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On the Division of Polynomial Matrices

by
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Abstract. The main purpose of this paper is to determine two new algorithms for the division of the polynomial matrix $B(s)\in\mathbb{R}[s]^{p\times q}$ by $A(s)\in\mathbb{R}[s]^{p\times p}$ a) based on the Laurent matrix expansion at $s=\infty$ of the inverse of $A(s)$ i.e. $A(s)^{-1}$ and b) in a similar way to the one presented in [4].

1. Introduction.

Let $A(s)\in\mathbb{R}[s]^{p\times p}$ $(A(s)\in\mathbb{R}[s]^{q\times q})$ and $B(s)\in\mathbb{R}[s]^{p\times q}$ be regular i.e.

$$B(s) = B_0 s^m + B_1 s^{m-1} + \cdots + B_m \in \mathbb{R}[s]^{p\times q} \quad (1a)$$

$$A(s) = A_0 s^n + A_1 s^{n-1} + \cdots + A_n \in \mathbb{R}[s]^{q\times q} (\in \mathbb{R}[s]^{p\times p}) \quad (1b)$$

with $A_0 \neq 0_{p\times p}$, $B_0 \neq 0_{p\times q}$ and $m \geq n$.

Definition 1. [4] The matrix polynomials $Q(s)\in\mathbb{R}[s]^{p\times q}$ $(\hat{Q}(s)\in\mathbb{R}[s]^{p\times q})$ and $R(s)\in\mathbb{R}[s]^{p\times q}$ $(\hat{R}(s)\in\mathbb{R}[s]^{p\times q})$ are the left (right) *quotient* and left (right) *remainder*, respectively, of B(s) on the left (right) division by A(s) if

$$B(s) = A(s)\, Q(s) + R(s) \quad (2a)$$

$$(B(s) = \hat{Q}(s)\, A(s) + \hat{R}(s)) \quad (2b)$$

and $A(s)^{-1}R(s)$ $(\hat{R}(s)A(s)^{-1})$ vanishes at $s=\infty$ (if A(s) is row (column) reduced then an equivalent condition is that the degree of the ith row (column) of R(s) is less than the respective ith row (column) degree of A(s)). The above division is always possible and unique. □

The problem of the determination of the quotient and the remainder of the above division was the main point of interest in a large number of recent papers [2],[5]–[8], because of the large number of its applications in linear system theory. [2], [6]–[8] use various forms of the polynomial matrix A(s) useful in the determination of Q(s) and R(s) in a recursive way. The division of a polynomial matrix B(s) by the matrix pencil (sE–A) has been studied by [5] in a complete different form based on the form of the Laurent expansion at $s=\infty$ of the matrix $(sE-A)^{-1}$. An extension of the results presented in [5] and thus of the generalised Bezout theorem for the case where both polynomial matrices are in general form is proposed in section 2. This result gives rise a) to a relative Cayley Hamilton theorem for polynomial matrices in terms of the fundamental matrix sequence $\{H_k\}$ of $A(s)^{-1}$ and b) to a finite expression for the relative resolvent matrix of $A(s)^{-1}$ in terms of the Tschirnhausen polynomials. An extension of the Generalised Bezout theorem for polynomial matrices is also presented in the same section in a quite similar way to the one presented in [4]. Without loss of generality we study in the sequel the left division of polynomial matrices i.e.when the left quotient and the left remainder are to be found.

2. Polynomial matrix division and other results.

Consider the polynomial matrices A(s), B(s) defined in (1). Then a first way for determining a left quotient Q(s) and left remainder R(s) is given by the following

Theorem 1. [9, p.37] When the polynomial matrix B(s) is divided on the left by the polynomial matrix A(s) then the left quotient Q(s) and the left remainder R(s) are given respectively by

$$Q(s) = \text{polynomial part of } A(s)^{-1}B(s) \quad (3a)$$

$$R(s) = B(s) - A(s)\, Q(s) \quad (3b)$$

Similar results holds for the right division of polynomial matrices. □

Regularity of the polynomial matrix A(s) implies the existence of the unique Laurent expansion

$$A(s)^{-1} = \sum_{k=-\mu}^{\infty} H_k\, s^{-k} \quad (4)$$

at $s=\infty$, with μ the greatest order of the infinite zeros of A(s) [9, p.196]. A method for finding the *relative fundamental matrix* $\{H_k\}$ in terms of the coefficient matrices of A(s) is given in [3].

Defining

$$B[H_k] = H_kB_0+H_{k-1}B_1+\cdots+H_{k-m}B_m \quad (5)$$

$$A[H_k] = A_0H_k+A_1H_{k-1}+\cdots+A_nH_{k-n} \quad (6a)$$

$$A^T[H_k] = H_kA_0+H_{k-1}A_1+\cdots+H_{k-n}A_n \quad (6b)$$

$$A[B[H_k]]=A_0B[H_k]+A_1B[H_{k-1}]+\cdots+A_nB[H_{k-n}] \quad (7)$$

$$B[A[H_k]]=A[H_k]B_0+A[H_{k-1}]B_1+\cdots+A[H_{k-m}]B_m \quad (8)$$

we have the following :

Lemma 1.

a) $A[B[H_k]] = B[A[H_k]]$

b) $A[H_i]=\begin{cases} 0 & i\neq n \\ I_q & i=n \end{cases} \quad A^T[H_i]=\begin{cases} 0 & i\neq n \\ I_q & i=n \end{cases}$

Proof.

a) $A[B[H_k]]=A_0[H_kB_0+H_{k-1}B_1+\cdots+H_{k-m}B_m] +$

$+ A_1[H_{k-1}B_0+H_{k-2}B_1+\cdots+H_{k-m-1}B_m] +$

$+ \cdots\cdots +$

$+A_n[H_{k-n}B_0+H_{k-n-1}B_1+\cdots+H_{k-m-n}B_m]=$

$= [A_0H_k+A_1H_{k-1}+\cdots + A_nH_{k-n}]\, B_0 +$

$+ [A_0H_{k-1}+A_1H_{k-2}+\cdots+A_nH_{k-n-1}]B_1 +$

$+ \cdots\cdots +$

$+[A_0H_{k-m}+A_1H_{k-m-1}+\cdots+A_nH_{k-m-n}]B_m=$

$= B[A[H_k]]$

b) The result can be proved by equating the coefficient powers of s of the left and right term in the equation $A(s)A(s)^{-1}=I_q(A(s)^{-1}A(s)=I_q)$.□

We can thus state the *relative Bezout theorem*

Theorem 2. When a matrix polynomial B(s) is divided on the left by the regular polynomial matrix A(s), the quotient and the remainder are given respectively by

$$Q(s) = \sum_{i=-\mu}^{m} B[H_i]s^{m-i} \quad (9a)$$

and

$$R(s) = \sum_{i=0}^{n-1}\left[\sum_{j=0}^{i} A_jB[H_{m+i+1-j}]\right]s^{n-1-i} \quad (9b)$$

Proof. Consider the difference :

$B(s) - A(s)Q(s) = (B_0s^m+B_1s^{m-1}+\cdots+B_m)$

$-(A_0s^n+A_1s^{n-1}+\cdots+A_n)([H_{-\mu}B_0]s^{\mu+m}+\cdots+$

$+[H_mB_0+H_{m-1}B_1+\cdots+H_0B_m]) =$

$=(B_0s^m+B_1s^{m-1}+\cdots+B_m)-\Big[A[B[H_{-\mu}]]s^{\mu+m+n}$

$+A[B[H_{-\mu+1}]]s^{\mu+m+n-1}+\cdots+A[B[H_{m+n}]]-$

$$-\sum_{i=0}^{n-1}\left[\sum_{j=0}^{i} A_jB[H_{m+i+1-j}]\right]s^{n-1-i}\Big]= \quad (10)$$

(Lemma 1a)

$= \quad (B_0s^m+B_1s^{m-1}+\cdots+B_m)-$

$-\Big[B[A[H_{-\mu}]]s^{\mu+m+n}+\cdots+ B[A[H_{m+n}]] -$

$$-\sum_{i=0}^{n-1}\left[\sum_{j=0}^{i} A_jB[H_{m+i+1-j}]\right]s^{n-1-i}\Big]$$

However from Lemma 1 (b), we have that :

$$B[A[H_{-\mu}]]=B[A[H_{-\mu+1}]]=\cdots=B[A[H_{n-1}]] = 0$$

$$B[A[H_{n+i}]] = B_{m-i} \quad \text{for } i=0,1,..,m \quad (11)$$

Thus from (10) and (11)

$R(s)=(B_0s^m+B_1s^{m-1}+\cdots+B_m)-\Big[B_0s^m+B_1s^{m-1}$

$$+\cdots+B_m-\sum_{i=0}^{n-1}\left[\sum_{j=0}^{i} A_jB[H_{m+i+1-j}]\right]s^{n-1-i}\Big]$$

$$= \sum_{i=0}^{n-1}\left[\sum_{j=0}^{i} A_jB[H_{m+i+1-j}]\right]s^{n-1-i} \quad (12)$$

We can check from (4) and (9b) that

polynomial part $[\, A(s)^{-1}R(s)\,] =$

$= H_{-\mu}\, A_0\, B[H_{m+1}]\, s^{\mu+n-1} +$

$+[H_{-\mu}\, [A_0B[H_{m+2}]+A_1B[H_{m+1}]] + H_{-\mu+1}\, A_0$

$B[H_{m+1}]]s^{\mu+n-2}+\cdots+[H_0[A_0B[H_{m+n}]+\cdots$

$+A_{n-1}B[H_{m+1}]] + \cdots + H_{n-1}\, A_0B[H_{m+1}]] =$

$= \{[H_{-\mu}A_0]\, B[H_{m+1}]\}\, s^{\mu+n-1} +$

$+ \{[H_{-\mu}A_1+H_{-\mu+1}\, A_0]B[H_{m+1}] +$

$+ [H_{-\mu}A_0]B[H_{m+2}]\}\, s^{\mu+n-2} +\cdots\cdots+$

$+ \{[H_0A_{n-1}+\cdots+H_{n-1}\, A_0]B[H_{m+1}]$

(6b)

$+\cdots+[H_0A_0]B[H_{m+n}]\} =$

$= A^T[H_{-\mu}]\, B[H_{m+1}]\, s^{\mu+n-1} +$

$+\{A^T[H_{-\mu+1}]B[H_{m+1}]+A^T[H_{-\mu}]\times$

$\times B[H_{m+2}]\} s^{\mu+n-2}+\cdots+\{A^T[H_{n-1}]B[H_{m+1}]+$

Lemma 1b

$+\cdots+A^T[H_0]B[H_{m+1}]\} \quad = \quad 0$

and thus $A(s)^{-1}R(s)$ vanishes at $s=\infty$. □

Corollary 1. In case where $A(s)=sE-A\equiv A_0s+A_1$ with det[E] not necessary equal to zero the left remainder on the division of B(s) by A(s) is according to Theorem 2 equal to

$$R(s) = A_0B[H_{m+1}] = EB[H_{m+1}] \qquad (13)$$

which coincides with the results in [5]. □

Corollary 2. The above Theorem is independent of the regularity of the coefficient matrix A_0 i.e. we may have a polynomial matrix A(s) as in (1) with $\det[A_0]=0$. In case however where $\det[A_0]\neq 0$ the leading coefficient matrix in the Laurent expansion of $A(s)^{-1}$ is not $-\mu$ but n [1] and thus the left quotient and remainder in the above devision are given respectively by

$$Q(s) = \sum_{i=n}^{m} B[H_i]s^{m-i} \quad (14a)$$

and

$$R(s)= \sum_{i=0}^{n-1}\left[\sum_{j=0}^{i} A_jB[H_{m+i+1-j}]\right]s^{n-1-i} \quad (14b) \; □$$

Corollary 3. A(s) is a left divisor of B(s) iff the left remainder of the division of B(s) by A(s) is zero or equivalently iff

$$R(s)= \sum_{i=0}^{n-1}\left[\sum_{j=0}^{i} A_jB[H_{m+i+1-j}]\right]s^{n-1-i}\equiv 0 \quad (15)$$

or equivalently iff the coefficients of the powers of s in (15) are equal to zero i.e.

$$\begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{n-1} & A_{n-2} & \cdots & A_0 \end{bmatrix} \times \begin{bmatrix} H_{m+1} & H_m & \cdots & H_1 \\ H_{m+2} & H_{m+1} & \cdots & H_2 \\ \vdots & \vdots & \ddots & \vdots \\ H_{m+n} & H_{m+n-1} & \cdots & H_n \end{bmatrix} \begin{bmatrix} B_0 \\ B_1 \\ \vdots \\ B_m \end{bmatrix} = 0 \quad (17)$$

Thus (17) is a necessary and sufficient condition for A(s) to be a left divisor of B(s). A similar statement holds for division on the right by A(s). In case where A_0 is nonsingular then the first matrix of the left term in (17) is nonsingular and thus (17) is equivalent to :

$$\begin{bmatrix} H_{m+1} & H_m & \cdots & H_1 \\ H_{m+2} & H_{m+1} & \cdots & H_2 \\ \vdots & \vdots & \ddots & \vdots \\ H_{m+n} & H_{m+n-1} & \cdots & H_n \end{bmatrix} \begin{bmatrix} B_0 \\ B_1 \\ \vdots \\ B_m \end{bmatrix} = 0 \quad (18)$$

□

By letting $B(s)=\Delta(s)\times I_q=\det[A(s)]\times I_q$ we arrive at the *relative Cayley–Hamilton theorem* in terms of the fundamental matrices $\{H_k\}$ defined in (4).

Theorem 3. Suppose that A(s) is regular with $\{H_k\}$ given by (4) and

$$\det[A(s)] = p_0s^{\ell}+p_1s^{\ell-1}+\cdots+p_{\ell} \quad (19)$$

where $nq\geq\ell\geq n$. Then

$$\Delta(H_k)=p_0H_k+p_1H_{k-1}+\cdots+p_{\ell}H_{k-\ell} = 0 \quad (20)$$

for $k>\ell$ & $k<\ell-\mu-n$.

Proof. We have that

$$\Delta(s)\, A(s)^{-1} = \mathrm{Adj}[A(s)^{-1}] \qquad (21)$$

Substituting $A(s)^{-1}$ from (4) and equating the coefficients a) of the negative powers of s in (21) and b) of the powers of s greater than $\mu+m$ (see (23)) we obtain (20). □

A(s) is a left divisor of $\Delta(s)$ and thus from Corollary 3

$$\begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{n-1} & A_{n-2} & \cdots & A_0 \end{bmatrix} \times \begin{bmatrix} H_{\ell+1} & H_{\ell} & \cdots & H_1 \\ H_{\ell+2} & H_{\ell+1} & \cdots & H_2 \\ \vdots & \vdots & \ddots & \vdots \\ H_{\ell+n} & H_{\ell+n-1} & \cdots & H_n \end{bmatrix} \begin{bmatrix} p_0I_q \\ p_1I_q \\ \vdots \\ p_{\ell}I_q \end{bmatrix} =0 \quad (22)$$

Comparison of (21) and (9a) also allow us to express the relative adjoint matrix in terms of the sequence $\{H_k\}$:

$$\mathrm{Adj}[A(s)^{-1}]= \qquad (23)$$

$$=[H_{-\mu}p_0]s^{\mu+n}+[H_{-\mu+1}p_0+H_{-\mu}p_1]s^{\mu+n-1}+$$

$$+\cdots+[H_np_0+H_{n-1}p_1+\cdots+H_0p_n]= \sum_{i=0}^{\mu+n} \Delta[H_{n-i}]s^i$$

or equivalently as

$$\mathrm{Adj}[A(s)^{-1}] =$$
$$= H_{-\mu}\,[p_0 s^{\mu+n} + p_1 s^{\mu+n-1} + \cdots + p_\ell s^{\mu+n-\ell}] +$$
$$H_{-\mu+1}[p_0 s^{\mu+n-1} + p_1 s^{\mu+n-2} + \cdot + p_\ell s^{\mu+n-\ell-1}]$$
$$+ \cdots\cdots + H_0\,[p_0 s^n + p_1 s^{n-1} + \cdots + p_n] +$$
$$+ \cdots\cdots + \tag{24}$$
$$+ H_{n-1}\,[p_0 s + p_1] + H_n\,[p_0] \equiv \sum_{i=-\mu}^{n} \Delta_i(s)\, H_i$$

Thus we have derived the following

Theorem 4. The relative resolvent matrix $A(s)$ is expressed in terms of the relative fundamental matrix $\{H_k\}$ by the relation

$$A(s)^{-1} = \frac{1}{\det[A(s)]}\left\{\sum_{i=-\mu}^{m} \Delta_i(s)\, H_i\right\} \tag{25}$$

where $\Delta_i(s)$, $i=-\mu,-\mu+1,..,m$ are the *Tschirnhausen polynomials* defined by (24). □

Example 1. Let

$$B(s)=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}s^2+\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}s+\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \equiv B_0 s^2 + B_1 s + B_2$$

and

$$A(s)=\begin{bmatrix}s & 1\\ 0 & s\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}s+\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\equiv A_0 s + A_1$$

where $\det[A_0]=2\neq 0$. Then we have that

$$A(s)^{-1}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\frac{1}{s}+\begin{bmatrix}0 & -1\\ 0 & 0\end{bmatrix}\frac{1}{s^2}\equiv H_1 s^{-1}+H_2 s^{-2}$$

Then the left quotient of the division of $B(s)$ by $A(s)$ is given according to Corollary 1 by

$$Q(s) = B[H_1]s+B[H_2] =$$
$$= [H_1B_0]\,s+[H_2B_0+H_1B_1] = \begin{bmatrix}s & 1\\ 1 & 0\end{bmatrix}$$

The left remainder on the division of $B(s)$ by $A(s)$ is given according to Corollary 1 by

$$R(s)=A_0B[H_3]=A_0[H_3B_0+H_2B_1+H_1B_2]=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$$ □

Example 2. Let

$$B(s)=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}s^2+\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}s+\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\equiv B_0 s^2 + B_1 s + B_2$$

and

$$A(s)=\begin{bmatrix}1 & s\\ 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}s+\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\equiv A_0 s + A_1$$

where $\det[A_0]=0$. Then we have that

$$A(s)^{-1}=\begin{bmatrix}1 & -s\\ 0 & 1\end{bmatrix}=\begin{bmatrix}0 & -1\\ 0 & 0\end{bmatrix}s+\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\equiv H_{-1}s+H_0$$

Then the left quotient of the division of $B(s)$ by $A(s)$ is given according to Theorem 2 by

$$Q(s)=B[H_{-1}]s^3+B[H_0]s^2+B[H_1]s+B[H_2] =$$
$$=[H_{-1}B_0]s^3+[H_0B_0+H_{-1}B_1]s^2+[H_1B_0+H_0B_1+$$
$$+H_{-1}B_2]s+[H_2B_0+H_1B_1+H_0B_2]=\begin{bmatrix}s^2 & -s^2+s\\ 0 & s\end{bmatrix}$$

The left remainder of the division of $B(s)$ by $A(s)$ is given according to Theorem 2 by

$$R(s)=A_0B[H_3]=A_0[H_3B_0+H_2B_1+H_1B_2]=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

It is easily seen that $Q(s)$, $R(s)$ of the form

$$\hat{Q}(s) = \begin{bmatrix}s^2+sL_{21}+L_{11} & -s^2+s(1+L_{22})+L_{12}\\ -L_{21} & s-L_{22}\end{bmatrix}$$
$$\hat{R}(s) = \begin{bmatrix}-L_{11} & -L_{12}\\ L_{21} & L_{22}\end{bmatrix} \tag{E.1}$$

where L_{ij} are arbitrary constant matrices are also satisfy the equation $B(s)=A(s)\hat{Q}(s)+\hat{R}(s)$. However we can check that $A(s)^{-1}\hat{R}(s)$ vanishes at $s=\infty$ iff $L_{ij}=0$ and thus $Q(s)$ and $R(s)$ are of the form (E.1). This example shows that condition (2) does not guarantee by itself the uniqueness of $Q(s)$ and $R(s)$ but the further condition of stricly properness of $A(s)^{-1}R(s)$ must be satisfied. □

Assume now that A_0 is a regular matrix. Without loss of generality we may assume that $A_0=I_q$ otherwise instead of making the left division of $B(s)$ by $A(s)$ we can make the division of $A_0^{-1}B(s)$ by $A_0^{-1}A(s)$ i.e.

$$A_0^{-1}B(s) = A_0^{-1}A(s)\hat{Q}(s) + \hat{R}(s) \tag{26}$$

and thus the left quotient and remainder of the division of $B(s)$ by $A(s)$ will be respectively $Q(s)=\hat{Q}(s)$ and $R(s)=A_0\hat{R}(s)$ respectively. Then we can state the relative Bezout theorem.

Theorem 5. If $A_0=I_q$ then the left quotient and remainder of the division of $B(s)$ by $A(s)$ are respectively :

$$Q(s)=Y_0 s^{m-n}+Y_1 s^{m-n-1}+\cdots+Y_{m-n} \tag{27a}$$
$$R(s)=Y_{m-n+1}s^{n-1}+Y_{m-n+2}s^{n-2}+\cdots+Y_m \tag{27b}$$

where Y_i are defined according to the following recursive way

$$Y_0 = B_0$$

$$Y_j = B_j - \sum_{i=\max(0,j-n)}^{\min(j-1,m-n)} A_{j-i} Y_i \quad (28)$$

Proof. To determine the right remainder we use the usual division scheme :

$$B(s) = A(s)B_0 s^{m-n} + (B_1 - A_1B_0)s^{m-1} +$$

$$(B_2 - A_2B_0)s^{m-2} + \cdots + B_m =$$

$$= A(s)\,[B_0 s^{m-n} + (B_1 - A_1B_0)s^{m-n-1}] +$$

$$[B_2 - A_2B_0 - A_1[B_1 - A_1B_0]]\, s^{m-2} + \cdots + B_m =$$

$$= \cdots\cdots = \quad (29)$$

$$= A(s)\,[Y_0 s^{m-n} + Y_1 s^{m-n-1} + \cdots + Y_{m-n}] +$$

$$[Y_{m-n+1}s^{n-1} + Y_{m-n+2}s^{n-2} + \cdots + Y_m]$$

However $\det[A_0] \neq 0$. Thus $A(s)$ is column (row) reduced and its ith row degree (n) is greater than the ith row degree of $R(s)$ which is at most $n-1$. Therefore $A(s)^{-1}R(s)$ vanishes at $s=\infty$ according to definition 1 which proves the Theorem. □

Example 3. Let $B(s)$ and $A(s)$ as in Example 1. Then

$$Y_0 = B_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \;;\; Y_1 = B_1 - A_1Y_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$Y_2 = B_2 - A_2Y_0 - A_1Y_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus according to Theorem 4

$$Q(s) = Y_1 s + Y_2 = \begin{bmatrix} s & 1 \\ 1 & 0 \end{bmatrix} \;;\; R(s) = Y_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \square$$

Further assuming that

$$B(s) = \Delta(s) = \det[A(s)] = p_0 s^{qn} + p_1 s^{qn-1} + \cdots + p_{qn}$$

and applying Theorem 5 we have that

$$R(s) = Y_{m-n+1}s^{n-1} + Y_{m-n+2}s^{n-2} + \cdots + Y_m \equiv 0 \quad (30)$$

or equivalently

$$Y_j = 0 \quad \text{for } j = m-n+1,\ m-n+2, \ldots, m \quad (31)$$

Substituting B_i with $p_i I_q$ and Y_j from relations (28) we have an alternative form of the generalised Cayley—Hamilton Theorem.

3. Conclusions.

The division of two polynomial matrices has been studied through two different algorithms. The first algorithm was an extension of the already known algorithm of Lewis [5] while the second one was an extension of the general Bezout theorem presented in Gantmacher [4]. Some interesting applications of the presented algorithm such as a) the relative Cayley Hamilton Theorem and b) the expression of the relative adjoint matrix in terms of the Tschirnhausen polynomials have also been presented.

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