_{jq}^{\infty}(t)=x_{j0}\delta^{(q)}(t)+x_{j1}\delta^{(q-1)}(t)+\cdots+x_{jq-1}\delta^{(1)}+x_{jq}\delta(t) \tag{4.21}$$

will be impulsive solutions of the differential equation (1.8).

Writing equations (4.18) in matrix form for $q=0,1,2,\dots,\hat{q}_j-1$ we have

$$
A(s)\psi_j(s)=G(s)M(-1)\underbrace{\begin{bmatrix}
x_{j1} & x_{j2} & \cdots & x_{j\hat{q}_j} \\
x_{j2} & x_{j3} & \cdots & x_{j\hat{q}_j+1} \\
\vdots & \vdots & & \vdots \\
x_{jq_1} & x_{jq_1+1} & \cdots & x_{j\hat{q}_j+q_1-1}
\end{bmatrix}}_{\hat{q}_j} q_1,
$$

$$j=k+1,\dots,r, \tag{4.22}$$

where $G(s):=[s^{q_1-1}I_r,\dots,sI_r,I_r]$ and $M\in\mathbb{R}^{q_1r\times q_1r}$ is the Toeplitz matrix in (4.18). The above clearly shows that for each appropriate choice of the initial conditions $\beta(0-)$, $\beta^{(1)}(0-),\dots,\beta^{(q_1-1)}(0-)$ as in (4.20) and for

$q=0,1,\ldots,\hat{q}_j-1$ the impulsive solutions

$$\left.\begin{aligned}\beta_{j0}^{\infty}(t)&=x_{j0}\delta(t),\\ \beta_{j1}^{\infty}(t)&=x_{j0}\delta^{(1)}(t)+x_{j1}\delta(t),\\ &\;\vdots\\ \beta_{j\hat{q}_j-1}^{\infty}(t)&=x_{j0}\delta^{(\hat{q}_j-1)}+x_{j1}\delta^{(\hat{q}_j-2)}(t)\\ &\quad+\cdots+x_{j\hat{q}_j-2}\delta^{(1)}+x_{j\hat{q}_j-1}\delta(t)\end{aligned}\right\}\quad j=k+1,\ldots,r, \tag{4.23}$$

are $\hat{q}_j$ $\mathbb{R}$-linearly independent impulsive solutions of the differential equation (1.8). Finally, and since the vectors $x_{j0}:=\tilde{u}_j(0)$, $j=k+1,\ldots,r$, as columns of the nonsingular matrix $\tilde{U}_R(w)$ evaluated at $w=0$, are $\mathbb{R}$-linearly independent, we have that the impulsive solutions in (4.23) for $j=k+1,\ldots,r$ form a set of $\hat{q}_{k+1}+\hat{q}_{k+2}+\cdots+\hat{q}_r$ $\mathbb{R}$-linearly independent impulsive solutions of equation (1.8).

If by X^{∞} we denote the set of all impulsive solutions, then from the linearity of the map $A(s)$ it simply follows that if $\beta_i^{\infty}(t)\in X^{\infty}$, $i\in\boldsymbol{\sigma}$, σ a positive integer, then $\sum_{i=1}^{\sigma}\alpha_i\beta_i^{\infty}(t)\in X^{\infty}$ and thus X^{∞} is an $\mathbb{R}$-linear vector space.

The above results constitute a proof of the next theorem which has been known for a number of years (e.g., see [3] and [12]) but a formal proof of which could not be traced by the authors.

Theorem 4.1. *Consider the linear, homogeneous matrix differential equation* (1.8). *Let* $A(s)\in\mathbb{R}[s]^{r\times r}$, $\operatorname{rank}_{\mathbb{R}(s)}A(s)=r$ *as in* (3.1) *and let* $S_{A(s)}^{\infty}(s)$ *be the Smith–McMillan form at* $s=\infty$ *of* $A(s)$ *as in* (3.4) *where*

$$q_1\geq q_2\geq\cdots\geq q_k\geq 1 \tag{4.24}$$

are the orders of its poles at $s=\infty$ *and*

$$\hat{q}_r\geq\hat{q}_{r-1}\geq\cdots\geq\hat{q}_{k+1}\geq 0 \tag{4.25}$$

are orders of its zeros at $s=\infty$. *Then*

$$\dim_{\mathbb{R}}X^{\infty}=\hat{q}_{k+1}+\hat{q}_{k+2}+\cdots+\hat{q}_r, \tag{4.26}$$

i.e., the dimension of the impulsive solution space X^{∞} *is equal to the total number of the zeros of* $A(s)$ *at* $s=\infty$ (*where orders of each zero at* $s=\infty$ *of* $A(s)$ *are accounted for*).

Example 4.1. Consider the set of differential equations

$$\left.\begin{aligned}\xi_1(t)+\frac{d^3\xi_2(t)}{dt^3}&=0,\\ \xi_2(t)+\frac{d\xi_3(t)}{dt}&=0,\\ \xi_3(t)&=0,\end{aligned}\right\}\quad t\geq 0-,$$

which can be written in matrix form as

$$\begin{bmatrix} 1 & \rho^3 & 0 \\ 0 & 1 & \rho \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \xi_1(t) \\ \xi_2(t) \\ \xi_3(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad t \geq 0-,$$

or as

$$A(\rho)\beta(t) = 0,$$

where $\beta(t) = [\xi_1(t), \xi_2(t), \xi_3(t)]^{\mathrm{T}}$ and

$$A(\rho) = A_3\rho^3 + A_2\rho^2 + A_1\rho + A_0 \in \mathbb{R}[\rho]^{3\times 3}$$

with

$$A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now with

$$U_L(s) = \begin{bmatrix} 1 & 0 & 0 \\ -1/s^3 & 1 & 0 \\ 1/s^4 & -1/s & 1 \end{bmatrix}, \qquad U_R(s) = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -1/s^3 \\ 0 & 1 & 1/s^4 \end{bmatrix} \in \mathbb{R}_{\mathrm{pr}}(s)^{3\times 3}$$

we have

$$U_L(s)A(s)U_R(s) = S^{\infty}_{A(s)}(s) = \begin{bmatrix} s^3 & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1/s^4 \end{bmatrix}$$

so that $A(s)$ has two poles at $s = \infty$ with orders $q_1 = 3$ and $q_2 = 1$, respectively, and one zero at $s = \infty$ with order $\hat{q}_3 = 4$. Notice that $\nu = 3 = q_1$ (Corollary 2.1). The "dual" polynomial matrix $\tilde{A}(w)$ is

$$\tilde{A}(w) = w^3 A\left(\frac{1}{w}\right) = A_3 + A_2 w + A_1 w^2 + A_0 w^3 = \begin{bmatrix} w^3 & 1 & 0 \\ 0 & w^3 & w^2 \\ 0 & 0 & w^3 \end{bmatrix} \in \mathbb{R}[w]^{3\times 3}$$

and from Proposition 3.1 the "local" Smith form $S^0_{\tilde{A}(w)}(w)$ of $\tilde{A}(w)$ at $w = 0$ is given by

$$S^0_{\tilde{A}(w)}(w) = w^{q_1} S^{\infty}_{A(s)}\left(\frac{1}{w}\right) = w^3 \begin{bmatrix} 1/w^3 & 0 & 0 \\ 0 & 1/w & 0 \\ 0 & 0 & w^4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & w^2 & 0 \\ 0 & 0 & w^7 \end{bmatrix}$$

so that $A(s)$ has two IEDs, one of degree $\mu_2 = q_1 - q_2 = 3 - 1 = 2$ which corresponds to the two poles at $s = \infty$ of $A(s)$ and one of degree $\mu_3 = q_1 + \hat{q}_3 = 3 + 4 = 7$ which corresponds to a pole and a *zero* at $s = \infty$. It is this IED which gives rise to a Jordan chain $x_{30} \neq 0$, x_{31}, x_{32}, x_{33}, x_{34}, x_{35}, x_{36}, of length $\mu_3 = 7$ that corresponds to the zero at $w = 0$ of $\hat{A}(w)$ which in turn gives rise to $\hat{q}_3 = 4$ linearly independent impulsive solutions $\beta_{3q}^{\infty}(t)$, $q = 0, 1, 2, 3 = \hat{q}_3 - 1$, of the differential equation if the initial conditions $\beta(0-)$, $\beta^{(1)}(0-)$, $\beta^{(2)}(0-)$ are chosen appropriately and as described by (4.20).

From Proposition 3.1

$$\tilde{U}_L(w) = U_L\left(\frac{1}{w}\right) = \begin{bmatrix} 1 & 0 & 0 \\ -w^3 & 1 & 0 \\ w^4 & -w & 1 \end{bmatrix},$$

$$\tilde{U}_R(w) = U_R\left(\frac{1}{w}\right) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -w^3 \\ 0 & 1 & w^4 \end{bmatrix}$$

so that

$$\tilde{u}_3(w) = \begin{bmatrix} 1 \\ -w^3 \\ w^4 \end{bmatrix}, \qquad \tilde{u}_3(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} =: x_{30},$$

$$\tilde{u}_3^{(1)}(w) = \begin{bmatrix} 0 \\ -3w^2 \\ 4w^3 \end{bmatrix}, \qquad \tilde{u}_3^{(1)}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} =: x_{31},$$

$$\tilde{u}_3^{(2)}(w) = \begin{bmatrix} 0 \\ -6w \\ 12w^2 \end{bmatrix}, \qquad \tilde{u}_3^{(2)}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} =: x_{32},$$

$$\tilde{u}_3^{(3)}(w) = \begin{bmatrix} 0 \\ -6 \\ 24w \end{bmatrix}, \qquad \tilde{u}_3^{(3)}(0) = \begin{bmatrix} 0 \\ -6 \\ 0 \end{bmatrix},$$

$$x_{33} := \frac{1}{3!}\tilde{u}_3^{(3)}(0) = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \qquad \tilde{u}_3^{(4)}(w) = \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix}, \qquad \tilde{u}_3^{(4)}(0) = \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix},$$

$$x_{34} := \frac{1}{4!}\tilde{u}_3^{(4)}(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \tilde{u}_3^{(5)}(w) = \tilde{u}_3^{(6)}(w) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = x_{35} = x_{36},$$

so that

$$\beta_{30}^{\infty}(t)=\begin{bmatrix}\delta(t)\\0\\0\end{bmatrix}$$

satisfies (4.22) if $\beta(0-)=-x_{31}$, $\beta^{(1)}(0-)=-x_{32}$, and $\beta^{(2)}(0-)=-x_{33}$, i.e., if $\xi_2^{(2)}(0-)=1$

$$\beta_{31}^{\infty}(t)=\begin{bmatrix}\delta^{(1)}(t)\\0\\0\end{bmatrix},$$

if $\beta(0-)=-x_{32}$, $\beta^{(1)}(0-)=-x_{33}$, and $\beta^{(2)}(0-)=-x_{34}$, i.e., if $\xi_2^{(1)}(0-)=1$ and $\xi_3^{(2)}(0-)=-1$

$$\beta_{32}^{\infty}(t)=\begin{bmatrix}\delta^{(2)}(t)\\0\\0\end{bmatrix},$$

if $\beta(0-)=-x_{33}$, $\beta^{(1)}(0-)=-x_{34}$, and $\beta^{(2)}(0-)=-x_{35}$, i.e., if $\xi_2(0-)=1$ and $\xi_3^{(1)}(0-)=-1$

$$\beta_{33}^{\infty}(t)=\begin{bmatrix}\delta^{(3)}(t)\\\delta(t)\\0\end{bmatrix},$$

if $\beta(0-)=-x_{34}$, $\beta^{(1)}(0-)=-x_{35}$, and $\beta^{(2)}(0-)=-x_{36}$, i.e., if $\xi_3(0-)=-1$.

In general the relation between the coefficient vectors $x_{j0}, x_{j1}, \ldots, x_{jq-1}, x_{jq}$ of the impulsive solutions $\beta_{jq}^{\infty}(t)$ in (4.21) and the initial conditions $\beta(0-), \beta^{(1)}(0-), \ldots, \beta^{(q_1-1)}(0-)$ can be obtained directly from equation (4.19) as follows. Denote by

$$Q(s) := [s^{q_1}I_r, \ldots, sI_r, I_r] \in \mathbb{R}^{r\times(q_1+1)r}, \qquad P := \begin{bmatrix}A_{q_1}\\\vdots\\A_1\\A_0\end{bmatrix} \in \mathbb{R}[s]^{r(q_1+1)\times r} \tag{4.27}$$

and write

$$A(s)=Q(s)P. \tag{4.28}$$

Then, since $\operatorname{rank}_{\mathbb{C}} Q(s)=r$, $\forall s\in\mathbb{C}$, from (4.28) we have

$$\operatorname{rank}_{\mathbb{R}(s)} A(s)=r \;\Rightarrow\; \operatorname{rank}_{\mathbb{R}} P=r \tag{4.29}$$

which in turn implies that if $T_q\in\mathbb{R}^{r(q_1+q+1)\times r(q+1)}$ and $L_q\in\mathbb{R}^{r(q_1+q+1)\times rq_1}$ are respectively the block Toeplitz coefficient matrices in (4.19), then

$$\operatorname{rank}_{\mathbb{R}} T_q=r(q+1) \tag{4.30}$$

and from equations (4.19) and (4.20) and using the above notation we have

$$\begin{bmatrix} x_{j0} \\ x_{j1} \\ \vdots \\ x_{jq} \end{bmatrix} = T_q^{\dagger} L_q \begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q_1-1)}(0-) \end{bmatrix}, \qquad q=0,1,2,\ldots,\hat{q}_j-1, \quad j=k+1,\ldots,r,$$

where $T_q^{\dagger} \in \mathbb{R}^{r(q+1)\times r(q_1+q+1)}$ is a left inverse of T_q which always exists due to (4.30).

5. Conclusions

In this paper we have examined the impulsive solution space of generalised singular (free) systems. By utilizing the concept of infinite elementary divisors of general polynomial matrices [11] we have established a link between impulsive solutions of linear homogeneous systems of differential equations and the theory of Jordan chains corresponding to infinite elementary divisors of the associated polynomial matrix.

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