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REACHABILITY OF POLYNOMIAL MATRIX DESCRIPTIONS (PMDS)*

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Abstract. We consider the concept reachability for Polynomial Matrix Descriptions (PMDs); i.e., systems of the form $\Sigma : A(\rho)\beta(t) = B(\rho)u(t),\ y(t) = C(\rho)\beta(t)$, where $\rho := d/dt$ the differential operator, $A(\rho) = A_0 + A_1\rho + \cdots + A_v\rho^v \in \mathbb{R}^{r\times r}[\rho]$, $A_i \in \mathbb{R}^{r\times r}$, $i = 0, 1, \ldots, v \geq 1$ with $\operatorname{rank}_{\mathbb{R}} A_v \leq r$, $B(\rho) = B_0 + B_1\rho + \cdots + B_\sigma\rho^\sigma \in \mathbb{R}^{r\times m}[\rho]$, $B_i \in \mathbb{R}^{r\times m}, i = 0, 1, \ldots, \sigma \geq 0, C(\rho) = C_0 + C_1\rho + \cdots + C_{\sigma_1}\rho^{\sigma_1} \in \mathbb{R}^{m_1\times r}[\rho], C_i \in \mathbb{R}^{m_1\times r}$, $i = 0, 1, \ldots, \sigma_1 \geq 0, \beta(t) : (0^-, \infty) \to \mathbb{R}^r$ is the pseudostate of (Σ), $u(t) : [0, \infty) \to \mathbb{R}^m$ is the control input to (Σ), and $y(t)$ is the output of the system (Σ). Starting from the fact that generalized state space systems, i.e., systems of the form $\Sigma_1 : Ex(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t)$, where $E \in \mathbb{R}^{r\times r}$, $\operatorname{rank}_{\mathbb{R}} E < r$, $A \in \mathbb{R}^{r\times r}$, $B \in \mathbb{R}^{r\times m}$, $C \in \mathbb{R}^{m_1\times r}$ represent a particular case of PMDs, we generalize various known results regarding the smooth and impulsive solutions of the homogeneous and the nonhomogeneous system (Σ_1) to the more general case of PMDs (Σ). Relying on the above generalizations we develop a theory regarding the reachability of PMDs using time-domain analysis, which takes into account finite and infinite zeros of the matrix $A(s) = L_-[A(\rho)]$. The present analysis extends in a general way many results previously known only for regular and generalized state space systems.

1. Introduction

Recently there has been a growing interest in the study of generalized state space or singular systems because of the many applications of this kind of system in electric circuits theory, economics, large-scale systems, etc. Generalized state space systems are systems described by:

$$E\dot{x}(t) = Ax(t) + Bu(t) \tag{1.1}$$

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$$y(t) = Cx(t) ,$$

where $E \in \mathbb{R}^{r\times r}$, $\text{rank}_{\mathbb{R}} E < r$, $A \in \mathbb{R}^{r\times r}$, $B \in \mathbb{R}^{r\times m}$, $C \in \mathbb{R}^{m_1\times r}$, whereas regular state space systems are the systems described by:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t) ,\end{aligned} \tag{1.2}$$

which represent a particular case of (1.1) with $E = I_r$, the identity matrix.

Regular state space systems were studied explicitly in the past decades; see for instance [23]. On the other hand, many researchers in recent years explored special properties of generalized state space systems and found many connection with regular systems. Originally in [24] generalized state space systems were studied in the frequency domain using an approach based on matrix pencil theory, and the system behavior was investigated in the light of its pole-zero structure at both finite and infinite frequencies. Controllability, observability, system equivalence, and other related topics were studied using algebraic analysis [29]–[35] or geometric analysis [21], [22]. Generalized state space systems were studied in the time domain (see [3]–[11], [18]). For a complete reference on singular systems see [19], [20].

Generalized state space systems such as (1.1) represent a particular case of Polynomial Matrix Descriptions (PMDs); i.e., physical systems whose dynamics can be described by a linear matrix differential equation having the form:

$$\begin{aligned}A(\rho)\beta(t) &= B(\rho)u(t)\\ y(t) &= C(\rho)\beta(t) ,\end{aligned} \tag{1.3}$$

where $\rho := d/dt$ the differential operator, $A(\rho) = A_0 + A_1\rho + \cdots + A_v\rho^v \in \mathbb{R}^{r\times r}[\rho]$, $A_i \in \mathbb{R}^{r\times r}$, $i = 0, 1, \ldots, v \geq 1$ with $\text{rank}_{\mathbb{R}} A_v \leq r$, $B(\rho) = B_0 + B_1\rho + \cdots + B_\sigma\rho^\sigma \in \mathbb{R}^{r\times m}[\rho]$, $B_i \in \mathbb{R}^{r\times m}$, $i = 0, 1, \ldots, \sigma \geq 0$, $C(\rho) = C_0 + C_1\rho + \cdots + C_{\sigma_1}\rho^{\sigma_1} \in \mathbb{R}^{m_1\times r}[\rho]$, $C_i \in \mathbb{R}^{m_1\times r}$, $i = 0, 1, \ldots, \sigma_1 \geq 0$, $\beta(t) : (0^-, \infty) \to \mathbb{R}^r$ is the *pseudo-state* of the system, $u(t) : [0, \infty] \to \mathbb{R}^m$ is the control input to the system, and $y(t)$ is the output of the system.

PMDs are governed by singular differential equations that endow the systems with many special features that are not found in regular state space systems. Among these are impulse terms and input derivatives in the free and forced pseudo-state responses, nonproperness of the transfer function matrix, noncausality between input and pseudo-state (or input and output), inconsistent and admissible initial conditions, and many others that make the study of PMDs more complicated than the study of the classical regular systems.

In recent papers [26], [27], [13] various known results regarding the smooth and impulsive solutions of homogeneous generalized state space systems have been translated to the more general case of PMDs. Also, relying heavily on the theory regarding the Smith–McMillan form of a rational matrix at infinity and applying it to the polynomial matrix $A(s) = L_-[A(\rho)]$, the theory of the Weierstrass canonical form of a regular matrix pencil Es–A under strict equivalence to the more general case of polynomial matrix $A(s)$ has been generalized [26].

In this paper, following the definitions of reachability of Yip and Sincovec [35] and Ozcaldiran [21], we consider the concept of reachability for PMDs and we develop a new theory using time-domain analysis, which takes into account the finite and infinite zero structure of the $A(s)$. Our analysis extends in a natural way many results previously known only for generalized state space systems. The paper is organized as follows. In Section 2, we give some preliminary results that are useful in the sequel. Then in Section 3 we find the admissible initial conditions for the homogeneous and the nonhomogeneous cases. In Section 4 the reachable subspace as well as some reachability tests or PMDs are examined. Finally in Section 5 an illustrative example is given.

2. Background and preliminary results

We give a number of results that are necessary in our development in the sequel.

Theorem 2.1 (Smith–McMillan form of a rational matrix at $s = \infty$; [28]). *Let $A(s) \in \mathbb{R}^{p\times m}(s)$, $\text{rank}_{\mathbb{R}}(s)A(s) = r$. Then $A(s)$ is equivalent at $s = \infty$ to a diagonal matrix having the form:*

$$S^{\infty}_{A(s)}(s) = block\ diag\left[s^{q_1}, s^{q_2}, \ldots, s^{q_k}, \frac{1}{s^{\hat{q}_{k+1}}}, \ldots, \frac{1}{s^{\hat{q}_r}}, 0_{p-r,m-r}\right], \tag{2.1}$$

where $1 \leq k \leq r$ and $\hat{q}_i = -q_i$, $i = k+1, \ldots, r$, so that $q_1 \geq q_2 \geq \cdots \geq q_k \geq 0$, $\hat{q}_r \geq \hat{q}_{r-1} \geq \cdots \geq \hat{q}_{k+1} \geq 0$.

Now we examine some relations between the Smith–McMillan form at $s = \infty$ of a general polynomial matrix $A(s) \in \mathbb{R}^{p\times m}[s]$ and its Laurent expansion at $s = \infty$. The next proposition generalizes to the matrix case some well-known properties of scalar rational functions.

Proposition 2.2 [26]. *Let $A(s) \in R^{p\times m}[s]$ and write it as a matrix polynomial; i.e., $A(s) = A_v s^v + A_{v-1}s^{v-1} + \cdots + A_1 s + A_0$ with $A_v \neq 0^n$ and let $S^{\infty}_{A(s)}$ be given by (2.1). Then $v = q_1$. Moreover, if $p = m = r$ and $|A(s)| \neq 0$ then if $A(s)$ has at least one zero at $s = \infty$, i.e., if $\hat{q}_r > 0$ and*

$$A(s)^{-1} = H_k s^k + H_{k-1}s^{k-1} + \cdots + H_1 s + H_0 + H_{-1}s^{-1} + H_{-2}s^{-2} + \cdots \tag{2.2}$$

is the Laurent expansion of $A(s)^{-1}$ at $s = \infty$ with $H_k \neq 0^n$, then $k = \hat{q}_r$.

Now we shall give some useful results concerning the spectral analysis of nonsingular polynomial matrices.

Definition 2.3 [16]. Let $A(s)$ be a polynomial matrix

$$A(s) = A_0 + A_1 s + \cdots + A_{q_1} s^{q_1} \in \mathbb{R}^{r\times r}[s] \tag{2.3}$$

$\operatorname{rank}_{\mathbb{R}[s]} A(s) = r$ and $C \in \mathbb{R}^{r\times n}$, $J \in \mathbb{R}^{n\times n}$ with $n := \deg|A(s)|$. Then the pair (C, J) is called *a finite Jordan pair of* $A(s)$ iff the matrices C, J satisfy the conditions:

$$A_{q_1} C J^{q_1} + A_{q_1^{-1}} C J^{q_1 - 1} + \cdots + A_0 C = 0 \tag{2.4}$$

$$Q_n := \begin{bmatrix} C \\ CJ \\ \vdots \\ CJ^{n-1} \end{bmatrix} \in \mathbb{R}^{rn\times n}, \quad \operatorname{rank} Q_n = n . \tag{2.5}$$

Crucial to what follows is the concept of an infinite Jordan pair of a nonsingular polynomial matrix, i.e., a Jordan pair C_∞, J_∞ that corresponds to its zeros at $s = \infty$. We start with:

Definition 2.4 [16.] Let $A(s)$ be as in (2.3), with $\operatorname{rank}_{\mathbb{R}[s]} A(s) = r$. Then the pair

$$C_\infty \in \mathbb{R}^{r\times v}, \quad J_\infty = \text{block diag}[J_{\infty 1}, J_{\infty 2}, \ldots, J_{\infty \zeta}] \in \mathbb{R}^{v\times v} \tag{2.6}$$

$$J_{\infty i} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{v_i\times v_i}, \quad i = 1, 2, \ldots, \zeta \tag{2.7}$$

$v = \sum_{i=1}^{\zeta} v_i$, $v_i, \zeta \in \mathbb{N}$, is called *an infinite Jordan pair of* $A(s)$ if it is a (finite) Jordan pair of the "dual" polynomial matrix

$$\tilde{A}(w) := w^{q_1} A\left(\frac{1}{w}\right) = A_0 w^{q_1} + A_1 w^{q_1 - 1} + \cdots + A_{q_1} \in \mathbb{R}[w]^{r\times r} \tag{2.8}$$

corresponding to its zero at $w = 0$, or equivalently iff (see Definition 2.3):

$$A_0 C_\infty J_\infty^{q_1} + A_1 C_\infty J_\infty^{q_1 - 1} + \cdots + A_{q_1} C_\infty = 0_{r,v} \tag{2.9}$$

$$Q_v^\infty := \begin{bmatrix} C_\infty \\ C_\infty J_\infty \\ \vdots \\ C_\infty J_\infty^{v-1} \end{bmatrix}, \quad \operatorname{rank}_{\mathbb{R}} Q_v^\infty = v . \tag{2.10}$$

Theorem 2.5 [26]. *Let* $A(s) \in \mathbb{R}^{r\times r}[s]$ *be as in (2.3) with Smith–McMillan form at* $s = \infty$, *given by (2.1). Write*

$$A(s)^{-1} = H_{\text{pol}}(s) + H_{\text{spr}}(s) \tag{2.11}$$

where $H_{\text{pol}}(s)\mathbb{R}[s]^{r\times r}$ *and* $H_{\text{spr}}(s)\mathbb{R}_{pr}^{r\times r}(s)$ *are strictly proper. Let* $n := \deg|A(s)|$*; then* $n = \delta_M(H_{\text{spr}}(s))$. *Let* $\mu = \sum_{i=k+1}^{r} (\hat{q}_i + 1)$*; then* $\delta_M(H_{\text{pol}}(s)) = \mu$. *Let* $C \in$

$\mathbb{R}^{r\times n}$, $J \in \mathbb{R}^{n\times n}$ $B \in \mathbb{R}^{n\times r}$ *be a minimal realization of* $H_{\rm spr}(s)$ *and* $C_\infty \in \mathbb{R}^{r\times\mu}$, $J_\infty \in \mathbb{R}^{\mu\times\mu}$, $B_\infty \in \mathbb{R}^{\mu\times r}$ *be a minimal realization of* $H_{\rm pol}(s)$. *Then* C, J *is a* finite Jordan pair of $A(s)$ *and* C_∞, J_∞ *is an infinite Jordan pair of* $A(s)$. *Furthermore,* $A(s)^{-1}$ *can be written:*

$$A(s)^{-1} = [C, C_\infty]\begin{bmatrix} sI_n - J & | & 0_{n,\mu} \\ --- & | & --- \\ 0_{\mu,n} & | & I_\mu - sJ_\infty \end{bmatrix}^{-1}\begin{bmatrix} B \\ \\ B_\infty \end{bmatrix}. \tag{2.12}$$

Consider now the following PMD:

$$\begin{aligned} A(\rho)\beta(t) &= B(\rho)u(t) \\ y(t) &= C(\rho)\beta(t)\,, \end{aligned} \tag{2.13}$$

where $A(\rho) = \sum_{i=0}^{q_1} A_i\rho^i \in \mathbb{R}^{r\times r}[\rho]$, $\operatorname{rank}_{\mathbb{R}(\rho)} A(\rho) = r$, $A_i \in \mathbb{R}^{r\times r}$, $i = 0, 1, \ldots, q_1$, $B(\rho) = \sum_{i=0}^{\sigma} B_i\rho^i \in \mathbb{R}^{r\times m}[\rho]$, $B_i \in \mathbb{R}^{r\times m}$, $i = 0, 1, \ldots, \sigma$, $C(\rho) = \sum_{i=0}^{\sigma_1} C_i\rho^i \in \mathbb{R}^{m_1\times r}[\rho]$, $C_i \in \mathbb{R}^{m_1\times r}$, $i = 0, 1, \ldots, \sigma_1$, $\beta(t) : (0^-, \infty) \to \mathbb{R}^r$ is the pseudo-state, $u(t) : [0, \infty] \to \mathbb{R}^r$ is the input, a piecewise sufficiently differentiable function, and $y(t)$ is the output. Consider the homogeneous matrix differential equation

$$A(\rho)\beta(t) = 0\,, \quad t \geq 0 \tag{2.14}$$

obtained from the first equation in (2.13) by setting $u(t) = 0$, $t \geq 0$. A closed formula for the solution of (2.14) in terms of the finite and infinite Jordan pairs of $A(s)$ was given in [26], [13]. Consider the Laplace transform of equation (2.14) and write

$$\hat{\beta}(s) = A(s)^{-1}\hat{a}(s) \in \mathbb{R}^{r\times 1}\,, \tag{2.15}$$

where $\hat{a}(s) \in \mathbb{R}^{r\times 1}[s]$ is the *initial condition vector* (see [2]) associated with the initial values of $\beta(t)$ and its $(q_1 - 1)$-derivatives at $t = 0^-$; i.e., $\beta(0^-)$, $\beta^{(1)}(0^-)$, $\ldots$, $\beta^{(q_1-1)}(0^-)$ and is given by:

$$\hat{a}(s) = \left[s^{q_1-1}I_r, s^{q_1-2}I_r, \ldots, sI_r, I_r\right] \times \begin{bmatrix} A_{q_1} & 0 & \cdots & 0 \\ A_{q_1-1} & A_{q_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ A_1 & A_2 & \cdots & A_{q_1} \end{bmatrix}\begin{bmatrix} \beta(0^-) \\ \beta^{(1)}(0^-) \\ \vdots \\ \beta^{(q_1-1)}(0^-) \end{bmatrix}. \tag{2.16}$$

Taking into account the form of $A(s)^{-1}$ from (2.12) and the form of $\hat{a}(s)$ from (2.16) after some matrix manipulations [26] we finally arrive at the following special form for $\hat{\beta}(s)$:

$$\hat{\beta}(s) = \hat{\beta}_{\mathrm{spr}}(s) + \hat{\beta}_{\mathrm{pol}}(s) = C\,[sI_n - J]^{-1}\, x_s(0^-) + C_\infty (sJ_\infty - I_\mu)^{-1} J_\infty x_f(0^-)\,, \tag{2.17}$$

where

$$x_f(0^-) := \left[B_\infty, J_\infty B_\infty, \ldots, J_\infty^{q_1-1} B_\infty\right] \times \begin{bmatrix} A_0 & A_1 & \cdots & A_{q_1-1} \\ 0 & A_0 & \cdots & A_{q_1-2} \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & A_0 \end{bmatrix} \begin{bmatrix} \beta(0^-) \\ \beta^{(1)}(0^-) \\ \vdots \\ \beta^{(q_1^{-1})}(0^-) \end{bmatrix} \in \mathbb{R}^{\mu} \tag{2.18}$$

and

$$x_s(0^-) := \left[J^{q_1-1}B, J^{q_1-2}B, \ldots, B\right] \times \begin{bmatrix} A_{q_1} & 0 & \cdots & 0 \\ A_{q_1^{-1}} & A_{q_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ A_1 & A_2 & \cdots & A_{q_1} \end{bmatrix} \begin{bmatrix} \beta(0^-) \\ B^{(1)}(0^-) \\ \vdots \\ \beta^{(q_1-1)}(0^-) \end{bmatrix} \in \mathbb{R}^{n}\,. \tag{2.19}$$

Definition 2.6 [26]. The vector

$$\begin{bmatrix} x_s(0^-) \\ x_f(0^-) \end{bmatrix} = \left[\begin{array}{c|c} J^{q_1-1}B, \ldots, B & 0_{n,q_1\mu} \\ \hline 0_{\mu,q_1 n} & B_\infty, \ldots, j_\infty^{q_1} B_\infty \end{array}\right] \times$$

$$\begin{bmatrix} A_{q_1} & 0 & \cdots & 0 \\ A_{q_1-1} & A_{q_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ A_1 & A_2 & \cdots & A_{q_1} \\ A_0 & A_1 & \cdots & A_{q_1-1} \\ 0 & A_0 & & A_{q_1-2} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A_0 \end{bmatrix} \begin{bmatrix} \beta(0^-) \\ \beta^{(1)}(0^-) \\ \vdots \\ \beta^{(q_1-1)}(0^-) \end{bmatrix} \tag{2.20}$$

$\in \mathbb{R}^{(m+\mu)}$ is defined as the *state at* $t = 0^-$ of the homogeneous matrix differential equation (2.14) ($x_s(0^-)$ is the *slow state at* $t = 0^-$ and $x_f(0^-)$ is the *fast state at* $t = 0^-$).

Taking the inverse Laplace transform of (2.17) we obtain

$$\beta^h(t) = L_-^{-1}\left[\hat{\beta}(s)\right] = [C\ C_\infty] \begin{bmatrix} e^{Jt} x_s(0^-) \\ -\sum_{i=1}^{\hat{q}_r} \delta^{(i-1)} J_\infty^i x_f(0^-) \end{bmatrix}. \tag{2.21}$$

Consider again the PMD (2.13). Now we shall present the solution of a nonhomogeneous matrix differential equation:

$$A(\rho)\beta(t) = B(\rho)u(t)\ . \tag{2.22}$$

Taking the L_- Laplace transform of (2.22) and assuming that the initial conditions are zero, i.e., $\beta^{(i)}(0^-) \equiv 0, i = 0, 1, \ldots, q_1 - 1, u^{(i)}(0^-) \equiv 0, i = 0, 1, \ldots, \sigma-1$, we obtain

$$A(s)\hat{\beta}(s) = B(s)\hat{u}(s)\ . \tag{2.23}$$

Hence in light of (2.12) we can write

$$A(s)^{-1}B(s) = C_\infty \left[I_\mu - sJ_\infty\right]^{-1} B_\infty B(s) + C\left[sI_n - J\right]^{-1} BB(s)\ , \tag{2.24}$$

which after some manipulations [26] can be written as

$$A(s)^{-1}B(s) = [C\ C_\infty] \begin{bmatrix} J^{\sigma-1}B, J^{\sigma-2}B, \ldots B & 0_{n,(\hat{q}_r+1)r} \\ 0_{\mu,\sigma r} & B_\infty, J_\infty B_\infty, \ldots, J_\infty^{\hat{q}_r} B_\infty \end{bmatrix} \times$$

$$\begin{bmatrix} B_\sigma & 0 & \cdots & 0 & \cdots & 0 \\ B_{\sigma-1} & B_\sigma & & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ B_1 & B_2 & \cdots & B_\sigma & \cdots & 0 \\ B_0 & B_1 & \cdots & B_{\sigma-1}\ B_\sigma & \cdots & 0 \\ 0 & B_0 & \cdots & B_{\sigma-2}\ B_{\sigma-1} & \ddots & 0 \\ \vdots & \vdots & & & & \vdots \\ 0 & 0 & \cdot\ B_0\ B_1 & \cdots & \cdot & B_\sigma \end{bmatrix} \begin{bmatrix} I_m \\ sI_m \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ s^{\hat{q}_r+\sigma} I_m \end{bmatrix}. \tag{2.25}$$

Taking the inverse Laplace transform of (2.25) and in light of (2.23) we obtain the solution of (2.22) [12]:

$$\beta^n(t) = [C\ C_\infty] \begin{bmatrix} \displaystyle\int_0^t e^{Jt}\Omega u(\tau)\,d\tau & +\displaystyle\sum_{i=0}^{\sigma-1} \Phi_{i+1}u^{(i)}(t) \\ \displaystyle\sum_{i=0}^{\hat{q}_r} J_\infty^i \bar{\Omega} u^{(\sigma+i)}(t) & +\displaystyle\sum_{i=0}^{\sigma-1} Z_i u^{(i)}(t) \end{bmatrix}, \tag{2.26}$$

where the superscript (i) means distributional derivative, σ is the maximum power of s in $B(s)$, and

$$\Omega = \sum_{i=0}^{\sigma} J^i BB_i = J^\sigma BB_\sigma + J^{\sigma-1}BB_{\sigma-1} + \cdots + BB_0 \tag{2.27}$$

$$\Phi_j = \sum_{i=0}^{\sigma-j} J^i BB_{i+j}\ , \quad j = 1, 2, \ldots, \sigma \tag{2.28}$$

$$\bar{\Omega} = \sum_{i=0}^{\sigma} J_\infty^i B_\infty B_{(\sigma-i)} = B_\infty B_\sigma + J_\infty B_\infty B_{\sigma-1} + \cdots + J_\infty^\sigma B_\infty B_0 \tag{2.29}$$

$$Z_{(\sigma-j)} = \sum_{i=0}^{\sigma} J_\infty^i B_\infty B_{(\sigma-j)-i}\ , \quad j = 1, 2, \ldots, \sigma \tag{2.30}$$

with $B_{(\sigma-j)-i} \equiv 0$ for $i, j : (\sigma - j) - i < 0$. In the case of generalized state space systems, i.e., with $q_1 = 1$ and $\sigma = 0$ the solution of (2.22) becomes from (2.26):

$$\beta^n(t) = [C\, C_\infty] \begin{bmatrix} \int_0^t e^{J\tau}\Omega\, u(\tau)\, d\tau \\ \sum_{i=0}^{\hat{q}_r} J_\infty^i \Omega\, u^{(i)}(t) \end{bmatrix}, \tag{2.31}$$

where $\Omega = B B_0$ and $\bar{\Omega} = B_\infty B_0$.

3. Admissible initial conditions for the homogeneous and the nonhomogeneous cases

We consider again the linear multivariable system Σ whose dynamic behavior is described by the PMD (2.13). First consider the homogeneous matrix differential equation

$$A(\rho)\beta(t) = 0 \tag{3.1}$$

with $A(\rho) = A_0 + A_1\rho + \cdots + A_{q_1}\rho^{q_1}$, $q_1 \geq 1$, $\text{rank}_{\mathbb{R}} A_{q_1} = k$, $1 \leq k < r$, $n = \deg|A(s)| < r$, $\mu = \sum_{i=k+1}^{r}(\hat{q}_i + 1)$ and assume that the polynomial matrix $A(s) = L_-[A(\rho)]$ has at least one zero at $s = \infty$; i.e., let $\hat{q}_r \geq 1$. The pseudo-state response of (3.1) is found to be (see (2.21))

$$\beta^h(t) = [C\, C_\infty] \begin{bmatrix} e^{Jt}x_s(0^-) \\ -\sum_{i=1}^{\hat{q}_r} \delta^{(i-1)} J_\infty^i x_f(0^-) \end{bmatrix}$$
$$= [C\, C_\infty] \begin{bmatrix} x_s^h(t) \\ x_f^h(t) \end{bmatrix} = \beta_{\text{spr}}^h(t) + \beta_{\text{pol}}^h(t)\ , \tag{3.2}$$

where the superscript (i) denotes the distributional derivative.

Definition 3.1. The vector

$$x^h(t) = \begin{bmatrix} x_s^h(t) \\ x_f^h(t) \end{bmatrix} \in \mathbb{R}^{(n+\mu)\times 1} \tag{3.3}$$

is called the *state of the homogeneous matrix differential equation (3.1)*. The $x_s^h(t)$ is called the *slow state* and the $x_f^h(t)$ is called *the fast or impulsive state*.

From (3.2) due to the fact that $\delta^{(i)}(0^-) = 0$, $i = 0, 1, \ldots,$ we obtain

$$\beta^{(i)}(0^-) = CJ^i x_s(0^-)\ , \quad i = 0, 1, \ldots, q_1\ . \tag{3.4}$$

If the slow state at $t = 0^-$ $x_s(0^-)$ is not equal to zero; i.e., $x_s(0^-) \neq 0$, then equations (3.4) give the algebraic constraints that the initial condition $\beta^{(i)}(0^-)$, $i = 0, 1, \ldots, q_1$, must satisfy in order for $\beta^h_{\text{spr}}(t) = Ce^{Jt}x_s(0^-)$ to be a solution of (3.1).

Proposition 3.2 [26]. *Let $x_s(0^-) \neq 0$ and assume that equations (3.4) are satisfied. Then $x_f(0^-) = 0$.*

Remark 3.3. From the above Proposition and equation (3.1) we have that if the constraints (3.4) are satisfied, then there is no impulsive behavior at $t = 0$.

It is clear from (3.2) that there are values of $x_f(0^-)$ that yield impulsive solutions $\beta^h(t)$. As discontinuous behavior is not desirable, we have the following:

Definition 3.4. The set of states at $t = 0^-$, x_0 with

$$x_0 = \begin{bmatrix} x_s(0^-) \\ x_f(0^-) \end{bmatrix} \in \mathbb{R}^{n+\mu} \tag{3.5}$$

that do not result in this impulsive behavior at $t = 0$ in (3.2) is called the set of *Admissible Initial Conditions (AIC)* for the system (3.1).

The set H_I of $x_0 \in \mathbb{R}^{n+\mu}$ that are AIC is given by

$$H_I = \left\{ x_0 = \begin{bmatrix} x_s(0^-) \\ x_f(0^-) \end{bmatrix} \in \mathbb{R}^{n+\mu} : \beta^h(t) = Ce^{Jt}x_s(0^-) \right\} \tag{3.6}$$

or equivalently

$$H_I = \left\{ x_0 = \begin{bmatrix} x_s(0^-) \\ x_f(0^-) \end{bmatrix} \in \mathbb{R}^{n+\mu} : \beta^h_{\text{pol}}(t) = -C_\infty \sum_{i=1}^{\hat{q}_r} \delta^{(i-1)}(t) J^i_\infty x_f(0^-) \equiv 0 \right\} . \tag{3.7}$$

From (3.7) it is obvious that the equality

$$C_\infty \left[\sum_{i=1}^{\hat{q}_r} \delta^{(i-1)}(t) J^i_\infty x_f(0^-) \right] = 0 \tag{3.8}$$

holds if

$$x_f(0^-) = 0 \tag{3.9}$$

or

$$J_\infty x_f(0^-) = 0 \Rightarrow x_f(0^-) \in \text{Ker}[J_\infty] . \tag{3.10}$$

If we now consider Remark 3.3 and Proposition 3.2 we finally obtain that the set H_I of $x_0 \in \mathbb{R}^{n+\mu}$ that are AIC is given by

$$H_I = \left\{ x_0 = \begin{bmatrix} x_s(0^-) \\ x_f(0^-) \end{bmatrix} / [x_s(0^-) \neq 0 \text{ s.t. } \beta^{(i)}(0^-) = CJ^i x_s(0^-) , \right.$$

$$\left. i = 0, \ldots, q_1] , [x_f(0^-) \in \ker[J_\infty] \right\} . \tag{3.11}$$

Consider now the nonhomogeneous case; i.e., consider again the PMD (2.13). Then the forced response of (2.13) is given by (2.26). From (2.21) and (2.26) we obtain that *the complete solution* of (2.13) is given by

$$\beta^c(t) = \beta^h(t) + \beta^n(t) = [C\ C_\infty] \times$$

$$\begin{bmatrix} e^{Jt} x_s(0^-) & + \int_0^t e^{J(t-\tau)} \Omega u(\tau)\, d\tau & + \sum_{i=0}^{\sigma-1} \Phi_{i+1} u^{(i)}(t) \\ - \sum_{i=1}^{\hat{q}_c} \delta^{(i-1)} J_\infty^i x_f(0^-) & + \sum_{i=0}^{\hat{q}_r} J_\infty^i \bar{\Omega} u^{(\sigma+i)}(t) & + \sum_{i=0}^{\sigma-1} Z_i u^{(i)}(t) \end{bmatrix}$$

$$= \beta_1^c(t) + \beta_2^c(t) , \tag{3.12}$$

where

$$\beta_1^c(t) = C \left[e^{Jt} x_s(0^-) + \int_0^t e^{J(t-\tau)} \Omega u(\tau)\, d\tau + \sum_{i=0}^{\sigma-1} \Phi_{i+1} u^{(i)}(t) \right] \tag{3.13}$$

and

$$\beta_2^c(t) = C_\infty \left[- \sum_{i=1}^{\hat{q}_r} \delta^{(i-1)}(t) J_\infty^i x_f(0^-) + \sum_{i=0}^{\hat{q}_r} J_\infty^i \bar{\Omega} j^{(\sigma+i)}(t) + \sum_{i=0}^{\sigma-1} Z_i u^{(i)}(t) \right] , \tag{3.14}$$

where the superscript (i) means distributional derivative. Let us now denote $u^{[i]}(t)$ the ith (ordinary) derivative of $u(t)$. Using the identity (see [4], p. 52]

$$u^{(i)}(t) = u^{[i]}(t) + \delta u^{[i-1]}(0) + \cdots + \delta^{[i-1]} u(0) , \quad i = 1, 2, \ldots \tag{3.15}$$

$\beta_1^c(t)$ and $\beta_2^c(t)$ can be written (see [12]):

$$\beta_1^c(t) = C \left[e^{Jt} x_s(0^-) + \int_0^t e^{J(t-\tau)} \Omega u(\tau)\, d\tau + \sum_{i=0}^{\sigma-1} \Phi_{i+1} u^{[i]}(t) + \sum_{i=0}^{\sigma-2} \delta^{[i]} \times \right.$$

$$\left. \left[\sum_{j=0}^{\sigma-2-i} \Phi_{j+2+i} u^{[j]}(0^-) \right] \right] \tag{3.16}$$

and

$$\begin{aligned}\beta_2^c(t) = &- C_\infty \sum_{i=1}^{\hat{q}_r} \delta^{(i-1)}(t) J_\infty^i x_f(0^-) \\ &+ C_\infty \left[\sum_{i=0}^{\sigma-1} Z_i u^{[i]}(t) + \sum_{i=0}^{\sigma-2} \delta^{(i)} \left(\sum_{j=0}^{\sigma-2-i} Z_{j+1+i} u^{[j]}(0) \right) \right] \\ &+ C_\infty \left[\sum_{i=0}^{\hat{q}_r} J_\infty^i \bar{\Omega} u^{[\sigma+i]}(t) + \sum_{i=0}^{\sigma-1} \delta^{(i)} \left(\sum_{j=0}^{\hat{q}_r} J_\infty^j \bar{\Omega} u^{[\sigma+j-i-1]}(0) \right) \right. \\ &+ \left. \sum_{i=\sigma}^{\sigma+\hat{q}_r-1} \delta^{(i)} \left(\sum_{j=i-(\sigma-1)}^{\hat{q}_r} J_\infty^j \bar{\Omega} u^{[j-1]}(0) \right) \right] . \end{aligned} \tag{3.17}$$

From the form of $\beta_1^c(t)$ and $\beta_2^c(t)$ as given by (3.16) and (3.17), respectively, and the fact that (see (3.12)) $\beta^c(t) = \beta_1^c(t) + \beta_2^c(t)$ it is obvious that the complete solution of (2.13) may have impulsive components. As discontinuous (impulsive) behavior is not desirable, we have the following:

Definition 3.5. A point $\beta_0^c \in \mathbb{R}^r$

$$\beta_0^c \equiv \beta^c(0^-) \tag{3.18}$$

is said to be *an Admissible Initial Condition (AIC)* for the system (2.13) if the solution $\beta^c(t; 0^-, \beta_0^c, u(t))$ is continuously differentiable on $[0, T]$ for some input $u(t)$ and for some $T > 0$, i.e., $\beta^c(t; 0^-, \beta_0^c, u(t))$ is impulse free.

It follows from (3.16) and (3.17) that a point β_0^c is an AIC if the following conditions hold:

$$C_\infty \sum_{i=1}^{\hat{q}_r} \delta^{(i-1)}(t) J_\infty^i x_f(0^-) = 0 \tag{3.19}$$

$$\sum_{j=0}^{\sigma-2-i} \Phi_{j+2+i} u^{[j]}(0^-) = 0 , \quad i = 0, 1, \ldots, \sigma - 2 \tag{3.20}$$

$$\sum_{j=0}^{\sigma-2-i} Z_{j+1+i} u^{[j]}(0^-) = 0 , \quad i = 0, 1, \ldots, \sigma - 2 \tag{3.21}$$

$$\sum_{j=0}^{\hat{q}_r} J_\infty^j \bar{\Omega} u^{[\sigma+j-i-1]}(0) = 0 , \quad \text{for } i = 0, 1, \ldots, \sigma - 1 \tag{3.22}$$

and finally

$$\sum_{j=i-(\sigma-1)}^{\hat{q}_r} J_\infty^j \bar{\Omega} u^{[j-1]}(0) = 0 , \quad \text{for } i = \sigma, \ldots, \sigma + \hat{q}_r - 1 . \tag{3.23}$$

Equation (3.19) holds if

$$J_\infty x_f(0^-) = 0 \Rightarrow x_f(0^-) \in \mathrm{Ker}[J_\infty]\,. \tag{3.24}$$

Equation (3.20) holds if

$$\begin{bmatrix} \Phi_2 & \Phi_3 & \cdots & \Phi_{\sigma-1} & \Phi_\sigma \\ \Phi_3 & \Phi_4 & \cdots & \Phi_\sigma & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \Phi_{\sigma-1} & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} u(0^-) \\ u^{[1]}(0^-) \\ \vdots \\ u^{[\sigma-2]}(0^-) \end{bmatrix} = 0\,. \tag{3.25}$$

Equation (3.21) holds if

$$\begin{bmatrix} Z_1 & Z_2 & \cdots & Z_{\sigma-2} & Z_{\sigma-1} \\ Z_2 & Z_3 & \cdots & Z_{\sigma-1} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ Z_{\sigma-1} & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} u(0^-) \\ u^{[1]}(0^-) \\ \vdots \\ u^{[\sigma-2]}(0^-) \end{bmatrix} = 0\,. \tag{3.26}$$

Similarly, equation (3.22) holds if

$$\begin{bmatrix} \bar\Omega & J_\infty\bar\Omega & \cdots & J_\infty^{\hat q_r}\bar\Omega & 0 & 0\cdots 0 \\ 0 & \bar\Omega & \cdots & & J_\infty^{\hat q_r}\bar\Omega & 0\cdots 0 \\ \vdots & \vdots & \ddots & & & \ddots \\ 0 & 0 & \cdots & \bar\Omega & J_\infty\bar\Omega & \cdots J_\infty^{\hat q_r}\bar\Omega \end{bmatrix} \begin{bmatrix} u(0^-) \\ u^{[1]}(0^-) \\ \vdots \\ u^{[\sigma+\hat q_r-1]}(0) \end{bmatrix} = 0\,. \tag{3.27}$$

And finally equation (3.23) holds if

$$\begin{bmatrix} J_\infty^{\hat q_r}\bar\Omega & 0 & \cdots & 0 \\ J_\infty^{\hat q_r-1}\bar\Omega & J_\infty^{\hat q_r}\bar\Omega & \cdots & 0 \\ \vdots & & \ddots & \\ J_\infty\bar\Omega & J_\infty^2\bar\Omega & \cdots & J_\infty^{\hat q_r}\bar\Omega \end{bmatrix} \begin{bmatrix} u(0^-) \\ u^{[1]}(0^-) \\ \vdots \\ u^{[\hat q_r-1]}(0^-) \end{bmatrix} = 0\,. \tag{3.28}$$

Remark 3.6. From the above it is obvious that the initial conditions of $u(t)$; i.e., $u^{[i]}(0^-)\quad i = 0, 1, \ldots, \sigma + \hat q_r - 1$ must satisfy the certain relations (3.24)–(3.28) in order for the system's solution to be impulse free.

The set of admissible states for $t \geq 0^-$ is given by

$$\beta^c(t) = [C\ C_\infty] \begin{bmatrix} e^{Jt}x_s(0^-) + \int_0^t e^{J(t-\tau)}\Omega\, u(\tau)\, d\tau + \sum_{i=0}^{\sigma-1} \Phi_{i+1} u^{[i]}(t) \quad (3.29) \\ \sum_{i=0}^{\hat q_r} J_\infty^i \bar\Omega u^{[\sigma+i]}(t) + \sum_{i=1}^{\sigma-1} Z_i u^{[i]}(t) \end{bmatrix}$$

$$= [C\ C_\infty] \begin{bmatrix} x_s^c(t) \\ x_s^c(t) \end{bmatrix}.$$

From (3.29) for $t = 0^-$ the set of AIC is

$$\beta^c(0^-) = [C\ C_\infty]\begin{bmatrix} x_s^c(0^-) \\ x_s^c(0^-) \end{bmatrix} \tag{3.30}$$

where

$$x_s^c(0^-) = x_s(0^-) + \sum_{i=0}^{\sigma-1} \Phi_{i+1}u^{[i]}(0^-) \in \mathbb{R}^n \tag{3.31}$$

$$x_f^c(0^-) = \sum_{i=0}^{\hat{q}_r} J_\infty^i \bar{\Omega} u^{[\sigma+i]}(0^-) + \sum_{i=0}^{\sigma-1} Z_i u^{[i]}(0^-) \in \mathbb{R}^\mu , \tag{3.32}$$

i.e., the set of AIC in this case is

$$H_{Iu} = \left\{ \beta^c(0^-) \in \mathbb{R}^r / \beta^c(0^-) \in \mathbb{R}^r / \beta^c(0^-) \quad = [C\ C_\infty] \right.$$
$$\left. \begin{bmatrix} x_s(0^-) & + \sum_{i=0}^{\sigma-1} \Phi_{i+1}u^{[i]}(0^-) \\ \sum_{i=0}^{\hat{q}_r} J_\infty^i \bar{\Omega} u^{[\sigma+i]}(0^-) & + \sum_{i=0}^{\sigma-1} Z_i u^{[i]}(0^-) \end{bmatrix} \right\} \tag{3.33}$$

or equivalently

$$H_{Iu} = \left\{ \beta^c(0^-) = [C\ C_\infty]\begin{bmatrix} x_s^c(0^-) \\ x_f^c(0^-) \end{bmatrix} / x_s^c(0^-) \in \mathbb{R}^n \text{ , and} \right.$$
$$\left. x_f^c(0^-) \in \sum_{i=0}^{\hat{q}_r} J_\infty^i \operatorname{Im} \bar{\Omega} + \sum_{i=0}^{\sigma-1} \operatorname{Im} Z_i + \operatorname{Ker} J_\infty \right\} . \tag{3.34}$$

Remark 3.7. Note that the zero vector 0 belongs to H_{Iu} because there exist $x_s(0^-) \equiv 0$ and input $u(t)$ such that $u^{[i]}(0^-) \equiv 0$ for $i = 0, 1, 2, \ldots, \hat{q}_r$ or $i = 0, 1, 2, \ldots, \sigma - 2$ in the case $\sigma - 2 > \hat{q}_r$.

4. The reachable subspace

In this section we generalize the notions of reachability given in [35], [21] to cover the case of PMDs as in (2.13).

Definition 4.1. Given a point $\beta_0^c = \beta^c(0^-) \in H_{Iu}$, we say that another point $\beta_T \in \mathbb{R}^r$ is *reachable* from β_0^c if there exists an input $u(t)$ and $T > 0$ such that $\beta^c(t) = \beta^c(t; 0^-, \beta_0^c, u(t))$ is impulse free on $[0^-, T]$ and holds:

$$\beta^c(T) = \beta_T . \tag{4.1}$$

Let $R(\beta_0^c)$ denote the set of reachable states from $\beta_0^c \in H_{Iu}$. $R(\beta_0^c) \neq \emptyset$ means that there exists an input that will make the solution $\beta^c(t)$ impulse free on $[0, T]$. We shall try to describe $R(\beta_0^c)$ in terms of its finite and infinite spectral data; i.e., the finite Jordan triple (C, J, B) and the infinite Jordan triple $(C_\infty, J_\infty, B_\infty)$ of the matrix $A(s)$ (see Section 2 and [15]).

We first assume that $\beta_0^c = 0 \in H_{Iu}$ and describe the set $R(0)$; i.e., the set of reachable states from $0 \in H_{Iu}$. We introduce the following notation:

$$\langle J/\operatorname{Im}\Omega\rangle := \operatorname{Im}\Omega + J\operatorname{Im}\Omega + \cdots + J^{n-1}\operatorname{Im}\Omega \tag{4.2}$$

$$\langle J_\infty/\operatorname{Im}\bar{\Omega}\rangle := \operatorname{Im}\bar{\Omega} + J_\infty\operatorname{Im}\bar{\Omega} + \cdots + J_\infty^{\hat{q}_r}\operatorname{Im}\bar{\Omega} \tag{4.3}$$

$$\operatorname{Im}\Omega = \{x/x \in \mathbb{R}^n \text{ and } \exists u \in \mathbb{R}^m : x = \Omega u\} \subset \mathbb{R}^n ,$$
$$\operatorname{Im}\bar{\Omega} = \left\{x/x \in \mathbb{R}^\mu \text{ and } \exists u \in \mathbb{R}^m : x = \bar{\Omega} u\right\} \subset \mathbb{R}^\mu , \tag{4.4}$$

where Ω and $\bar{\Omega}$ are given by (2.27), (2.29), respectively. We shall need the following Lemmas:

Lemma 4.2 [35]. *For any polynomial $g(s) \in \mathbb{R}[s]$ not identical to zero and for any $z \in \mathbb{R}^n$, define $W(g, t) : \mathbb{R}^n \to \mathbb{R}^n$ by*

$$W(g, t)z = \int_0^t \left[g(\tau)e^{J\tau}\Omega\,\Omega^T e^{J^T\tau} g(\tau)\right] z\, d\tau . \tag{4.5}$$

Then

$$\operatorname{Im}[W(g, t)] = \langle J/\operatorname{Im}\Omega\rangle . \tag{4.6}$$

Lemma 4.3 [25]. *For any h-vectors $\beta_i \in \mathbb{R}^n$, $i = 0, 1, \ldots, h-1$, and $t_1 > 0$, there always exists a vector polynomial $f(t) \in \mathbb{R}^r(t)$ whose degree is $(h-1)$ such that*

$$f^{[i]}(t_1) = \beta_i , \quad i = 0, 1, \ldots, h-1 . \tag{4.7}$$

Remark 4.4. The above Lemma 4.3 is the formal definition of the *Taylor interpolation polynomial*, which gives the values of a specified function and its derivatives *at only one point*; i.e., at $t = 0$ or $t = T$, but not at both points. The Taylor interpolation polynomial with degree equal to $\hat{q}_r$ is given by

$$p(t) = \sum_{i=0}^{\hat{q}_r} \frac{(t-T)^i}{i!} F^{[i]}(T) . \tag{4.8}$$

When we want the interpolation polynomial to take the same value as a specified function and its derivatives *but we also* want to have this at *a number of points* (many points) then we use the *Hermite interpolation polynomial* [25] shown below.

Lemma 4.5 [25]. *Let $x_i, y_i \in \mathbb{R}^{m\times 1}$, $i = 0, 1, \ldots, \hat{q}_r$, be arbitrary vectors and $T > 0$. Then there exists a vector polynomial $u(t) \in \mathbb{R}^{m\times 1}$ of degree $(2\hat{q}_r + 1)$ such that*

$$u^{[i]}(0) = x_i \quad \text{and} \quad u^{[i]}(T) = y_i , \quad i = 0, 1, \ldots, \hat{q}_r . \tag{4.9}$$

Theorem 4.6.

$$R(0) = [C\ C_\infty]\begin{bmatrix} \langle J/\operatorname{Im}\Omega\rangle & +\sum_{i=0}^{\sigma-1}\operatorname{Im}\Phi_{i+1} \\ \langle J_\infty/\operatorname{Im}\bar{\Omega}\rangle & +\sum_{i=0}^{\sigma-1}\operatorname{Im} Z_i \end{bmatrix},$$

where Z_i, $i = 0, 1, \ldots, \sigma-1$, is given by (2.30) and Φ_j, $j = 1, \ldots, \sigma$, is given by (2.28).

Proof. (i) We first prove that $R(0) \subset [C\ C_\infty]\begin{bmatrix} \langle J/\operatorname{Im}\Omega\rangle & +\sum_{i=0}^{\sigma-1}\operatorname{Im}\Phi_{i+1} \\ \langle J_\infty/\operatorname{Im}\bar{\Omega}\rangle & +\sum_{i=0}^{\sigma-1}\operatorname{Im} Z_i \end{bmatrix}$.

Let $\beta^c(t)$ be as in (3.29) and $\beta^c(t) \in R(0)$. This means that $\beta^c(0^-) = 0$; i.e., $x_s(0^-) \equiv 0$ and $u^{[i]}(0^-) \equiv 0$ and $\beta^c(t)$ is impulse free for $t > 0$. From (3.29) for $t = T$ we have

$$\beta^c(T) = [C\ C_\infty]\begin{bmatrix} \int_0^T e^{J(T-\tau)}\Omega\, u(\tau)\, d\tau & +\sum_{i=0}^{\sigma-1}\Phi_{i+1}u^{[i]}(T) \\ \sum_{i=0}^{\hat{q}_r} J_\infty^i\bar{\Omega}\, u^{[i]}(T) & +\sum_{i=0}^{\sigma-1} Z_i u^{[i]}(T) \end{bmatrix}. \tag{4.10}$$

From (4.10) it is clear that

$$\sum_{i=0}^{\hat{q}_r} J_\infty^i\bar{\Omega}\, u^{[i]}(T) \in \langle J_\infty/\operatorname{Im}\bar{\Omega}\rangle \tag{4.11}$$

and

$$\sum_{i=0}^{\sigma-1} Z_i u^{[i]}(T) \in \sum_{i=0}^{\sigma-1}\operatorname{Im} Z_i\ , \quad \sum_{i=0}^{\sigma-1}\Phi_{i+1}u^{[i]}(T) \in \sum_{i=0}^{\sigma-1}\operatorname{Im}\Phi_{i+1}\ . \tag{4.12}$$

We also have

$$\int_0^t e^{J(T-\tau)}\Omega\, u(\tau)\, d\tau = \sum_{i=0}^{n-1} J^i\Omega\int_0^T \Psi_i(T-\tau)u(\tau)\, d\tau \in \langle J/\operatorname{Im}\Omega\rangle \tag{4.13}$$

in which we made use of the Cayley–Hamilton theorem. Now from (4.10)–(4.13) we obtain

$$\beta^c(T) \in [C\ C_\infty]\begin{bmatrix} \langle J/\operatorname{Im}\Omega\rangle & +\sum_{i=0}^{\sigma-1}\operatorname{Im}\Phi_{i+1} \\ \langle J_\infty/\operatorname{Im}\bar{\Omega}\rangle & +\sum_{i=0}^{\sigma-1}\operatorname{Im} Z_i \end{bmatrix} \tag{4.14}$$

and from the fact that $\beta^c(T) \in R(0)$ we finally arrive at

$$R(0) \subset [C\ C_\infty] \begin{bmatrix} \langle J/\operatorname{Im}\Omega\rangle & +\sum_{i=0}^{\sigma-1} \operatorname{Im}\Phi_{i+1} \\ \langle J_\infty/\operatorname{Im}\bar{\Omega}\rangle & +\sum_{i=0}^{\sigma-1} \operatorname{Im} Z_i \end{bmatrix}. \tag{4.15}$$

(ii) Now we shall prove that

$$R(0) \supset [C\ C_\infty] \begin{bmatrix} \langle J/\operatorname{Im}\Omega\rangle & +\sum_{i=0}^{\sigma-1} \operatorname{Im}\Phi_{i+1} \\ \langle J_\infty/\operatorname{Im}\bar{\Omega}\rangle & +\sum_{i=0}^{\sigma-1} \operatorname{Im} Z_i \end{bmatrix}.$$

Assume that

$$\beta = [C\ C_\infty]\begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix} \in [C\ C_\infty] \begin{bmatrix} \langle J/\operatorname{Im}\Omega\rangle & +\sum_{i=0}^{\sigma-1} \operatorname{Im}\Phi_{i+1} \\ \langle J_\infty/\operatorname{Im}\bar{\Omega}\rangle & +\sum_{i=0}^{\sigma-1} \operatorname{Im} Z_i \end{bmatrix}$$

i.e., $0 \neq \beta_1 \in \langle J/\operatorname{Im}\Omega\rangle + \sum_{i=0}^{\sigma-1} \operatorname{Im}\Phi_{i+1}$ and $0 \neq \beta_2 \in \langle J_\infty/\operatorname{Im}\bar{\Omega}\rangle + \sum_{i=0}^{\sigma-1} \operatorname{Im} Z_i$. We have to find an input $u(t)$ such that β_T is a solution of (2.13), i.e., $\beta^c(t) = \beta$ for some $T > 0$. We choose $u(t)$ to be of the form

$$u(t) = u_1(t) + u_2(t)\,. \tag{4.16}$$

Then

$$\beta^c(T) = [C\ C_\infty] \begin{bmatrix} \sum_{i=0}^{\sigma-1} \Phi_{i+1} u_1^{[i]}(T) & + \int_0^T e^{J(T-\tau)}\Omega\, u_1(\tau)\, d\tau \\ \sum_{i=0}^{\sigma-1} Z_i u_1^{[i]}(T) & + \sum_{i=0}^{\hat{q}_r} J_\infty^i \bar{\Omega}\, u_1^{[i]}(T) \end{bmatrix}$$
$$+ [C\ C_\infty] \begin{bmatrix} + \sum_{i=0}^{\sigma-1} \Phi_{i+1} u_2^{[i]}(T) & + \int_0^T e^{J(T-\tau)}\Omega\, u_2(\tau)\, d\tau \\ + \sum_{i=0}^{\sigma-1} Z_i u_2^{[i]}(T) & + \sum_{i=0}^{\hat{q}_r} J_\infty^i \bar{\Omega}\, u_2^{[i]}(T) \end{bmatrix}. \tag{4.17}$$

We first consider the choice of $u_1(t)$. We choose $T > 0$ arbitrarily and choose $u_1(t)$ to be of the form

$$u_1(t) = t^{\hat{q}_r+1+\sigma}(t-T)^{\hat{q}_r+1+\sigma}\psi(t) \tag{4.18}$$

for some $\psi(t) \in \mathbb{R}^m[t]$. Thus, $u_1^{[i]}(0) = u_1^{[i]}(T) = 0$ for $i < (\hat{q}_r + 1 + \sigma)$ and $u_1(t)$ contributes nothing to the bottom part of (4.17) at $t = 0$ and at $t = T$.

We write $\beta_1 \in \langle J/\operatorname{Im}\Omega\rangle + \sum_{i=0}^{\sigma-1} \operatorname{Im}\Phi_{i+1}$ as $\beta_1 = \beta_{10} + \beta_{11}$, where

$$\beta_{10} = \int_0^t e^{J(t-\tau)}\Omega\, u_1(\tau)\, d\tau + \int_0^t e^{J(t-\tau)}\Omega\, u_2(\tau)\, d\tau \in \langle J/\operatorname{Im}\Omega\rangle \tag{4.19}$$

and

$$\beta_{11} = \sum_{i=0}^{\sigma-1} \Phi_{i+1} u_1^{[i]}(t) + \sum_{i=0}^{\sigma-1} \Phi_{i+1} u_2^{[i]}(t) \in \sum_{i=0}^{\sigma-1} \operatorname{Im}\Phi_{i+1}\,. \tag{4.20}$$

We choose $\psi(t)$ to satisfy the following equation:

$$\begin{aligned}\int_0^T e^{J(T-\tau)}\Omega\, u_1(\tau)\, d\tau &= \int_0^T e^{J(T-\tau)}\Omega \left[\tau^{\hat{q}_r+1+\sigma}(\tau - T)^{\hat{q}_r+1+\sigma}\psi(\tau)\right] d\tau \\ &= \beta_{10} - \left[\int_0^T e^{J(T-\tau)}\Omega\, u_2(\tau)\, d\tau\right] = \tilde{\beta}_{10} \in \langle J/\operatorname{Im}\Omega\rangle\,.\end{aligned} \tag{4.21}$$

From Lemma 4.2 we have that there exists a $z \in \mathbb{R}^n$ such that $W(g, t)z = \tilde{\beta}_{10}$. Let

$$g(\tau) = \tau^{\hat{q}_r+1+\sigma}(\tau - T)^{\hat{q}_r+1+\sigma} \tag{4.22}$$

and

$$\psi(\tau) = g(\tau)\Omega^T e^{J^T(T-\tau)} z\,. \tag{4.23}$$

Then from (4.18) we have

$$u_1(\tau) = [g(\tau)]^2 \Omega^T e^{J^T(T-\tau)} z\,. \tag{4.24}$$

Thus

$$\begin{aligned}\int_0^T e^{J(T-\tau)}\Omega\, u_1(\tau)\, d\tau &= \int_0^T \left[g(\tau)\, e^{J(T-\tau)}\Omega\,\Omega^T e^{J^T(T-\tau)} g(\tau)\right] z\, d\tau \\ &= W(g, t)z = \tilde{\beta}_{10}\end{aligned}$$

and from (4.21) we have

$$\int_0^T e^{J(T-\tau)}\Omega\, u_1(\tau)\, d\tau = \tilde{\beta}_{10} = \beta_{10} - \int_0^T e^{J(T-\tau)}\Omega\, u_2(\tau)\, d\tau\,,$$

i.e.,

$$\int_0^T e^{J(T-\tau)}\Omega\, u_1(\tau)\, d\tau + \int_0^T e^{J(T-\tau)}\Omega\, u_2(\tau)\, d\tau = \beta_{10}\,. \tag{4.25}$$

Furthermore, from (4.20) for $t = T$ we obtain

$$\beta_{11} = \sum_{i=0}^{\sigma-1} \Phi_{i+1} u_1^{[i]}(T) + \sum_{i=0}^{\sigma-1} \Phi_{i+1} u_2^{[i]}(T) = 0 + \sum_{i=0}^{\sigma-1} \Phi_{i+1} u_2^{[i]}(T)$$

— note that $u_1^{[i]}(T) = 0$, see (4.18) — and from (4.25) and (4.26) we finally obtain

$$\int_0^T e^{J(T-\tau)}\Omega\, u_1(\tau)\, d\tau + \int_0^T e^{J(T-\tau)}\Omega\, u_2(\tau)\, d\tau + \sum_{i=0}^{\sigma-1} \Phi_{i+1} u_2^{[i]}(T) =$$

$$\beta_{10} \quad + \quad \beta_{11} \quad = \quad \beta_1 \, .$$

Let us consider the choice of $u_2(t)$. We consider two different approaches that depend on whether Lemma 4.3 or Lemma 4.5 is used.

Case A. Let $\beta_{2,i}$, $i = 0, 1, \ldots, \hat{q}_r + \sigma$, i.e., $\beta_{20}, \beta_{21}, \ldots, \beta_{2,\hat{q}_r+\sigma}$ such that

$$\beta_2 = \sum_{i=0}^{\sigma-1} Z_i \beta_{2i} + \sum_{i=\sigma}^{\hat{q}_r+\sigma} J_\infty^{(i-\sigma)} \bar{\Omega} \beta_{2i} \, . \tag{4.27}$$

Now from Lemma 4.3 there exists a polynomial $u_2(t)$ of degree $(\hat{q}_r + 1)$ such that $u_2^{[i]}(T) = \beta_{2i}$, $i = 0, 1, \ldots, \hat{q}_r + \sigma$. The polynomial $u_2(t)$ is of the form

$$u_2(t) = \beta_{20} + (t-T)\beta_{21} + \frac{1}{2!}(t-T)^2\beta_{22} + \cdots + \frac{1}{(\hat{q}_r+\sigma)!}(t-T)^{\hat{q}_r+\sigma} \beta_{2,\hat{q}_r+\sigma} \, . \tag{4.28}$$

Then

$$u_2^{[1]}(t) = \beta_{21} + (t-T)\beta_{22} + \cdots + \frac{1}{(\hat{q}_r+\sigma-1)!}(t-T)^{\hat{q}_r+\sigma-1} \beta_{2,\hat{q}_r+\sigma-1}$$

$$u_2^{[2]} = \beta_{22} + (t-T)\beta_{23} + \cdots + \frac{1}{(\hat{q}_r+\sigma-2)!}(t-T)^{\hat{q}_r+\sigma-2} \beta_{2,\hat{q}_r+\sigma-2}$$

$$\vdots$$

$$u^{[\hat{q}_r+\sigma]}(t) = \beta_{2,\hat{q}_r+\sigma} \, . \tag{4.29}$$

Then for some $T > 0$ we shall have from (4.17) (using (4.26) and recalling that $u_1^{[i]}(T) = 0$ for $i < (\hat{q}_r + 1 + \sigma)$

$$\beta^c(T) = [C\ C_\infty] \begin{bmatrix} \beta_1 \\ \sum_{i=0}^{\sigma-1} Z_i u_2^{[i]}(T) + \sum_{i=\sigma}^{\hat{q}_r+\sigma} J_\infty^{(i-\sigma)} \bar{\Omega}\, u_2^{[i]}(T) \end{bmatrix} \quad \text{(using (4.29))}$$

$$= [C\ C_\infty] \begin{bmatrix} \beta_1 \\ \sum_{i=0}^{\sigma-1} Z_i \beta_{2i} + \sum_{i=\sigma}^{\hat{q}_r+\sigma} J_\infty^{(i-\sigma)} \bar{\Omega} \beta_{2i} \end{bmatrix} \quad \text{(using (4.27))}$$

$$= [C\ C_\infty] \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \beta \, , \tag{4.30}$$

i.e.,

$$\beta^c(T; 0^-, 0, u(t)) = \beta \quad \text{for some} \quad T > 0 \, . \tag{4.31}$$

Case B. The input $u_2(t)$ must satisfy the following two conditions:

$$\sum_{i=0}^{\sigma-1} Z_i u_2^{[i]}(0^-) + \sum_{i=\sigma}^{\hat{q}_r+\sigma} J_\infty^{(i-\sigma)} \bar{\Omega} u_2^{[i]}(0^-) = 0 \quad \text{or} \quad u_2^{[i]}(0^-) = 0 \tag{4.32}$$

$$\sum_{i=0}^{\sigma-1} Z_i u_2^{[i]}(T) + \sum_{i=\sigma}^{\hat{q}_r+\sigma} J_\infty^{(i-\sigma)} \bar{\Omega} u_2^{[i]}(T) \neq 0 \quad \text{or} \quad u_2^{[i]}(T) \neq 0 \quad \text{for some } t > 0\,. \tag{4.33}$$

$\beta_2 \in \langle J_\infty / \operatorname{Im} \bar{\Omega}\rangle + \sum_{i=0}^{\sigma-1} \operatorname{Im} Z_i$ implies that there exists $\beta_{2i} \in \mathbb{R}^{m\times 1}$, $i = 0, 1, \ldots, \hat{q}_r + \sigma$, i.e., $\exists\ \beta_{20}, \beta_{21}, \ldots, \beta_{2,\hat{q}_r+\sigma}$ such that

$$\beta_2 = \sum_{i=0}^{\sigma-1} Z_i \beta_{2i} + \sum_{i=\sigma}^{\hat{q}_r+\sigma} J_\infty^{(i-\sigma)} \bar{\Omega} \beta_{2i}\,. \tag{4.34}$$

Now from Lemma 4.5 there exists a polynomial $u_2(t) \in \mathbb{R}^{m\times 1}$ of degree $2(\hat{q}_r + \sigma) + 1$ such that

$$u_2^{[i]}(0^-) = 0 \quad \text{and} \quad u_2^{[i]}(T) = \beta_{2i}\,, \quad i = 0, 1, \ldots, \hat{q}_r + \sigma\,, \tag{4.35}$$

which means that the conditions (4.32) and (4.33) are satisfied. We shall find the form of $u_2(t)$ later. First we shall show that $\beta^c(T) = \beta$. To this end we have

$$\begin{aligned}
\beta^c(T) &= [C\ C_\infty] \begin{bmatrix} \beta_1 \\ \sum_{i=0}^{\sigma-1} Z_i u_2^{[i]}(T) + \sum_{i=\sigma}^{\hat{q}_r+\sigma} J_\infty^{(i-\sigma)} \bar{\Omega} u_2^{[i]}(T) \end{bmatrix} \\
&= [C\ C_\infty] \begin{bmatrix} \beta_1 \\ \sum_{i=0}^{\sigma-1} Z_i \beta_{2i} + \sum_{i=\sigma}^{\hat{q}_r+\sigma} J_\infty^{(i-\sigma)} \bar{\Omega} \beta_{2i} \end{bmatrix} = [C\ C_\infty] \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \beta\,.
\end{aligned} \tag{4.36}$$

It remains now to find the form of $u_2(t)$. Let us take

$$u_2(t) = \tilde{u}_0 t^{\hat{q}_r+\sigma+1} + \tilde{u}_1 t^{\hat{q}_r+\sigma+2} + \cdots + \tilde{u}_{\hat{q}_r+\sigma} t^{2(\hat{q}_r+\sigma)+1}\,, \tag{4.37}$$

where $\tilde{u}_i$, $i = 0, 1, \ldots, \hat{q}_r + \sigma$, are m-dimensional unspecified vectors such that

$$\begin{aligned}
u_2^{[1]}(t) =& (\hat{q}_r + \sigma + 1)\tilde{u}_0 t^{\hat{q}_r+\sigma} + (\hat{q}_r + \sigma + 2)\tilde{u}_1 t^{\hat{q}_r+\sigma+1} \\
&+ \cdots + 2(\hat{q}_r + \sigma + 1)\tilde{u}_{\hat{q}_r+\sigma} t^{2(\hat{q}_r+\sigma)} \\
u_2^{[2]}(t) =& (\hat{q}_r + \sigma)(\hat{q}_r + \sigma + 1)\tilde{u}_0 t^{\hat{q}_r+\sigma-1} \\
&+ \cdots + [2(\hat{q}_r + \sigma)][2(\hat{q}_r + \sigma) + 1]\tilde{u}_{\hat{q}_r+\sigma} t^{2(\hat{q}_r+\sigma)-1} \\
&\vdots \\
u_2^{[\hat{q}_r+\sigma]}(t) =& \frac{(\hat{q}_r + \sigma + 1)!}{1!}\tilde{u}_0 t + \frac{(\hat{q}_r + \sigma + 2)!}{2!}\tilde{u}_1 t^2
\end{aligned}$$

$$+\cdots+\frac{(2(\hat{q}_r+\sigma)+1)!}{(\hat{q}_r+\sigma+1)!}\tilde{u}_{\hat{q}_r+\sigma}t^{\hat{q}_r+\sigma+1}\,. \tag{4.38}$$

The previous equations (4.38) can also be written in compact form as

$$\begin{aligned} u_2^{[1]}(t) &= \sum_{i=0}^{\hat{q}_r+\sigma}\frac{(\hat{q}_r+\sigma+1+i)!}{(\hat{q}_r+\sigma+i)!}\tilde{u}_i t^{\hat{q}_r+\sigma+i} \\ u_2^{[2]}(t) &= \sum_{i=0}^{\hat{q}_r+\sigma}\frac{(\hat{q}_r+\sigma+1+i)!}{(\hat{q}_r+\sigma+i-1)!}\tilde{u}_i t^{\hat{q}_r+\sigma+i-1} \\ &\vdots \end{aligned} \tag{4.39}$$

The general form for the derivatives of $u_2(t)$ is

$$u_2^{[j]}(t)=\sum_{i=0}^{\hat{q}_r+\sigma}\frac{(\hat{q}_r+\sigma+1+i)!}{(\hat{q}_r+\sigma+i-j+1)!}\tilde{u}_i t^{\hat{q}_r+\sigma+i-j+1}\,, \tag{4.40}$$

for $j=1,2,\ldots,\hat{q}_r+\sigma$.

Equations (4.37) and (4.40) can be written in matrix form for $t=T$:

$$\begin{bmatrix} u_2(T) \\ u_2^{[1]}(T) \\ \vdots \\ u_2^{[\hat{q}_r+\sigma]}(T) \end{bmatrix} = \begin{bmatrix} T^{\hat{q}_r+\sigma+1} & T^{\hat{q}_r+\sigma+2} & \cdots & T^{2(\hat{q}_r+\sigma)+1} \\ \frac{(\hat{q}_r+\sigma+1)!}{(\hat{q}_r+\sigma)!}T^{\hat{q}_r+\sigma} & \frac{(\hat{q}_r+\sigma+2)!}{(\hat{q}_r+\sigma+1)!}T^{\hat{q}_r+\sigma+1} & \cdots & \frac{[2(\hat{q}_r+\sigma)+1]!}{[2(\hat{q}_r+\sigma)]!}T^{2(\hat{q}_r+\sigma)} \\ \vdots & \vdots & & \vdots \\ \frac{(\hat{q}_r+\sigma+1)!}{1!}T & \frac{(\hat{q}_r+\sigma+2)!}{2!}T^2 & \cdots & \frac{[2(\hat{q}_r+\sigma)+1]!}{(\hat{q}_r+\sigma+1)!}T^{\hat{q}_r+\sigma+1} \end{bmatrix} \begin{bmatrix} \tilde{u}_0 \\ \tilde{u}_1 \\ \vdots \\ \tilde{u}_{\hat{q}_r+\sigma} \end{bmatrix} \tag{4.41}$$

or in compact form

$$\begin{bmatrix} u_2(T) \\ u_2^{[1]}(T) \\ \vdots \\ u_2^{[\hat{q}_r+\sigma]}(T) \end{bmatrix} = S(T)\begin{bmatrix} \tilde{u}_0 \\ \tilde{u}_1 \\ \vdots \\ \tilde{u}_{\hat{q}_r+\sigma} \end{bmatrix}, \tag{4.42}$$

where $S(T)$ is a $(\hat{q}_r+\sigma+1)\times(\hat{q}_r+\sigma+1)$-nonsingular matrix. If we now demand that

$$u_2^{[i]}(T)=\beta_{2i}\,,\quad i=0,1,\ldots,\hat{q}_r+\sigma \tag{4.43}$$

then from (4.42) and (4.43) we shall have

$$\begin{bmatrix} \beta_{20} \\ \beta_{21} \\ \vdots \\ \beta_{2,\hat{q}_r+\sigma} \end{bmatrix} = S(T) \begin{bmatrix} \tilde{u}_0 \\ \tilde{u}_1 \\ \vdots \\ \tilde{u}_{\hat{q}_r+\sigma} \end{bmatrix} \Rightarrow \begin{bmatrix} \tilde{u}_0 \\ \tilde{u}_1 \\ \vdots \\ \tilde{u}_{\hat{q}_r+\sigma} \end{bmatrix} = S^{-1}(T) \begin{bmatrix} \beta_{20} \\ \beta_{21} \\ \vdots \\ \beta_{2,\hat{q}_r+\sigma} \end{bmatrix} \tag{4.44}$$

and the values of the $(m \times 1)$-vectors $\tilde{u}_i$, $i = 0, 1, \ldots, \hat{q}_r + \sigma$, can be calculated easily from (4.44) (*end of case B*).

Hence, we have shown how to choose an input $u(t)$ such that

$$\beta^c(T) = \beta \quad \text{for some} \quad T > 0\,, \tag{4.45}$$

i.e.,

$$R(0) \supset [C\ C_\infty] \begin{bmatrix} \langle J/\operatorname{Im}\Omega\rangle & +\sum_{i=0}^{\sigma-1} \operatorname{Im}\Phi_{i+1} \\ \langle J_\infty/\operatorname{Im}\bar{\Omega}\rangle & +\sum_{i=0}^{\sigma-1} \operatorname{Im} Z_i \end{bmatrix}. \tag{4.46}$$

Finally from (4.15) and (4.46) we obtain

$$R(0) = [C\ C_\infty] \begin{bmatrix} \langle J/\operatorname{Im}\Omega\rangle & +\sum_{i=0}^{\sigma-1} \operatorname{Im}\Phi_{i+1} \\ \langle J_\infty/\operatorname{Im}\bar{\Omega}\rangle & +\sum_{i=0}^{\sigma-1} \operatorname{Im} Z_i \end{bmatrix}. \tag{4.47}$$

□

So far we have examined the structure of $R(0)$. We shall now examine the structure of $R(\beta_1)$ with $\beta_1 \equiv \beta^c(0^-) \neq 0 \in \mathbb{R}^r$. To this end consider the following two sets of admissible initial conditions (taken from (3.34)):

(i) AIC with $x_2^c(0^-) = 0 \in \mathbb{R}^n$ and $x_f^c(0^-) \neq 0$, i.e.,

$$H_2 = \left\{ \beta^c(0^-) = [C\ C_\infty] \begin{bmatrix} x_s^c(0^-) \\ x_f^c(0^-) \end{bmatrix} \in \mathbb{R}^r / x_s^c(0^-) = 0 \in \mathbb{R}^n \quad \text{and} \right.$$
$$\left. x_f^c(0^-) \in \sum_{i=0}^{\hat{q}_r} J_\infty^i \operatorname{Im}\bar{\Omega} + \sum_{i=0}^{\sigma-1} \operatorname{Im} Z_i + \operatorname{Ker} J_\infty \right\} \tag{4.48}$$

or equivalently

$$H_2 = \left\{ \beta^c(0^-) = [C\ C_\infty] \begin{bmatrix} x_s^c(0^-) \\ x_f^c(0^-) \end{bmatrix} \right.$$
$$\left. = C_\infty \left[\sum_{i=0}^{\hat{q}_r} J_\infty^i \bar{\Omega} u^{[\sigma+i]}(0^-) + \sum_{i=0}^{\sigma-1} Z_i u^{[i]}(0^-) \right] \right\} \tag{4.49}$$

and

(ii) AIC with $x_s^c(0^-) \neq 0$ and $x_f^c(0^-) = 0$; i.e.,

$$H_3 = \left\{ \beta^c(0^-) = [C\ C_\infty] \begin{bmatrix} x_s^c(0^-) \\ x_f^c(0^-) \end{bmatrix} : x_s^c(0^-) \neq 0 \in \mathbb{R}^n \quad \text{and} \right.$$
$$\left. x_f^c(0^-) = 0 \in \mathbb{R}^\mu \right\} \tag{4.50}$$

or equivalently

$$H_3 = \left\{ \beta^c(0^-) = [C\ C_\infty] \begin{bmatrix} x_s^c(0^-) \\ x_f^c(0^-) \end{bmatrix} = C \left[x_s(0^-) + \sum_{i=0}^{\sigma-1} \Phi_{i+1} u^{[i]}(0^-) \right] \right\} . \tag{4.51}$$

The case $x_s(0^-) \neq 0$ and $x_f(0^-) \neq 0$ is not valid because of Proposition 3.2 and its following Remark 3.3.

The complete set of AIC can be written as follows:

$$H_{Iu} = \left\{ \beta^c(0^-) \in \mathbb{R}^r / \beta^c(0^-) \subset [C\ C_\infty] \begin{bmatrix} \mathbb{R}^n \\ \langle J_\infty / \operatorname{Im} \bar{\Omega} \rangle + \sum_{i=0}^{\sigma-1} \operatorname{Im} Z_i \end{bmatrix} \right\} \tag{4.52}$$

or equivalently from (4.48)–(4.51) and Remark 3.7

$$H_{Iu} = H_2 \cup H_3 \cup \{0\} , \tag{4.53}$$

where $\{0\}$ denotes the zero vector corresponding to $x_s(0^-) \equiv 0$ and to an input $u(t)$ such that $u^{[i]}(0^-) \equiv 0$, $i = 1, 2, \ldots, \hat{q}_r + 1 + \sigma$. Now the complete set of reachable states $\beta^c(T) \in \mathbb{R}^r$ from $\beta_1 \in H_{Iu}$ is

$$\tilde{R} = \bigcup_{\beta_1 \in H_{Iu}} R(\beta_1) , \tag{4.54}$$

which can also be written as

$$\tilde{R} = R(0) = \cup R(\beta_2) \cup R(\beta_3) , \tag{4.55}$$

where $R(0)$ is the set of reachable states from $0 \in H_{Iu}$, $R(\beta_2)$ is the set of reachable states from $\beta_2 \in H_2$; i.e., from all AIC that have $x_s^c(0^-) = 0$ and $x_f^c(0^-) \neq 0$ and

$$R(\beta_3) = \left\{ \beta_3(t) \in H_3 / \beta_3(t) = [C\ C_\infty] \begin{bmatrix} x_s^c(t) \\ x_f^c(t) \end{bmatrix} / x_s^c(t) = e^{Jt} x_s(0^-) \right.$$
$$\left. + \sum_{i=0}^{\sigma-1} \Phi_{i+1} u^{[i]}(t) \in \mathbb{R}^n , \quad x_f(t) \equiv 0 \in \mathbb{R}^\mu \quad \forall t > 0 \right\} \tag{4.56}$$

which represents the free-state reachable set from starting point (state) $\beta_3(0^-) = Cx_s(0^-) + \sum_{i=0}^{\sigma-1} \Phi_{i+1}u^{[i]}(0^-)$. From Theorem 4.6 we have that

$$R(0) = [C\ C_\infty]\begin{bmatrix} \langle J / \operatorname{Im}\Omega\rangle & +\sum_{i=0}^{\sigma-1}\operatorname{Im}\Phi_{i+1} \\ \langle J_\infty / \operatorname{Im}\bar{\Omega}\rangle & +\sum_{i=0}^{\sigma-1}\operatorname{Im}Z_i \end{bmatrix}. \tag{4.57}$$

From the form of $R(\beta_3)$ in (4.56) we have

$$R(\beta_3) \in [C\ C_\infty][\mathbb{R}^n \oplus \{0\}]\,. \tag{4.58}$$

Hence it remains only to find $R(\beta_2)$, where $\beta_2 \in H_2$. We shall show the following:

Proposition 4.7. *Let $\beta_2 \in H_2$ as in (4.49). Then*

$$R(\beta_2) = [C\ C_\infty]\begin{bmatrix} \langle J / \operatorname{Im}\Omega\rangle & +\sum_{i=0}^{\sigma-1}\operatorname{Im}\Phi_{i+1} \\ \langle J_\infty / \operatorname{Im}\bar{\Omega}\rangle & +\sum_{i=0}^{\sigma-1}\operatorname{Im}Z_i \end{bmatrix} \equiv R(0)\,. \tag{4.59}$$

Proof. For any $\beta_2 = \beta^c(0^-) \in H_2$ we have that (see (4.48)) $x_s^c(0^-) = 0 \in \mathbb{R}^n$ and $x_f^c(0^-) \in \sum_{i=0}^{\hat{q}_r} J_\infty^i \operatorname{Im}\bar{\Omega} + \sum_{i=0}^{\sigma-1}\operatorname{Im}Z_i + \operatorname{Ker}J_\infty$. Then from (4.10) we obtain

$$\beta^c(T) \in [C\ C_\infty]\begin{bmatrix} \langle J / \operatorname{Im}\Omega\rangle & +\sum_{i=0}^{\sigma-1}\operatorname{Im}\Phi_{i+1} \\ \langle J_\infty / \operatorname{Im}\bar{\Omega}\rangle & +\sum_{i=0}^{\sigma-1}\operatorname{Im}Z_i \end{bmatrix}. \tag{4.60}$$

Applying Theorem 4.6 we obtain (4.59). □

Taking into account that $\langle J / \operatorname{Im}\Omega\rangle + \sum_{i=0}^{\sigma-1}\operatorname{Im}\Phi_{i+1} \subset \mathbb{R}^n$ and $\{0\} \subset \langle J_\infty / \operatorname{Im}\bar{\Omega}\rangle + \sum_{i=0}^{\sigma-1}\operatorname{Im}Z_i$ from (4.55) and (4.57), (4.58), and (4.59), we obtain that the complete set of reachable states from any $\beta_1 \in H_{Iu}$ is given by

$$\bar{R} = \bigcup_{\beta_1 \in H_{Iu}} R(\beta_1) = [C\ C_\infty]\begin{bmatrix} \mathbb{R}^n \\ \langle J_\infty / \operatorname{Im}\bar{\Omega}\rangle + \sum_{i=0}^{\sigma-1}\operatorname{Im}Z_i \end{bmatrix}. \tag{4.61}$$

Remark 4.8. Taking into account that $\langle J/\operatorname{Im}\Omega\rangle+\sum_{i=0}^{\sigma-1}\operatorname{Im}\Phi_{i+1}\subset\mathbb{R}^n$ we obtain that every point y in R, where

$$R := [C\ C_\infty]\begin{bmatrix} \langle J/\operatorname{Im}\Omega\rangle & +\sum_{i=0}^{\sigma-1}\operatorname{Im}\Phi_{i+1} \\ \langle J_\infty/\operatorname{Im}\bar{\Omega}\rangle & +\sum_{i=0}^{\sigma-1}\operatorname{Im} Z_i \end{bmatrix} \tag{4.62}$$

is reachable (according to Definition (4.1)) from every point x in R.

We have the following definition:

Definition 4.9. The system (2.13) is called *reachable* if every point $\beta_T\in\mathbb{R}^r$ is reachable from every point $\beta_0\in H_{Iu}$.

Proposition 4.10. *The system (2.13) is reachable iff*

$$R=\mathbb{R}^r\ . \tag{4.63}$$

Proof. (i) From Remark (4.8) we have that every point in R is reachable from every other point in R. Now if $R=\mathbb{R}^r$ then in light of Definition (4.9) the system is reachable.

(ii) Let us assume that the system is reachable. Then according to Definition (4.9) every point $\beta_T\in\mathbb{R}^r$ is reachable from a point $\beta_0=H_{Iu}$. Without loss of generality we assume that $\beta_0=0\in H_{Iu}$. Then the set of reachable states from $\beta_0=0$ is $R(0)$ and it holds that $R(0)=\mathbb{R}^r$ because the system is reachable. But from Theorem (4.6) we have that $R(0)=[C\ \ C_\infty]\begin{bmatrix} \langle J/\operatorname{Im}\Omega\rangle & +\sum_{i=0}^{\sigma-1}\operatorname{Im}\Phi_{i+1} \\ \langle J_\infty/\operatorname{Im}\bar{\Omega}\rangle & +\sum_{i=0}^{\sigma-1}\operatorname{Im} Z_i \end{bmatrix}$ and it is clearly equal to R. Hence, $R=\mathbb{R}^r$. □

Definition 4.11. The set R as given in (4.62) is called a *reachable subspace* of the system (2.13).

Now we shall give some useful reachability tests for PMDs, which are natural extensions of the corresponding tests for generalized state space systems. We start with the following. Let the subspace

$$R_s := \langle J/\operatorname{Im}\Omega\rangle+\sum_{i=0}^{\sigma-1}\operatorname{Im}\Phi_{i+1}\subset\mathbb{R}^n\ , \tag{4.64}$$

where R_s is spanned by the linearly independent columns of the matrix:

$$Q_s=[\Omega\ ,\ J\Omega\ ,\ldots,\ J^{n-1}\Omega\ ,\ \Phi_1\ ,\ \Phi_2\ ,\ldots,\ \Phi_\sigma]\in\mathbb{R}^{n\times(n+\sigma)m}\ . \tag{4.65}$$

Let also the subspace

$$R_f := \langle J_\infty / \operatorname{Im} \bar{\Omega} \rangle + \sum_{i=0}^{\sigma-1} \operatorname{Im} Z_i \subset \mathbb{R}^\mu , \tag{4.66}$$

where R_f is spanned by the linearly independent columns of the matrix:

$$Q_f = [\bar{\Omega} \,,\; J_\infty \bar{\Omega} \,, \ldots ,\; J_\infty^{\hat{q}_r} \bar{\Omega} \,,\; Z_0 \,,\; Z_1 \,, \ldots ,\; Z_{\sigma-1}] \in \mathbb{R}^{\mu \times (\hat{q}_r + 1 + \sigma) m} . \tag{4.67}$$

From the form of R in (4.62) and (4.64)–(4.67) we have the following definition:

Definition 4.12. The reachable subspace R is spanned by the linearly independent columns of the matrix

$$Q = [C \; C_\infty] \begin{bmatrix} Q_s & | & 0 \\ & | & \\ 0 & | & Q_f \end{bmatrix} \mathbb{R}^{r \times (n + \hat{q}_r + 1 + 2\sigma) m} , \tag{4.68}$$

which is called the *pseudo-state reachability matrix* of (2.13).

Combining (4.68) with Proposition 4.10 we can state the obvious:

Theorem 4.13. *Every* $\beta_T \in \mathbb{R}^r$ *is* reachable *iff*

$$R \equiv \mathbb{R}^r \tag{4.69}$$

or equivalently

$$\operatorname{rank}[Q] = r . \tag{4.70}$$

Remark 4.14. We have the following:

$$[C \; C_\infty] \in \mathbb{R}^{r \times (n+\mu)} \quad \text{and} \quad \operatorname{rank}[C \; C_\infty] = r \tag{4.71}$$

$$n + \mu = r + \sum_{i=1}^{k} (q_i - 1) . \tag{4.72}$$

Hence generally it holds that

$$n + \mu > r . \tag{4.73}$$

From Theorem 4.13 (see equation (4.70) and Remark 4.14 (see relation (4.73)) we can state the following:

Corollary 4.15. *The system (2.13) is* reachable *iff*

$$\operatorname{rank}[C \; C_\infty] = r \tag{4.74}$$

and

$$\operatorname{rank} \begin{bmatrix} Q_s & | & 0 \\ & | & \\ 0 & | & Q_f \end{bmatrix} \geq r . \tag{4.75}$$

Remark 4.16. If the case of internally proper PMDs [17] the matrix $J_\infty \in 0_{\mu,\mu}$ and we obtain

$$\Phi_j = 0 \quad \text{for} \quad j = 2, 3, \ldots, \sigma \quad \text{and} \quad Z_k = 0 \quad \text{for} \quad k = 1, 2, \ldots, \sigma - 1 .$$

As a consequence the reachable subspace can be written:

$$R_{\text{int}} = \langle J / \operatorname{Im} \Omega \rangle + \operatorname{Im} E \ , \tag{4.76}$$

where

$$E := C\Phi_1 + C_\infty Z_0 \in \mathbb{R}^{r \times m} . \tag{4.77}$$

The reachable subspace R_{int} is spanned by the linearly independent columns of the following matrix:

$$Q_{\text{int}} = [C\Omega \ , \ CJ\Omega \ , \ldots , \ CJ^{n-1}\Omega \ , \ E] \in \mathbb{R}^{r \times (n+1)m} . \tag{4.78}$$

The above results first appeared in [26]. Analytical expressions for various forms of internally proper PMDs as well as controllability/observability and other structural properties of internally proper PMDs are given explicitly in [14].

5. Illustrative example

Let $A(s) = \begin{bmatrix} s+1 & s^2 \\ 0 & 1 \end{bmatrix}$ be a polynomial matrix with Smith–McMillan form at $s = \infty : S^{\infty}_{A(s)}(s) = \begin{bmatrix} s^2 & 0 \\ 0 & 1/s \end{bmatrix}$ and $r = 2, n = 1, \mu = 2$; hence $n + \mu = 1 + 2 = 3 > 2 = r$.

Let also $C = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $J = [-1]$, $B = [1 \ \ -1]$ be a minimal realization of the strictly proper part of $A^{-1}(s)$ and $C_\infty = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$, $J_\infty = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B_\infty = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$ be a minimal realization of the polynomial part of $A^{-1}(s)$. Then

$$\operatorname{rank}[C \ C_\infty] = \operatorname{rank} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = 2 = r .$$

Hence the first condition (4.74) of Corollary 4.15 holds true.

Case A.

Let $B(s) = B_0 + B_1 s = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} s = \begin{bmatrix} s+1 & 0 \\ 0 & s+1 \end{bmatrix}$ i.e., $\sigma = 1$.

Then

$$\Omega = JBB_1 + BB_0 = [0\,0] \ , \quad \Phi_1 = BB_1 = [1, \ -1]$$

$$\bar{\Omega} = B_\infty B_1 + J_\infty B_\infty B_0 = \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix} , \quad Z_0 = B_\infty B_0 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

(i) $$\operatorname{rank}[Q_s] = \operatorname{rank}[\Omega, \ \Phi_1] = \operatorname{rank}[0, \ 0, \ 1, \ -1] = 1$$

$$\text{(ii)}\operatorname{rank}[Q_f] = \operatorname{rank}[\bar{\Omega},\ J_\infty\bar{\Omega},\ Z_0] = \operatorname{rank}\begin{bmatrix} 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 \end{bmatrix} = 2$$

and

$$\operatorname{rank}\left[\begin{array}{c|c} Q_s & \\ & \\ 0 & Q_f \end{array}\right] = \operatorname{rank}\left[\begin{array}{cccc|cccccc} 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \end{array}\right]$$
$$= 3 > r\ ,$$

i.e., the system is *reachable* according to Corollary 4.15.

Case B.

Let $B(s) = B_0 + B_1 s = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} s = \begin{bmatrix} s \\ 0 \end{bmatrix}$ i.e., $\sigma = 1$. Then

$$\Omega = JBB_1 + BB_0 = [-1]\ , \quad \Phi_1 = BB_1 = [1]$$

$$\bar{\Omega} = B_\infty B_1 + J_\infty B_\infty B_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad Z_0 = B_\infty B_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{(i)} \qquad \operatorname{rank}[Q_s] = \operatorname{rank}[\Omega,\ \Phi_1] = \operatorname{rank}[-1,\ 1] = 1$$

$$\text{(ii)} \qquad \operatorname{rank}[Q_f] = \operatorname{rank}[\bar{\Omega},\ J_\infty\bar{\Omega},\ Z_0] = \operatorname{rank}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

and

$$\operatorname{rank}[Q] = \operatorname{rank}\left[\begin{array}{c|c} Q_s & \\ & \\ 0 & Q_f \end{array}\right] = \operatorname{rank}\left[\begin{array}{cc|ccc} -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] = 1 < r\ ,$$

i.e., the system is *not reachable* because the condition (4.75) does not hold.

6. Conclusion

The concept of reachability for Polynomial Matrix Descriptions (PMDs) has been considered. After generalizing various known results regarding the smooth and impulsive solutions of generalized state space systems (which represent a particular case of PMDs) we have developed a theory regarding reachability properties of PMDs using time-domain analysis. This analysis extends in a general way a number of results previously known only for regular and generalized state space systems.

To be more precise, we note that our definition of reachability (Definition 4.1) was motivated by the works of Yip and Sincovec [35] and Ozcaldiran [21], [22]. We extend the notions of Admissible Initial Conditions (AIC) proposed in [35], [21] in a way to cover the more general case of PMDs. We treat the differential equations that give rise to PMD as in (1.3) using ordinary (regular) derivatives and we generalize the results of [6], [35], evaluating the complete solutions of PMDs (see (3.29)). We remark here that the complete solution of a PMD lacks impulsive components because we assume that the set of the initial conditions of our system belongs to the set of AICs (see (3.33) and (3.34)).

Cobb in his research papers [7]–[11] using time-domain analysis and no admissible initial conditions considers the distributional solution of a singular system of the form (1.1). We extend his theory and show how to cover the more general case evaluating the complete solution of a PMD (see (3.12)–(3.14)) using distributional derivatives. In [8] is also proposed a closed-loop control that eliminates the impulsive components of the solution of generalized state space systems (1.1) using linear feedback. In [21] an open-loop control has been described that achieves the same result as the closed-loop method. We extend this latter method in a way to obtain an open-loop control $u(t)$ such that the complete solutions of a PMD as in (1.3) have no impulsive terms without using linear feedback. We introduce also the notion of a reachable subspace for PMDs and provide a precise form for all the future (reachable) states of our system for $t \geq 0^-$. Furthermore, the proof of Theorem 4.6 shows how the future states of our system may be reached in any short period with a suitably chosen input $u(t) = u_1(t) + u_2(t)$, where $u_1(t)$ is given by (4.24) and $u_2(t)$ is given by (4.28) or (4.37).

Finally, we have to point out that our definition of reachability is an equivalent and natural generalization of the notions of controllability [7], C-controllability [35], and reachability [21]. However, the way that our theory is related to further aspects such as the notions of strong controllability [29], observability, and duality for the case of PMDs are topics for further research.

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