



Technical Communique

A note on the action of constant pseudostate feedback on the internal properness of an ARMA model[☆]

E. N. Antoniou, A. I. G. Vardulakis*

Department of Mathematics, Aristotle University of Thessaloniki, Thessaloniki, 54 006, Greece

Received 20 November 1998; received in final form 3 February 1999

Abstract

In this note we study the effect of constant pseudostate feedback on the internal properness of a linear multivariable system, described by an ARMA model. It is shown that the existence of a constant pseudostate feedback control law which makes the closed-loop system internally proper is equivalent to the absence of decoupling zeros at infinity of the open-loop system, a well-known result from the theory of descriptor systems. © 1999 Elsevier Science Ltd. All rights reserved.

Keywords: ARMA models; Impulsive behaviour; Internal properness; Constant output feedback

1. Introduction

We consider systems described by ARMA models having the form

$$A(\rho)\zeta(t) = B(\rho)u(t), \quad (1.1)$$

where $A(\rho) = \sum_{i=1}^q A_i \rho^i \in R[\rho]^{r \times r}$, $B(\rho) = \sum_{i=1}^p B_i \rho^i \in R[\rho]^{r \times m}$, $\zeta(t)$ is the ‘pseudostate’ vector and $u(t)$ is the input vector, ρ stands either for the differential operator d/dt in the continuous time or the time advance operator $\rho \zeta(t) = \zeta(t+1)$ in the discrete-time case. We assume that Eq. (1.1) is *regular*, i.e. $\det A(\rho) \neq 0$ for almost every ρ , which guarantees the uniqueness of the solution given the initial conditions and the input. The term ‘pseudostate’ is justified by the fact that $\zeta(t)$ can be considered as the vector of internal or ‘latent’ variables of the system (see Willems, 1991).

Consider now Eq. (1.1) together with the pseudostate feedback control law

$$u(t) = K\zeta(t) + v(t), \quad (1.2)$$

where $K \in R^{r \times m}$ and $v(t)$ is a new input. The objective of this note is to derive a necessary and sufficient condition, under which the closed-loop system described by Eqs. (1.1) and (1.2) is *internally proper*.

The significance of *internal properness* of a continuous-time, linear system arises from the fact that its absence gives rise to impulsive behavior, either because of inconsistent initial conditions or due to the presence of discontinuous input signals. In general, impulsive behavior is an undesirable feature for a system. Furthermore, many polynomial design techniques require systems that are already internally proper (see for example Callier and Desoer, 1982; Vardulakis, 1991, Chapter 7).

The action of constant pseudostate (output) feedback K on the (external) properness of the closed-loop transfer function matrix of Eq. (1.1) has been studied in Scott and Anderson (1976), where it is shown that generically there exists a K which makes the closed-loop transfer function matrix proper. Furthermore, in Pugh (1984) conditions for K giving rise to a non-proper closed-loop transfer function have been derived.

The theory of descriptor systems has developed conditions under which a generalized state space system can be made internally proper by the use of state (descriptor) feedback. Such results can be found in Cobb (1980; 1983a, b) and Lewis (1986), where a geometric characterization of impulse controllability and its role on the existence of a state feedback that makes the system ‘impulse free’, is given. For instance, the elimination of

*Corresponding author. E-mail: antoniou@ccf.auth.gr. Tel.: +3031-997951; fax: +3031-997951; avardula@ccf.auth.gr(vardulakis)

[☆]This research is supported by the Greek General Secretariat of Research and Technology under the PENED project. This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Editor Peter Dorato.

impulsive behavior is a necessary step in order to apply the LQ control techniques on a descriptor system in Cobb (1983a). The related problem of controllability at infinity is discussed in Kucera and Zagalak (1988), Yamada and Luenberger (1986) and Verghese et al. (1981). In particular, in Kurcera and Zagalak (1988) it is shown that controllability (finite and infinite) is one of the necessary conditions for arbitrary pole placement via generalized state feedback.

In what follows, R, C denote the field of real and complex numbers, respectively, $R[\rho]$ the ring of polynomials with coefficients in R , $R(\rho)$ the field of real rational functions and $R_{pr}(\rho)$ the ring of (real) proper rational functions. The superscripts in the above symbols denote sets of matrices or vectors having their elements in the corresponding ring or field.

2. Main results

We give first a definition of the McMillan degree of a general rational matrix (see e.g. Vardulakis, 1991):

Definition 1. The McMillan degree of a rational matrix $A(\rho) \in R(\rho)^{p \times m}$ is defined as the total number of poles in $C \cup \{\infty\}$ of $A(\rho)$, i.e.

$$\delta_M(A(\rho)) := \{ \# \text{ of poles in } C \cup \infty \}.$$

We notice that in case $A(\rho)$ is a polynomial matrix, its McMillan degree equals the total number of poles at $\rho = \infty$, since there are no finite poles.

The definition of internal properness given below is a direct consequence of the definitions given in Callier and Desoer (1982, p. 114) where internally proper systems are termed ‘well formed’ or in Vardulakis (1991, p. 240). The above-mentioned definitions are a bit more general since they involve an output vector as well. However, in our case the pseudostate vector can be considered as the output of the system.

Definition 2. System (1.1), is said to be internally proper iff

- (i) for every initial value $\xi(0-)$ and its derivatives $\xi^{(i)}(0-)$, $i = 1, 2, \dots, q - 1$
- (ii) for every ‘impulse free’ input $u(t)$ with $u^{(i)}(0-) = 0$, $i = 0, 1, 2, \dots$ the pseudostate $\xi(t)$ is ‘impulse free’, i.e. it does not contain Dirac impulses $\delta(t)$ and its derivatives $\delta^{(i)}(t)$.

We now give some results regarding the internal properness of Eq. (1.1).

Lemma 3. The following statements are equivalent:

- (i) The system described by Eq. (1.1) is internally proper.

- (ii) $A^{-1}(\rho) \in R_{pr}^{r \times r}(\rho)$, $A^{-1}(\rho)B(\rho) \in R_{pr}^{r \times m}(\rho)$.
- (iii) The polynomial matrix

$$R(\rho) := \begin{bmatrix} A(\rho) & B(\rho) \\ 0 & I_m \end{bmatrix}$$

has no zeros at $\rho = \infty$.

- (iv) $\deg|A(\rho)| = \delta_M[A(\rho), B(\rho)]$.

Proof. The equivalence of statements (i), (ii) and (iv) is a direct consequence of Callier and Desoer (1982, Theorem 52, p. 115), while the equivalence of (ii) and (iii) is established after some trivial manipulations in Vardulakis (1991, Theorem 4.90, p. 240). \square

Condition (ii) in the above lemma states that the properness of the transfer function matrix $A^{-1}(\rho)B(\rho)$ cannot guarantee impulse free behavior for the system. This is due to the fact that even if the input-pseudostate transfer function is proper, there might still be impulsive behavior in the free pseudostate response due to appropriate initial conditions $\xi^{(i)}(0-)$, $i = 0, 1, \dots, q - 1$.

Remark 1. The above lemma can be considered as a generalization of a well-known result from the theory of descriptor systems. If we set $A(\rho) = \rho E - F$ and $B(\rho) = G$ then condition (iv) of Lemma 3 becomes.

The system described by $(\rho E - F)x(t) = Gu(t)$ is internally proper \Leftrightarrow

$$(\rho E - F)^{-1} \in R_{pr}^{r \times r}(\rho) \Leftrightarrow \deg|\rho E - F| = \tag{2.1}$$

$$\delta_M([\rho E - F, G]) = \delta_M(\rho E - F) = \text{rank } E. \tag{2.2}$$

This result occurs in several studies (see for example, Yamada and Luenberger, 1985; Lewis, 1986).

The following definition can be considered as a special case of the definition of input decoupling zeros at infinity of a general polynomial matrix description of a system, which appears in Verghese (1978).

Definition 4. The decoupling zeros at $\rho = \infty$ of (1.1) are the zeros at $\rho = \infty$ of $[A(\rho), \beta(\rho)]$.

Consider now Eq. (1.1) together with the following pseudostate feedback

$$u(t) = K\xi(t) + v(t), \tag{2.3}$$

where $K \in R^{r \times m}$ and $v(t)$ is a new input. Then the closed-loop system is described by

$$[A(\rho) + B(\rho)K]\xi(t) = B(\rho)v(t). \tag{2.4}$$

Definition 5. The pseudostate feedback law K is called admissible iff Eq. (2.4) is regular i.e. iff $\det[A(\rho) + B(\rho)K] \neq 0$ for almost every ρ .

Consider now a left coprime at $\rho = \infty$ proper rational matrix fractional representation of $[A(\rho), B(\rho)]$, i.e. let

$$[A(\rho), B(\rho)] = D(\rho)^{-1} [N_A(\rho), N_B(\rho)], \quad (2.5)$$

where $D(\rho) \in R_{pr}^{r \times r}(\rho)$, $N_A(\rho) \in R_{pr}^{r \times r}(\rho)$, $N_B(\rho) \in R_{pr}^{r \times m}(\rho)$ and $\text{rank} [D(\infty), N_A(\infty), N_B(\infty)] = r$. Then we have

Fact 1. *The zeros at $\rho = \infty$ of $[A(\rho), B(\rho)]$ are the zeros at $\rho = \infty$ of $[N_A(\rho), N_B(\rho)]$ (Vardulakis, 1991).*

Consider also the polynomial matrix

$$R_K(\rho) := \begin{bmatrix} A(\rho) + B(\rho)K & B(\rho) \\ 0 & I_m \end{bmatrix} \\ = \begin{bmatrix} D(\rho) & 0 \\ 0 & I_m \end{bmatrix}^{-1} \begin{bmatrix} N_A(\rho) & N_B(\rho) \\ -K & I_m \end{bmatrix} \begin{bmatrix} I_r & 0 \\ K & I_m \end{bmatrix}. \quad (2.6)$$

It is easy to see that

$$\text{rank} \begin{bmatrix} D(\infty) & 0 & N_A(\infty) & N_B(\infty) \\ 0 & I_m & -K & I_m \end{bmatrix} = r + m$$

so that Eq. (2.6) is a left coprime at $\rho = \infty$ proper rational matrix fractional representation of $R_K(\rho)$ and therefore we have

Fact 2. *The zeros of $R_K(\rho)$ at $\rho = \infty$ are the zeros at $\rho = \infty$ of*

$$N_K(\rho) := \begin{bmatrix} N_A(\rho) & N_B(\rho) \\ -K & I_m \end{bmatrix}.$$

We now state our main result.

Theorem 6. *The following statements are equivalent:*

(i) *There exists an admissible pseudostate feedback as in Eq. (2.3) such that the closed-loop system (2.4) is internally proper*

(ii) $N_K(\rho) := \begin{bmatrix} N_A(\rho) & N_B(\rho) \\ -K & I_m \end{bmatrix}$

has no zeros at $\rho = \infty$

(iii) *System (1.1) has no decoupling zeros at $\rho = \infty$.*

Proof. (i) \Rightarrow (ii): Assume that there exists a K as in (i). Then from Lemma 3, $R_K(\rho)$ has no zeros at $\rho = \infty$ which in view of Fact 2 implies (ii).

(ii) \Rightarrow (iii): $N_K(\rho)$ has no zeros at $\rho = \infty$ implies $\text{rank} N_K(\infty) = r + m$ which in turn implies that $\text{rank} [N_A(\infty), N_B(\infty)] = r$ which due to Fact 1 implies that $[A(\rho), B(\rho)]$ has no zeros at $\rho = \infty$ or from Definition 4 implies (iii).

(iii) \Rightarrow (i): Assume that the system (1.1) has no decoupling at $\rho = \infty$ or equivalently that $[A(\rho), B(\rho)]$

has no zeros at $\rho = \infty$. From fact 1 this implies that $[N_A(\rho), N_B(\rho)]$ has no zeros at $\rho = \infty$ or that $\text{rank} [N_A(\infty), N_B(\infty)] = r$ or equivalently that $\text{rank} [sI_r - N_A(\infty), N_B(\infty)] = r$, for $s = 0$. In other words, the pair of matrices $(sI_r - N_A(\infty), N_B(\infty))$ has no input decoupling zeros at $s = 0$. This guarantees the existence of an appropriate (state) feedback K which assigns the (possible) zero eigenvalues of $N_A(\infty)$ to any arbitrary position in the C -plane, i.e. such that $\det [N_A(\infty) + N_B(\infty)K] \neq 0$. Now, it is easy to see that $\det N_K(\infty) = \det [N_A(\infty) + N_B(\infty)K] \neq 0$. Thus, there exists a K such that $\text{rank} N_K(\infty) = r + m$, i.e. $N_K(\rho) \in R_{pr}^{(r+m) \times (r+m)}(s)$ is biproper which implies:

- (a) $\det N_K(\rho) \neq 0$ for almost every ρ which implies that $\det R_K(\rho) \neq 0$ for almost every ρ which, from Eq. (2.6), implies that $\det [A(\rho) + B(\rho)K] \neq 0$ for almost every ρ i.e. closed-loop system (2.4) is regular or equivalently that the pseudostate feedback law K is admissible and
- (b) $R_K(\rho)$ has no zeros at $\rho = \infty$, which from Lemma 3(iii) implies that system (1.1) is internally proper. \square

The proof of the above theorem suggests a way to obtain K

- Calculate a coprime at $\rho = \infty$ proper rational matrix fractional representation of $[A(\rho), B(\rho)]$ as in Eq. (2.5).
- Find a K such that $\det [N_A(\infty) + N_B(\infty)K] \neq 0$.

We illustrate this result via the following:

Example 7. Consider the system described by

$$\underbrace{\begin{bmatrix} \rho^2 - 1 & \rho + 1 \\ \rho - 2 & 1 \end{bmatrix}}_{A(\rho)} \xi(t) = \underbrace{\begin{bmatrix} \rho - 1 \\ \rho^2 \end{bmatrix}}_{B(\rho)} u(t). \quad (2.7)$$

Since

$$A(\rho)^{-1} = \begin{bmatrix} \frac{1}{\rho + 1} & -1 \\ -\frac{\rho - 2}{\rho + 1} & \rho - 1 \end{bmatrix} \notin R_{pr}^{2 \times 2}(\rho),$$

$$A(\rho)^{-1} B(\rho) = \begin{bmatrix} -\frac{(\rho^3 + \rho^2 - \rho + 1)}{\rho + 1} \\ \frac{(\rho - 1)(\rho^3 + \rho^2 - \rho + 2)}{\rho + 1} \end{bmatrix} \notin R_{pr}^{2 \times 1}(\rho).$$

the system is not internally proper. The Smith-McMillan form of $[A(\rho), B(\rho)]$ at $\rho = \infty$ is

$$S_{[A(\rho), B(\rho)]}^\infty = \begin{bmatrix} \rho^2 & 0 & 0 \\ 0 & \rho^2 & 0 \end{bmatrix}. \quad (2.8)$$

Obviously $[A(\rho), B(\rho)]$ has no zeros at $\rho = \infty$, i.e. Eq. (2.7) has no decoupling zeros at $\rho = \infty$, which implies the existence of a constant feedback $K = [k_1, k_2]$ such that the closed-loop system (2.4) is internally proper. We calculate a coprime at $\rho = \infty$ proper rational matrix fractional representation of $[A(\rho), B(\rho)]$ with

$$D(\rho) = \begin{bmatrix} 1/\rho^2 & 0 \\ 0 & 1/\rho^2 \end{bmatrix},$$

$$[N_A(\rho), N_B(\rho)] = \begin{bmatrix} 1 - \frac{1}{\rho^2} & \frac{(\rho+1)}{\rho^2} & \frac{\rho-1}{\rho^2} \\ \frac{\rho-2}{\rho^2} & \frac{1}{\rho^2} & 1 \end{bmatrix}. \quad (2.9)$$

Now, it is enough to find a K such that

$$\det[N_A(\infty) + N_B(\infty)K] = \det \begin{bmatrix} 1 & 0 \\ k_1 & k_2 \end{bmatrix} \neq 0. \quad (2.10)$$

Thus, any K with $k_2 \neq 0$ can make the closed-loop system internally proper. For simplicity, choose $k_1 = 0$ and $k_2 = 1$. With this feedback closed-loop system (2.4) is given by

$$\begin{bmatrix} \rho^2 - 1 & 2\rho \\ \rho - 2 & 1 + \rho^2 \end{bmatrix} \zeta(t) = \begin{bmatrix} \rho - 1 \\ \rho^2 \end{bmatrix} v(t) \quad (2.11)$$

and thus from the facts that

$$[A(\rho) + B(\rho)K]^{-1} =$$

$$\begin{bmatrix} \frac{1 + \rho^2}{\rho^4 - 1 - 2\rho^2 + 4\rho} & \frac{-2\rho}{\rho^4 - 1 - 2\rho^2 + 4\rho} \\ \frac{\rho - 2}{\rho^4 - 1 - 2\rho^2 + 4\rho} & \frac{\rho^2 - 1}{\rho^4 - 1 - 2\rho^2 + 4\rho} \end{bmatrix} \in R_{\text{pr}}^{2 \times 2}(\rho),$$

$$[A(\rho) + B(\rho)K]^{-1}B(\rho) =$$

$$\begin{bmatrix} \frac{-\rho + 1 + \rho^3 + \rho^2}{\rho^4 - 1 - 2\rho^2 + 4\rho} \\ \frac{-2\rho^2 + 3\rho - 2 + \rho^4}{\rho^4 - 1 - 2\rho^2 + 4\rho} \end{bmatrix} \in R_{\text{pr}}^{2 \times 1}(\rho).$$

Eq. (2.11) is internally proper. \square

3. Conclusions

In this note we have proposed a method for the elimination of the undesirable impulsive behavior of a linear system described by an ARMA representation, using constant pseudostate feedback. It has been shown that a necessary and sufficient condition for the existence of such a feedback, is the absence of decoupling zeros at infinity of the open-loop system. This condition appears to be a generalization of known results from the theory of descriptor systems and particularly the ones regarding the impulse controllability of a generalized state space system.

References

- Callier, F. M., & Desoer, C. A. (1982). *Multivariable feedback systems*, New York: Springer.
- Cobb, D. (1980). *Descriptor variable and generalized singularly perturbed systems: A geometric approach*. Ph.D. Thesis, University of Illinois at Urbana-Champaign.
- Cobb, D. (1983a). Descriptor variable systems and optimal state regulation. *IEEE Transactions on Automatic Control*, AC-28(5), 601–611.
- Cobb, D. (1983b). Controllability, observability and duality in singular systems. *IEEE Transactions on Automatic Control*, AC-29(12), 1076–1082.
- Kucera, V., & Zagalak, P. (1988). Fundamental theorem of state feedback for singular systems. *Automatica*, 24(5), 653–658.
- Lewis, F. L. (1986). A survey of linear singular systems. *Circuits, Systems & Signal Processing*, 5(1).
- Pugh, A. C. (1984). The occurrence of non-properness in closed-loop systems and some implications. In S.G. Tzefestas, *Multivariable Control – New Concepts and tools*. Dordrecht: D. Reidel Publishing Company. (pp. 43–63).
- Scott, R. W., & Anderson, B. D. O. (1976). Comments on conditions for a feedback transfer matrix to be proper. *IEEE Transactions Automatic Control*, 632–634.
- Vardulakis, A. I. G. (1991). *Linear multivariable control — algebraic analysis and synthesis methods*. New York: Wiley.
- Verghese, G. C. (1978). *Infinite frequency behaviour in dynamical systems*. Ph.D. Dissertation, Department of Electrical Engineering, Stanford University.
- Verghese, G. C., & Levy, B. C., Kailath, T. (1981). A generalized state-space for singular systems. *IEEE Transactions on Automatic Control*, AC-26(4), 811–831.
- Willems, J. C. (1991) Paradigms and puzzles in the theory of dynamical systems. *IEEE Transactions Automatic Control*, 36(3), 259–294.
- Yamada, T., & Luenberger, D. G. (1985). Generic controllability theorems for descriptor systems. *IEEE Transactions on Automatic Control*, AC-30(2), 144–152.