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An Extension of Wolovich's Definition of Equivalence of Linear Systems

N. P. Karampetakis, A. C. Pugh, A. I. Vardulakis, and G. E. Hayton

Abstract— **Wolovich's classical definition of equivalence for linear systems is extended to the generalized study of linear systems. It is shown that the resulting equivalence is an alternative characterization of the notion of full system equivalence underlying its fundamental role in the generalized study of linear systems.**

I. Introduction

The conventional theory of linear systems deals with the finite frequency (exponential and sinusoidal) behavior of such systems. In this theory the transformation of strict system equivalence originally proposed by Rosenbrock [9] plays a central role. This transformation does indeed possess the property of preserving the finite frequency structure of any polynomial matrix description to which it is applied. Another notion of "equivalence," proposed by Wolovich [13], was based on the intuitive idea that two general linear systems should be deemed equivalent in case any state-space reductions of them are related by the usual change of basis in the state space or in system matrix terms, system similarity [9]. Pernebo [5] has shown that strict system equivalence in Rosenbrock's sense is equivalent to the existence of a certain bijective mapping between the sets of solutions to the differential equations describing the systems. A consequence of this proposition was that Wolovich's definition and strict system equivalence are seen as identical notions of equivalence.

The generalized theory of linear systems seeks a more complete study of linear system behavior by considering additionally the possible impulsive motion. This necessitates treating the system's infinite frequency behavior on an equal basis to its finite frequency behavior, and in this respect the above transformations do not suffice since they do not preserve the infinite frequency properties of the system. Within this spirit of an integrated study [1], [7] proposed the transformation of complete system equivalence for generalized state-space (g.s.s.) systems, while [3] proposed the transformation of full system equivalence for general linear systems. Recently, [8] has given a characterization of full system equivalence in the manner of [5], where the existence of a certain bijective mapping between the sets of finite and infinite solutions of the differential equations describing the system is critical. These transformations do indeed have the property of simultaneously preserving the finite and infinite frequency structure of systems to which they are applied.

This paper follows Wolovich [13], and a notion of "equivalence" between two general system descriptions is attributed on the basis of equivalence of their underlying g.s.s. models. This represents a natural extension of the Wolovich idea since g.s.s. systems are the most simple form of system equations which can simultaneously exhibit finite and infinite frequency behavior. The connection between this notion of equivalence and full system equivalence is considered.

Manuscript received April 9, 1993; revised May 19, 1995. This work was supported in part by the Greek National Foundation, the British Council Office of Thessaloniki, and the Loughborough University Research Committee.

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Publisher Item Identifier S 0018-9286(96)00976-2.

II. Preliminary Results

Consider a linear time invariant multivariable system Σ described by

$$A(\rho)\beta(t) = B(\rho)u(t) \tag{1a}$$

$$y(t) = C(\rho)\beta(t) + D(\rho)u(t) \tag{1b}$$

where $\rho = d/dt, A(\rho) \in \mathbf{R}[\rho]^{r\times r}$ with $|A(\rho)| \neq 0, B(\rho) \in \mathbf{R}[\rho]^{r\times m}$, $C(\rho) \in \mathbf{R}[\rho]^{p\times r}, D(\rho) \in \mathbf{R}[\rho]^{p\times m}, \beta(t) : (0-, \infty) \to \mathbf{R}^r$ the pseudostate of Σ , $u(t) : (0\text{-}, \infty) \to \mathbf{R}^m$ the control input, and $y(t)$ the output of Σ, and let its Rosenbrock system matrix be

$$P(s) = \begin{bmatrix} A(s) & B(s) \\ -C(s) & D(s) \end{bmatrix} \in \mathbf{R}[s]^{(r+p)\times(r+m)}. \tag{2}$$

Σ is in generalized state-space g.s.s. form if it takes the form

$$E\dot{x}(t) = Ax(t) + Bu(t) \tag{3a}$$

$$y(t) = Cx(t) + Du(t) \tag{3b}$$

where $E \in \mathbf{R}^{q\times q}$, $A \in \mathbf{R}^{q\times q}$, $B \in \mathbf{R}^{q\times m}$, $C \in \mathbf{R}^{p\times q}$, and $D \in \mathbf{R}^{p\times m}$. Consider the set $P(p, m)$ of $(r+p)\times(r+m)$ polynomial matrices where the integer $r \geq \max\{-p, -m\}$.

Definition 1 [2]: Two matrices $T_1(s), T_2(s) \in P(p, m)$ are said to be fully equivalent (f.e.) in case there exist polynomial matrices $M(s), N(s)$ of appropriate dimensions such that

$$[M(s) \quad T_2(s)]\begin{bmatrix} T_1(s) \\ -N(s) \end{bmatrix} = 0 \tag{4}$$

where the compound matrices in (4) satisfy the following:

1) They have full normal rank. (5a)
2) They have no finite nor infinite zeros. (5b)
3) The following McMillan degree conditions hold:

$$\delta_M([M(s) \quad T_2(s)]) = \delta_M(T_2(s))$$

$$\delta_M\left(\begin{bmatrix} T_1(s) \\ -N(s) \end{bmatrix}\right) = \delta_M(T_1(s)). \tag{5c}$$

□

Let $P_0(p, m)$ be the set of $(r+p)\times(r+m)$ Rosenbrock system matrices (2), then the result is shown below.

Definition 2 [3]: $P_1(s), P_2(s) \in P_0(p, m)$ are said to be full system equivalent (f.s.e.) if $\exists$ polynomial matrices $M(s)$, $N(s)$, $X(s)$, $Y(s)$ s.t. (6), shown at the bottom of the next page, where (6) is an f.e. transformation. □

If $M(s), N(s), X(s), Y(s)$ in (6) are constant and $P_1(s), P_2(s)$ are in g.s.s. form, then $P_1(s)$, $P_2(s)$ are termed completely system equivalent (c.s.e.). If the complete (finite and infinite) solution space of (1) for a fixed control $u(t)$ is denoted by $\mathcal{X}_u$, then a mapping interpretation of f.s.e. and c.s.e. follows.

Definition 3 [1], [8]: Let Σ_1, Σ_2 be two general dynamical systems of the form (1) (respectively, g.s.s. systems) with respective solution spaces $\mathcal{X}_u^1, \mathcal{X}_u^2$ for a given fixed control $u(t)$. Σ_1 and Σ_2 are said to be fundamentally equivalent if and only if the two following conditions hold:

1) If $\beta_i(t)$ is the pseudostate of Σ_i (i=1,2), $\exists$ a bijective map between $\mathcal{X}_u^1$, $\mathcal{X}_u^2$ of the form

$$\begin{pmatrix}\beta_2(t)\\ u(t)\end{pmatrix} = \begin{pmatrix}L(\rho) & W(\rho)\\ 0 & I\end{pmatrix}\begin{pmatrix}\beta_1(t)\\ u(t)\end{pmatrix}$$
$$\left(\begin{pmatrix}\beta_2(t)\\ u(t)\end{pmatrix} = \begin{pmatrix}L & W\\ 0 & I\end{pmatrix}\begin{pmatrix}\beta_1(t)\\ u(t)\end{pmatrix}\right). \tag{7}$$

2) Σ_1 and Σ_2 have the same output. □

Why (7) should be constant in the case of g.s.s. systems will be addressed in the sequel. The formal connection between f.s.e. and fundamental equivalence is stated in Theorem 1.

Theorem 1: Let Σ_1, Σ_2 be two general dynamical systems (respectively, g.s.s. systems). Σ_1, Σ_2 are fundamentally equivalent iff their corresponding Rosenbrock system matrices are f.s.e. (c.s.e.). Further, with f.s.e. of the form (6), then

$$\begin{pmatrix}\beta_2(t)\\ u(t)\end{pmatrix} = \begin{pmatrix}N(\rho) & Y(\rho)\\ 0 & I\end{pmatrix}\begin{pmatrix}\beta_1(t)\\ u(t)\end{pmatrix} \tag{8}$$

is a bijective mapping between $\mathcal{X}_u^1$ and $\mathcal{X}_u^2$.

Proof: See Pugh *et al.* [8]. □

The importance of f.s.e., and hence fundamental equivalence, is that it leaves invariant the finite (exponential, sinusoidal) and impulsive behavior of the systems [3], [4].

III. An Extension of the Wolovich Definition of Equivalence

The extension of the well-known Wolovich definition of equivalence [13] proposed here relates to the complete solution space of the generalized dynamical system (1), not simply its finite solution space. The notion of a normalized form of the system equations, or what is the same thing—the associated normalized system matrix, permits consistent definitions of finite and infinite frequency system properties to be given [11]. Thus it facilitates the integrated study of the finite frequency and impulsive behaviors of the system. The initial definitions given here therefore relate to normalized forms of the representation. Consider the normalized form Σ^N of the system Σ of (1), i.e.,

$$\mathcal{T}(\rho)\xi(t) = \mathcal{U}u(t) \tag{9a}$$
$$y(t) = \mathcal{V}\xi(t) \tag{9b}$$

where

$$\mathcal{T}(\rho) = \begin{bmatrix} A(\rho) & B(\rho) & 0\\ -C(\rho) & D(\rho) & -I_m\\ 0 & I_p & 0\end{bmatrix} \in \mathbf{R}[\rho]^{(r+p+m)\times(r+p+m)}$$
$$\mathcal{U} = \begin{bmatrix}0\\0\\I_p\end{bmatrix} \in \mathbf{R}^{(r+p+m)\times p} \tag{10}$$
$$\mathcal{V} = [0 \quad 0 \quad I_m] \in \mathbf{R}^{m\times(r+p+m)}$$

and

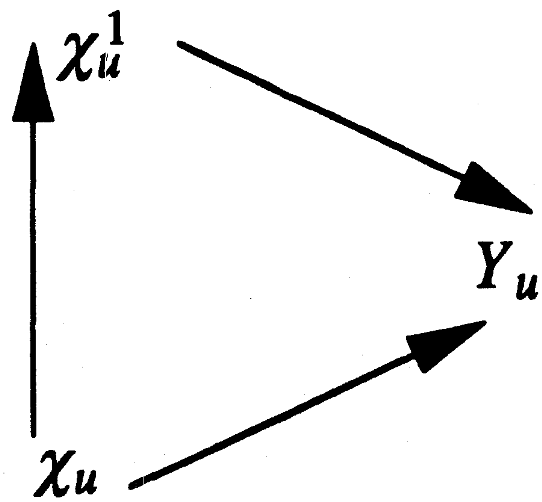

Fig. 1. Y_u is the set of outputs corresponding to u.

$$\xi(t) = [\beta(t)^T, -u(t)^T, y(t)^T]^T. \tag{11}$$

It was shown in [3] that two Rosenbrock system matrices are f.s.e. if and only if their corresponding normalized forms are so related. It can also be shown that any Rosenbrock system matrix is f.s.e. with its normalized form. It should be noted that f.s.e. defines an equivalence relation on $P_0(p,m)$ [12]. Following Wolovich [13], the equivalence of two general dynamical systems in (1) will be defined in two parts. The first step establishes the notion of the equivalence of Σ to a generalized state-space form, while the second step defines the equivalence of two such g.s.s. forms. With regard to the first step we propose the following definition.

Definition 4: Systems (1) and (3) are "equivalent" iff the following hold:

1) There is a constant bijective mapping

$$\begin{bmatrix}\xi(t)\\u(t)\end{bmatrix} = \begin{bmatrix}C_0 & D_0\\ 0 & I\end{bmatrix}\begin{bmatrix}x(t)\\u(t)\end{bmatrix} \tag{12}$$

between the set of solutions $\mathcal{X}_u^1$ and $\mathcal{X}_u$ of (9) and (3) for each $u(t)$.

2) The systems (9) and (3) have the same output for the given $u(t)$. □

Note that the equivalence is defined in terms of Σ^N, the normalized form of Σ, and not directly in terms of Σ itself. Inserting (12) in (9b) we obtain that

$$\begin{aligned} y(t) &= \mathcal{V}(C_0x(t) + D_0u(t)) = \mathcal{V}C_0x(t) + \mathcal{V}D_0u(t)\\ &\equiv Cx(t) + Du(t). \end{aligned} \tag{13}$$

Condition 2) of Definition 4 means that $C \equiv \mathcal{V}C_0$ and $D \equiv \mathcal{V}D_0$ which indicates, on taking into account Condition 1), that Fig. 1 commutes.

Notice that in the Definition 4, as in the original Wolovich definition, (12) is taken to be constant seemingly without any justification. An explanation of why this can be done results from the following lemma.

$$\begin{bmatrix}M(s) & 0\\ X(s) & I\end{bmatrix}\begin{bmatrix}A_1(s) & B_1(s)\\ -C_1(s) & D_1(s)\end{bmatrix} = \begin{bmatrix}A_2(s) & B_2(s)\\ -C_2(s) & D_2(s)\end{bmatrix}\begin{bmatrix}N(s) & Y(s)\\ 0 & I\end{bmatrix} \tag{6}$$

Lemma 1: [8] Consider (1) and the following relation:

$$\begin{pmatrix} \xi_1(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} N(\rho) & Y(\rho) \\ 0 & I \end{pmatrix} \begin{pmatrix} \xi(t) \\ u(t) \end{pmatrix} \tag{14}$$

where $\xi_1(t) : (0-, \infty) \to \mathbf{R}^{q\times(r+p+m)}$. Then a necessary and sufficient condition for (14) to be a map in the formal sense (of being a many-one relation) is

$$\delta_M \begin{pmatrix} A(\rho) & B(\rho) \\ -C(\rho) & D(\rho) \\ N(\rho) & Y(\rho) \\ 0 & I \end{pmatrix} = \delta_M \begin{pmatrix} A(\rho) & B(\rho) \\ -C(\rho) & D(\rho) \end{pmatrix}. \tag{15}$$

□

Based on Lemma 1 we can now prove the following theorem.

Theorem 2: Consider (9) and (3). Let

$$\begin{pmatrix} \xi(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} N(\rho) & Y(\rho) \\ 0 & I \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \tag{16}$$

be a relation between the solution/input space $(\xi(t)^\top, u(t)^\top)^\top$ of (9) and $(x(t)^\top, u(t)^\top)^\top$ of (3), where $N(\rho) = N_q\rho^q + \cdots + N_1\rho + N_0$ and $Y(\rho) = Y_q\rho^q + \cdots + Y_1\rho + Y_0$ (where at least one of N_q, Y_q is nonzero). Then (16) is a map iff it is independent of ρ.

Proof: By Lemma 1, a necessary and sufficient condition for (16) to be a map is shown in (17), at the bottom of the page, for some constant matrix H. Thus

$$\begin{aligned}\begin{pmatrix} \xi(t) \\ u(t) \end{pmatrix} &= \begin{pmatrix} N_0 + HE\rho & Y_0 \\ 0 & I \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \\ &= \begin{pmatrix} N_0 + HE\dot{x}(t) + Y_0u(t) \\ u(t) \end{pmatrix} \\ &= \begin{pmatrix} N_0 + H(Ax(t) + Bu(t)) + Y_0u(t) \\ u(t) \end{pmatrix} \\ &= \begin{pmatrix} N_0 + HA & Y_0 + HB \\ 0 & I \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \end{aligned} \tag{18}$$

and so the theorem is proven. □

Note that Theorem 2 confirms our intuition that the physical system variables $\xi(t)$ are constant linear combinations of the generalized state variables $x(t)$. The theorem also justifies why the definition of fundamental equivalence for g.s.s. systems [1] should be based on constant maps as follows.

Corollary 1: Fundamental equivalence of Definition 3 reduces to the definition of fundamental equivalence for g.s.s. systems given in [1] when the systems are taken to be in g.s.s. form.

Proof: If there exists a bijective map of (16) between two g.s.s. systems, then according to Theorem 2 this map is a constant map as required. □

The solution sets of $\mathcal{X}_0^1$ and $\mathcal{X}_0$ of the homogeneous systems (9a) and (3a) form vector spaces with dimensions equal to the generalized order $f_1 = \delta_M(\mathcal{T}(s))$ and $f = \delta_M(\rho E - A)$ [10], respectively. Equation (12) is then a vector space isomorphism between $\mathcal{X}_0^1$ and $\mathcal{X}_0$, and so in particular it preserves this generalized order, i.e., $\delta_M(\mathcal{T}(s)) = \delta_M(\rho E - A)$. Also, (12) has the property of preserving, obviously, the controllability subspaces of (9) and (3) since it is a bijection between the solution/input pairs of these systems which encode the controllability properties. Less obviously, (12) also preserves the observability subspaces of (9) and (3) (see [8]). It is thus reasonable to call (9) and (3) "equivalent." Additional properties (which relate to Wolovich [13]) arise from the above definition. The first such result establishes invariants of an external nature.

Theorem 3: The "equivalent" (1) and (3) are:

1) partial state/input transfer matrix equivalent;
2) input/output transfer matrix equivalent.

Proof:

1) Laplace transforming (9a) and (3a) and ignoring the initial conditions gives

$$\mathcal{T}(s)\bar{\xi}(s) = \mathcal{U}\bar{u}(s)$$
$$(sE - A)\bar{x}(s) = B\bar{u}(s).$$

Thus from Condition 1) of Definition 4 it follows that

$$\bar{\xi}(s) = C_0\bar{x}(s) + D_0\bar{u}(s) = C_0(sE - A)^{-1}B\bar{u}(s) + D_0\bar{u}(s)$$

but $\bar{\xi}(s) = \mathcal{T}^{-1}(s)\mathcal{U}\bar{u}(s)$, and so necessarily

$$C_0(sE - A)^{-1}B + D_0 \equiv \mathcal{T}^{-1}(s)\mathcal{U}. \tag{19}$$

2) From (9)

$$\bar{y}(s) = \mathcal{V}\mathcal{T}^{-1}(s)\mathcal{U}\bar{u}(s)$$

while from (3)

$$\bar{y}(s) = (C(sE - A)^{-1}B + D)\bar{u}(s).$$

By Condition 2) of Definition 4, these outputs are the same for any given input $\bar{u}(t)$, and so necessarily

$$C(sE - A)^{-1}B + D = \mathcal{V}\mathcal{T}^{-1}(s)\mathcal{U}(= C(s)A^{-1}(s)B(s) + D(s)). \tag{20}$$

□

From an internal point of view we have the following results from Definition 4.

$$\delta_M \begin{pmatrix} \rho E - A & B \\ -C & 0 \\ N(\rho) & Y(\rho) \\ 0 & I \end{pmatrix} = \delta_M \begin{pmatrix} \rho E - A & B \\ -C & 0 \end{pmatrix} \overset{[6]}{\Leftrightarrow}$$

$$\mathrm{rank}_{\mathbf{R}} \begin{pmatrix} E & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ N_q & 0 & \cdots & 0 & Y_q & \cdots & 0 \\ N_{q-1} & N_q & \cdots & 0 & Y_{q-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ N_1 & N_2 & \cdots & N_q & Y_1 & \cdots & Y_q \end{pmatrix} = \mathrm{rank}_R \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} \Leftrightarrow$$

$$N_i = 0, \quad ; \quad Y_i = 0 \quad i = 2, \ldots, q \quad ; \quad N_1 = HE \quad ; \quad Y_1 = 0 \tag{17}$$

Theorem 4:

1) Definition 4 reduces to the definition of c.s.e. when (1), which underlies (9), is in g.s.s. form.
2) The g.s.s. system formed from (3a) and (12), i.e.,

$$E\dot{x}(t) = Ax(t) + Bu(t) \tag{21a}$$
$$\xi(t) = C_0 x(t) + D_0 u(t) \tag{21b}$$

is strongly observable, i.e., $((\rho E - A)^\top \quad C_0^\top)^\top$ has no finite nor infinite zeros.

Proof:

1) Consider the case where (1) is in generalized state-space form, i.e., $A(\rho) = E_1\rho - A_1$, $B(\rho) = B_1$, $C(\rho) = C_1$ and $D(\rho) = D_1$ and so (9) is also in the g.s.s. form

$$\bar{E}\dot{\mathbf{x}}(t) = \bar{A}\mathbf{x}(t) + \bar{B}u(t) \tag{22a}$$
$$y(t) = \bar{C}\mathbf{x}(t) \tag{22b}$$

where

$$\bar{E} = \begin{bmatrix} E_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \bar{A} = \begin{bmatrix} A_1 & -B_1 & 0 \\ C_1 & -D_1 & -I \\ 0 & I & 0 \end{bmatrix}$$
$$\bar{B} = \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}, \quad \bar{C} = [0 \quad 0 \quad I]$$

and

$$\mathbf{x}(t) = [x(t)^T, -u(t)^T, y(t)^T]^T. \tag{23}$$

If (22) and (3) are "equivalent" under Definition 4, then they are fundamentally equivalent under a constant map, and so by Theorem 1, (22) and (3) are c.s.e. Additionally note that (1) is c.s.e. to its normalized form [3], and so from the transivity property of c.s.e., (1) and (3) will be c.s.e.
2) If we recall that (12) is a bijective mapping, then (21) is strongly observable [11]. □

To complete the definition of equivalence in the Wolovich manner, it is necessary for every general dynamical system in normalized form to possess an equivalent (in the sense of Definition 4) g.s.s. representation. In fact it is always possible to construct this "equivalent" g.s.s. system as follows.

Theorem 5: Every general dynamical system of (1) has an equivalent (in the sense of Definition 4) g.s.s. system representation.

Proof: Verghese [11] proposed a reduction method which forms a strongly irreducible realization $\{C_\infty, J_\infty, B_\infty\}$ of the denominator matrix of the normalized form (9) s.t.

$$\mathcal{T}(s) = C_\infty(I_\mu - sJ_\infty)^{-1}B_\infty. \tag{24}$$

Consider the g.s.s. system

$$\left[\begin{array}{cc|c} I_\mu - \rho J_\infty & -B_\infty & 0 \\ C_\infty & 0 & \mathcal{U} \\ \hline 0 & \mathcal{V} & 0 \end{array}\right] \begin{bmatrix} z(t) \\ \xi(t) \\ -u(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ y(t) \end{bmatrix}. \tag{25}$$

We shall show that (25) is an "equivalent" model for (1). Note that

$$\xi(t) = [0 \quad I]\begin{bmatrix} z(t) \\ \xi(t) \end{bmatrix} := C_0 \begin{bmatrix} z(t) \\ \xi(t) \end{bmatrix} \tag{26}$$

is a mapping between (9a) and (25). However, the compound matrix $((sE - A)^\top \quad C_0^\top)^\top$ satisfies the McMillan degree condition in (15) and has no finite nor infinite zeros because the realization $\{C_\infty, J_\infty, B_\infty\}$ is strongly irreducible. Thus (26) is an injective mapping. From the form of C_0, (26) is clearly surjective and hence is a bijection. Thus (25) satisfies the first condition of Definition 4 of equivalence. We have also that

$$y(t) = \mathcal{V}\xi(t) \overset{(26)}{=} \mathcal{V}C_0 \begin{bmatrix} z(t) \\ \xi(t) \end{bmatrix} = [0 \quad \mathcal{V}] \begin{bmatrix} z(t) \\ \xi(t) \end{bmatrix} \tag{27}$$

which is the output from the g.s.s. representation in (25). Thus the second condition of equivalence in Definition 4 is also fulfilled, and so the theorem is proven. □

The definition of equivalence given in Definition 4 is a special case of fundamental equivalence of Definition 3, and thus will possess, by Theorem 1, a formulation as an f.e. transformation. However, more than this can be said.

Theorem 6: (1) and (3) are equivalent in the sense of Definition 4 if and only if they are f.s.e.

Proof: ($\Leftarrow$) Suppose (1) and (3) are f.s.e. It has already been noted that any system Σ is f.s.e. to its normalized form Σ^N, and so from the transitivity of f.s.e., (9) and (3) are f.s.e. Hence $\exists$ polynomial matrices $M(\rho), N(\rho), X(\rho), Y(\rho)$ such that

$$\left[\begin{array}{cc|cc} M & 0 & \mathcal{T} & \mathcal{U} \\ X & I & -\mathcal{V} & 0 \end{array}\right] \left[\begin{array}{cc} \rho E - A & B \\ -C & D \\ \hline -N & -Y \\ 0 & -I \end{array}\right] = 0 \tag{28}$$

where (28) is an f.s.e. transformation. According to the McMillan degree conditions on the compound polynomial matrices in (28), we obtain that $Y(\rho) = Y_0$ is a constant matrix and $N(\rho) = N_0 + HE\rho$, and so (28) may be rewritten as

$$\left[\begin{array}{cc|cc} M & 0 & \mathcal{T} & \mathcal{U} \\ X & I & -\mathcal{V} & 0 \end{array}\right] \left[\begin{array}{cc} \rho E - A & B \\ -C & D \\ \hline -N_0 - HE_\rho & -Y_0 \\ 0 & -I \end{array}\right] = 0. \tag{29}$$

Under constant elementary operations this becomes

$$\left[\begin{array}{cc|cc} M - \mathcal{T}H & 0 & \mathcal{T} & \mathcal{U} \\ X + \mathcal{V}H & I & -\mathcal{V} & 0 \end{array}\right] \left[\begin{array}{cc} \rho E - A & B \\ -C & D \\ \hline -N_0 - HA & -Y_0 + HB \\ 0 & -I \end{array}\right] = 0 \tag{30}$$

which, according to Theorem 1, gives that

$$\begin{bmatrix} \xi(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} N_0 + HA & Y_0 - HB \\ 0 & I \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \tag{31}$$

is a bijective mapping. Full system equivalent also has the property of leaving invariant the transfer function matrix, and so (1) and (3) will have the same output which is the second condition of Definition 4.

($\Rightarrow$)If (9) and (3) are equivalent in the sense of Definition 4, then it is obvious that they will be fundamental equivalent according to Definition 3. Hence according to Theorem 1, they are f.s.e. Again since any system is f.s.e. to its normalized form it follows from the transitivity property of f.s.e. that (1) and (3) will be so related. □

Corollary 2: The general dynamical system (1) and g.s.s. system (25) are f.s.e. □

Theorem 7: Two g.s.s. systems are f.s.e. iff they are c.s.e.

Proof: ($\Rightarrow$) In the case where (9) is in g.s.s. form, we obtain from Theorem 6 that if the two g.s.s. systems are f.s.e., then there exists a constant bijective mapping between their solution sets of the form

$$\begin{bmatrix} x_2(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} N & Y \\ 0 & I \end{bmatrix} \begin{bmatrix} x_1(t) \\ u(t) \end{bmatrix} \tag{32}$$

and so the two g.s.s. systems are fundamentally equivalent or by Theorem 1, c.s.e.

($\Leftarrow$) It is obvious that c.s.e. is a special case of f.s.e. □

It is now possible to complete the definition of equivalence between two general systems of (1). Whereas in the original Wolovich definition system similarity [9] plays a key role, in this study it will be the transformation of c.s.e.

Definition 5: Two general dynamical systems Σ_1 and Σ_2 of the form (1) are equivalent if and only if their equivalent g.s.s. systems are c.s.e. □

Theorem 6 may now be extended to include systems in (1) as follows.

Theorem 8: Two general dynamical systems Σ_1 and Σ_2 are equivalent in the sense of Definition 5 if and only if they are f.s.e.

Proof: ($\Rightarrow$) Consider two equivalent general dynamical systems Σ_1 and Σ_2. Then by Definition 5, their equivalent (in the sense of Definition 4) g.s.s. representations S_1, S_2 of the form of (25) are c.s.e. From Theorem 6 it follows that Σ_1 and S_1 (respectively, Σ_2 and S_2) are f.s.e. Further, since c.s.e. is a special case of f.s.e., we have the following relation:

$$\Sigma_1 \overset{\text{f.s.e.}}{\sim} S_1 \overset{\text{f.s.e.}}{\sim} S_2 \overset{\text{f.s.e.}}{\sim} \Sigma_2. \tag{33}$$

Under the transitivity property of f.s.e., Σ_1 and Σ_2 are therefore f.s.e.

($\Leftarrow$) Consider two f.s.e. general dynamical systems Σ_1 and Σ_2, and let S_1, S_2, respectively, be their equivalent (in the sense of Definition 4) g.s.s. systems. By Theorem 6, Σ_1 and S_1 (respectively, Σ_2 and S_2) are f.s.e. and so

$$S_1 \overset{\text{f.s.e.}}{\sim} \Sigma_1 \overset{\text{f.s.e.}}{\sim} \Sigma_2 \overset{\text{f.s.e.}}{\sim} S_2. \tag{34}$$

Thus by the transitivity property of f.s.e., S_1 and S_2 are f.s.e. or further (by Theorem 7) S_1 and S_2 are c.s.e. Thus Σ_1, Σ_2 are equivalent in the sense of Definition 5. □

IV. Conclusions

An extension of the Wolovich definition of equivalence, to encompass the generalized theory of linear systems, has been given. The extension is based on the notion that a general dynamical system has an equivalent g.s.s. reduction. In fact, several reductions are available, but the one selected here is that proposed by Verghese [11]. The basis of the definition is then that two general dynamical systems are equivalent in the case where their g.s.s. reductions are completely system equivalent. Of course in the generalized study of linear systems, the g.s.s. system is seen to play the same role as the state-space model in the conventional study, while complete system equivalence is seen to replace system similarity in this context. Overall the definition is seen to coincide with the previously defined transformation of f.s.e. and so has the property of simultaneously preserving the system's finite and infinite frequency behavior. As such, this extension of the Wolovich notion of equivalence provides some neat explanations of certain features of the transformation of f.s.e. and underlines its important role in the generalized study of linear systems.

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A Family of Nonlinear H^∞-Output Feedback Controllers

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***Abstract*—State-space formulas are derived for a family of controllers solving the nonlinear H^∞-output feedback control problem. The formulas given are expressed in terms of the solutions to two Hamilton–Jacobi inequalities in n independent variables. These controllers are obtained by interconnecting the "central controller" with an asymptotically stable, free system having L^2-gain $\leq \gamma$. All proofs given are simple and clear and provide deeper insight in the synthesis of the corresponding linear H^∞ controllers.**

I. Introduction

Over the past decade, there has been an increasing interest in linear H^∞-control theory since Zames' original work [22] appeared (see, e.g., [7]–[9] and the references quoted therein). An important breakthrough in this line of research was the derivation of state-space solutions to the standard linear H^∞-output feedback control problem in terms of the solutions to two Riccati equations [8]. A parameterization of all H^∞-(sub)optimal output feedback controllers was also given in [8].

Recently, there has been much attention given to the extensions of the results of linear H^∞-control theory to nonlinear settings [2]–[5], [13]–[15], [17], [18], [20]. The notion of dissipativity which is being stressed most recently was first proposed by Willems [19] and later generalized and fully applied to stability analysis of nonlinear systems by Hill and Moylan [10]–[12]. The conditions under which a nonlinear system can be rendered passive rather than dissipative

Manuscript received August 26, 1994; revised April 18, 1995 and August 29, 1995. This work was supported in part by the National Science Council of the Republic of China under Grant NSC-84-2121-M-019-001-MS.

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Publisher Item Identifier S 0018-9286(96)00991-9.