A new notion of equivalence for discrete time AR Representations

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Abstract

We present a new equivalence trasformation termed *divisor equivalence*, that has the property of preserving both the finite and the infinite elementary divisor structures of a square nonsingular polynomial matrix. This equivalence relation extends the known notion of strict equivalence [1], which dealt only with matrix pencils, to the general polynomial matrix case. It is proved that divisor equivalence characterizes in a closed form relation the equivalence classes of polynomial matrices that give rise to fundamentally equivalent discrete time auto-regressive representations as defined in [2].

1 Introduction

The problem of equivalence of polynomial matrices has been studied extensively. The primary target of these studies was the preservation of the finite elementary divisors structure of the polynomial matrices involved in the equivalence transformations examined. An equivalence relation for matrix pencils, termed *strict equivalence,* was initially introduced in [1] where it was shown to have the property of preserving the finite elementary divisor structure of (strictly) equivalent pencils. Strict equivalence of matrix pencils has been extended to the general polynomial matrix case, in [3] by *unimodular equivalence*. However, both *strict equivalence* and *unimodular equivalence* can only be applied to matrices of the same dimension. In [4], *extended unimodular equivalence* was introduced as a closed form relation between polynomial matrices of possibly different dimensions, preserving the finite elementary divisor structure of the polynomial matrices involved.

Infinite elementary divisors (IED's) of matrix pencils where initially defined in [1] where it was shown that IED's remain invariant under the transformation of strict equivalence. Furthermore, existing work on strict equivalence of matrix pencils makes no distinction between the preservation of their infinite zeros and infinite elementary divisors structures, since in the case of matrix pencils, the orders of the IEDs are related to the orders of zeros at infinity by the "plus one property" [5]. *Complete equivalence* proposed in [6] for matrix pencils of possibly different dimensions, which preserves both finite and infinite zero (elementary divisor) structures of matrix pencils. [7] extended the definition of the IEDs to the polynomial matrix case and showed that the IED structure gives a complete description of the total pole-zero structure at infinity of a polynomial matrix and not simply that associated with the zeros at infinity. Thus, although *full equivalence* presented later in [8], preserves the infinite zero structure of polynomial matrices, it does not preserve the IED structure, apart of the special class of polynomial matrices having the same order. *Strict equivalence* transformation of matrix pencils was generalized for the case of regular, i.e. square and nonsingular, polynomial matrices in [2] where it was shown to preserve both the FED and IED structures of the polynomial matrices involved. This notion of equivalence, characterizes strict equivalent regular polynomial matrices through their first order representations using strict equivalence of matrix pencils [1], providing no closed form equivalence relation, something that is its main drawback. Many authors applied the algebraic results of the above studies in order to define equivalence relations between state space and descriptor systems in both the continuous and discrete time cases. In this paper we present a *closed form equivalence relation* between polynomial matrices of possibly different degrees and dimensions, which leaves invariants both the finite and infinite elementary divisor structures, thus extending in these terms the notion of strict equivalence of matrix pencils. This work was motivated by the study of discrete time autoregressive representations (DTARR), where the finite and infinite elementary divisor structure of the polynomial matrix describing a DTARR was proved to be of crucial importance ([9], [10], [11]).

Consider a linear, homogeneous, matrix difference equation

$$A(\sigma)\beta(k) = 0, \ \ k \in [0, N] \tag{1}$$

$$A(\sigma) = A_q\sigma^q + A_{q-1}\sigma^{q-1} + \ldots + A_0 \in \mathbb{R}[\sigma]^{r\times r} \tag{2}$$

where $A_i \in \mathbb{R}^{r\times r}, i = 0, 1, ..., q$, σ denotes the forward shift operator: $\sigma\beta(k) = \beta(k+1)$ and $\beta(k) : [0, N] \longrightarrow \mathbb{R}^r$ is a vector valued sequence. Following the terminology of Willems ([12], [13], [14]), we call the set of equations (1) an AR-Representation of $B_{A(\sigma)}$ (behavior) where $B_{A(\sigma)}$ is defined as:

$$B_{A(\sigma)} := \{\beta(k) : [0, N] \to \mathbb{R}^r \ \ : \ \ (1) \ \text{ is satisfied } \forall k \in [0, N]\}$$

In the particular case when the polynomial matrix $A(\sigma)$ is a matrix pencil $A(\sigma) = \sigma E - A$, the discrete-time AR-Representations exhibit noncausal behavior and the natural framework for their study appears to be over a finite-time interval $[0, N]$ where the system can be essentially decomposed into a purely causal and a purely anticausal part (see [15], [16], [10], [17], [18], [19]). The causal and anticausal behaviors of such systems, has been proved to be associated respectively with the finite and infinite elementary divisor structure of the polynomial matrix $A(\sigma)$ describing the system (see [10] for descriptor systems or [9] for higher order systems). In the more general case when the matrix pencil is singular, then its causal and anticausal behavior is due to the FED, IED and the right minimal indices of the pencil [20]. The causal behavior of (1) for the case when $A(\sigma)$ is a regular polynomial matrix has been extensively studied in [21]. These results have been extended in [9], where both causal and anticausal behavior has been studied. The more general case where $A(\sigma)$ is a nonregular polynomial matrix, has been studied in [22], where it is shown that additionally the right null space of the polynomial matrix $A(\sigma)$ plays a crucial role to the causal and anticausal behavior of (1).

[2] defined two AR-representations as *fundamentally equivalent* (FE) if their solution spaces or behaviors (both causal and anti-causal) are isomorphic in a particular way. Motivated by the fact that the behavior of the AR-representation (1), when considered over a *finite* time interval $[0, N]$, depends on the algebraic structure of both the *finite* and the *infinite elementary divisors* of the polynomial matrix $A(\sigma)$ associated with (1), they showed that this structure is identical to the corresponding structure of a block companion matrix pencil $\sigma\overline{E} - \overline{A} \in \mathbb{R}[\sigma]^{rq\times rq}$ which constitutes a *linearization* of the polynomial matrix [9] and consequently the AR associated with $\sigma\overline{E} - \overline{A}$ constitutes the natural first-order *fundamentally equivalent* representation (realization) of (1). Thus, they proposed a generalization of the concept of *strict equivalence* (SE) of regular matrix pencils [1] to the case of general nonsingular polynomial matrices and showed that two AR-representations described by regular polynomial matrices of possibly different degrees and dimensions are *fundamentally equivalent* if and only if these polynomial matrices are strict equivalent. This *fundamental equivalence* transformation proposed in [2] generalizes the behavior homomorphism presented in [23], for non-proper discrete time AR-representations of the form (1) over a finite time interval. However the main disadvantage of this transformation is that it does not provide a closed formula relating the polynomial matrices that describe the *fundamentally equivalent* discrete time AR-representations. Our main interest in this paper is to propose that missing closed formula by presenting a new equivalence transformation named *divisor equivalence.* We shall prove that divisor equivalence preserves both the finite and infinite elementary divisor structure of equivalent polynomial matrices and thus extends the *strict equivalence* presented in [1] for matrix pencils to the general polynomial matrix case.

The paper is organized as follows. Section 2 provides the necessary mathematical background for the consequent sections. In Section 3 a generalization of strict equivalence to the polynomial matrix case, named divisor equivalence, is introduced and certain properties of divisor equivalence are obtained. In Section 4 we show the connection between the divisor equivalence transformation and the strict equivalence transformation that has been presented by [1] and [2]. Finally, in Section 5 we summarize our results and propose directions for further research on the subject.

2 Preliminary results

In what follows $\mathbb{R}, \mathbb{C}$ denote respectively the fields of real and complex numbers and $\mathbb{Z}, \mathbb{Z}^+$ denote respectively the integers and non negative integers. By $\mathbb{R}[s]$ and $\mathbb{R}[s]^{p\times m}$ we denote the sets of polynomials and $p \times m$ polynomial matrices respectively with real coefficients and indeterminate $s \in \mathbb{C}$.

Definition 1 *Let $A(s) \in \mathbb{R}[s]^{p\times m}$ with $rank_{\mathbb{R}(s)} A(s) = r \leq \min(p, m)$. The values $\lambda_i \in \mathbb{C}$ that satisfy the condition $rank_{\mathbb{C}} A(\lambda_i) < r$ are called finite zeros of $A(s)$. Assume that $A(s)$ has l distinct zeros $\lambda_1, \lambda_2, \ldots, \lambda_l \in \mathbb{C}$, and let*

$$S_{A(s)}^{\lambda_i}(s) = \begin{bmatrix} diag\{(s-\lambda_i)^{m_{i1}}, ..., (s-\lambda_i)^{m_{ir}}\} & 0_{r,m-r} \\ 0_{p-r,r} & 0_{p-r,m-r} \end{bmatrix}$$

be the local Smith form of $A(s)$ at $s = \lambda_i$, $i = 1, 2, ..., l$ where $m_{ij} \in \mathbb{Z}^+$ and $0 \leq m_{i1} \leq m_{i2} \leq ... \leq m_{ir}$. The terms $(s-\lambda_i)^{m_{ij}}$ are called the finite elementary divisors (f.e.d.) of $A(s)$ at $s = \lambda_i$. The total number n of the f.e.d. of $A(s)$ is $n := \sum_{i=1}^{l} \sum_{j=1}^{r} m_{ij}$.

Definition 2 *[21], [24] The dual matrix of $A(s) = A_q s^q + A_{q-1} s^{q-1} + \cdots + A_0 \in \mathbb{R}[\sigma]^{p\times m}$, is defined as $\tilde{A}(s) := s^q A(\frac{1}{s}) = A_0 s^q + A_1 s^{q-1} + \ldots + A_q$. Since $rank\tilde{A}(0) = rankA_q$ the dual matrix $\tilde{A}(s)$ of $A(s)$ has zeros at $s = 0$ iff $rankA_q < r$. Let $rankA_q < r$ and let*

$$S_{\tilde{A}(s)}^{0}(s) = \begin{bmatrix} diag\{s^{\mu_1}, ..., s^{\mu_r}\} & 0_{r,m-r} \\ 0_{p-r,r} & 0_{p-r,m-r} \end{bmatrix} \quad (3)$$

be the local Smith form of $\tilde{A}(s)$ at $s = 0$ where $\mu_j \in \mathbb{Z}^+$ and $0 \leq \mu_1 \leq \mu_2 \leq ... \leq \mu_r$. The infinite elementary divisors (i.e.d.) of $A(s)$ are defined as the f.e.d. σ^{μ_j} of its dual $\tilde{A}(s)$ at $s = 0$. The total number μ of the i.e.d. of $A(s)$ is $\mu = \sum_{i=1}^{r} \mu_i$.

It is easily seen from Definition 2, that $A(s) \in \mathbb{R}[s]^{p\times m}$ has no i.e.d. iff $rankA_q = r$. The total number of f.e.d. and i.e.d. is connected with the dimension and the highest degree of all entries of a polynomial matrix as follows.

Theorem 3 *[21][2] The total number $n + \mu$ of f.e.d. and i.e.d. (orders accounted for) of a square polynomial matrix $A(s) \in \mathbb{R}[s]^{r\times r}$ is equal to rq, where q is the highest degree of all the entries of $A(s)$.*

Define by $P(m, l)$ the class of $(r + m) \times (r + l)$ polynomial matrices where l and m are fixed integers and r ranges over all integers which are greater than $\max(-m, -l)$. In the following, we will introduce some essential transformations between polynomial matrices.

Definition 4 *[4] $A_1(s), A_2(s) \in P(m,l)$ are said to be* ***extended unimodular equivalent*** *(e.u.e) if there exist polynomial matrices $M(s), N(s)$ such that*

$$\begin{bmatrix} M(s) & A_2(s) \end{bmatrix} \begin{bmatrix} A_1(s) \\ -N(s) \end{bmatrix} = 0 \qquad (4)$$

where the compound matrices

$$\begin{bmatrix} M(s) & A_2(s) \end{bmatrix} \quad ; \quad \begin{bmatrix} A_1(s) \\ -N(s) \end{bmatrix} \qquad (5)$$

have full rank $\forall s \in \mathbb{C}$.

In [4], it was shown that the transformations of strict system equivalence (s.s.e.) [25], and Furhmann system equivalence (f.s.e.) [26] and [27], is a specialization to system matrices of the extended unimodular equivalence. E.u.e. allows matrices of different dimensions to be related and preserves the f.e.d. of the polynomial matrices involved. Extending the notion of *strict equivalence* for matrix pencils, defined in [1], to the case of square polynomial matrices we propose the following definition.

Definition 5 *Let $A_1(s), A_2(s) \in \mathbb{R}[s]^{m\times l}$ be two polynomial matrices of the same degree. They are said to be* ***strictly equivalent*** *(s.e.) if there exist nonsingular $M \in \mathbb{R}^{m\times m}, N \in \mathbb{R}^{l\times l}$ such that $MA_1(s)N = A_2(s)$.*

S.e. is actually a unimodular and bicausal equivalence transformation and thus has the property of preserving both the f.e.d. and i.e.d. of polynomial matrices. Note however that it relates matrices of the same dimensions and degree.

Definition 6 *[28] $A_1(s), A_2(s) \in P(m,l)$ are said to be* ***{0}-equivalent*** *if there exist rational matrices $M(s), N(s)$, having no poles at $s = 0$, such that (4) is satisfied and where the compound matrices in (5) have full rank at $s = 0$.*

{0}-equivalence preserves only the f.e.d. of $A_1(s), A_2(s) \in P(m,l)$ of the form $s^i, i > 0$.

3 A new notion of equivalence between square and regular polynomial matrices

Considering Theorem 3, a necessary condition for two polynomial matrices $A_1(s) \in \mathbb{R}[\sigma]^{r_1\times r_1}, A_2(s) \in \mathbb{R}[\sigma]^{r_2\times r_2}$ of degrees q_1, q_2 respectively, to have the same f.e.d. and i.e.d., is that $r_1q_1 = r_2q_2$. Therefore we define the following set of polynomial matrices

$$\mathbb{R}_c[s] := \{A(s) \text{ defined in (2) with } c = rq,\ r \geq 2\} \qquad (6)$$

To clarify the definition of the set $\mathbb{R}_c[s]$, we give the following example.

Example 7

$$A_1(s) = \begin{bmatrix} 1 & s^2 \\ 0 & s+1 \end{bmatrix} \in \mathbb{R}_4[s] \;\; ; \;\; A_2(s) = \begin{bmatrix} 1 & s^3 \\ 0 & s+1 \end{bmatrix} \in \mathbb{R}_6[s]$$

$$A_3(s) = \begin{bmatrix} s & 0 & -1 & 0 \\ 0 & s & 0 & -1 \\ 1 & 0 & 0 & s \\ 0 & 1 & 0 & 1 \end{bmatrix} \in \mathbb{R}_4[s]$$

Before we continue to the main results of this section, we will present a simple example to demonstrate the limitations of the extended unimodular transformation to take into account both finite and infinite elementary divisor structures.

Example 8 *Consider the polynomial matrices*

$$A_1(s) = \begin{bmatrix} 1 & s^2 \\ 0 & s+1 \end{bmatrix} \in \mathbb{R}_4[s] \;\; ; \;\; A_2(s) = \begin{bmatrix} 1 & s^3 \\ 0 & s+1 \end{bmatrix} \in \mathbb{R}_6[s]$$

and the following e.u.e. transformation between them

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{M(s)} \underbrace{\begin{bmatrix} 1 & s^2 \\ 0 & s+1 \end{bmatrix}}_{A_1(s)} = \underbrace{\begin{bmatrix} 1 & s^3 \\ 0 & s+1 \end{bmatrix}}_{A_2(s)} \underbrace{\begin{bmatrix} 1 & s^2 - s^3 \\ 0 & 1 \end{bmatrix}}_{N(s)}$$

As the Smith forms of $A_1(s), A_2(s)$ are $S^{\mathbb{C}}_{A_2(s)}(s) = S^{\mathbb{C}}_{A_1(s)}(s) = diag \begin{bmatrix} 1 & s+1 \end{bmatrix}$, both $A_1(s), A_2(s)$ have the same f.e.d. $(s+1)$. As the Smith forms of the duals of $A_1(s), A_2(s)$ are $S^0_{\tilde{A}_1(s)}(s) = diag \begin{bmatrix} 1 & s^3 \end{bmatrix}$ and $S^0_{\tilde{A}_2(s)}(s) = diag \begin{bmatrix} 1 & s^5 \end{bmatrix}$ respectively, they have different i.e.d. s^3 and s^5 respectively.

The above example indicates that in order to ensure that the associated transformation will preserve both the f.e.d. and i.e.d. of $A_1(s)$ and $A_2(s)$, further restrictions have to be imposed on the compound matrices $\begin{bmatrix} M(s) & A_2(s) \end{bmatrix}$ and $\begin{bmatrix} A_1(s) \\ -N(s) \end{bmatrix}$. In this respect we propose the following transformation.

Definition 9 *$A_1(s), A_2(s) \in \mathbb{R}_c[s]$ are said to be **divisor equivalent** (d.e.) if there exist polynomial matrices $M(s), N(s)$, such that (4) is satisfied and the compound matrices in (5) satisfy the following two conditions*
(i) they have full rank over $\mathbb{R}(s)$ and no f.e.ds.,
(ii) they have no i.e.ds.

Note that condition (i) is equivalent to the relative primeness requirements of e.u.e. while (from the remark after definition 2) condition (ii) amounts to the requirement that the coefficient matrices corresponding to the highest degree terms in each of the compound matrices in (5) have respectively full row (column) rank over $\mathbb{R}$. The two conditions taken together are equivalent to the requirement that the compound matrices in (5) are *minimal bases* [24] of the rational vector spaces spanned by their respective rows (columns). D.e. is a special case of e.u.e.. Our main interest is to prove that d.e. leaves invariant the finite and infinite elementary divisor structure of the polynomial matrices involved. However, before we proceed with the main theorem of this paper, we need to prove some preliminary results. Firstly we need the following partial transitivity result.

Lemma 10 *If $A_1(s), A_2(s) \in \mathbb{R}_c[s]$ are d.e. and $A_2(s), A_3(s) \in \mathbb{R}_c[s]$ are s.e., then $A_1(s), A_3(s) \in \mathbb{R}_c[s]$ are d.e.*

Proof. See Appendix 1. ■

Given a polynomial matrix $A(s)$, we can construct a matrix pencil $sE - A$ as in (7) ([21], page 186) which possesses the same f.e.d. [21] and i.e.d. [29], [2] with $A(s)$. In the following lemma it is shown that the proposed pencil $sE - A$ is actually d.e. to the polynomial matrix $A(s)$.

Lemma 11 *With $c = rq$, $r \geq 2$ the polynomial matrix $A(s) \in \mathbb{R}_c[s]$ defined in (2) and the matrix pencil*

$$sE - A := \begin{bmatrix} sI_r & -I_r & 0 & \cdots & 0 & 0 \\ 0 & sI_r & -I_r & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & sI_r & -I_r \\ A_0 & A_1 & A_2 & \cdots & A_{q-2} & sA_q + A_{q-1} \end{bmatrix} \in \mathbb{R}_c[s] \qquad (7)$$

are d.e. .

Proof. See Appendix 2. ■

The required conditions of d.e. give rise to certain degree conditions on the left and right transforming matrices as we can see in the following lemma.

Lemma 12 *a) Let $A_1(s), A_2(s) \in \mathbb{R}_c[s]$ with dimensions $m \times m$ and $(m+r) \times (m+r)$ respectively where $r \neq 0$. Then the two conditions of d.e. imply the following degree conditions: $\deg M(s) \leq \deg A_2(s)$ and $\deg N(s) \leq \deg A_1(s)$, where $\deg[\cdot]$ denotes the highest degree among the polynomial entries of the indicated matrix.*

b) Let $A_1(s), A_2(s) \in \mathbb{R}_c[s]$ have the same dimensions $r \times r$ and therefore, since they belong to the same class $\mathbb{R}_c[s]$, the same degree q. If $A_1(s), A_2(s)$ are d.e. then $\deg M(s) = \deg N(s)$.

Proof. See Appendix 3. ■

We can always increase the degree of the polynomial matrices $A_1(s), A_2(s)$ involved in the d.e. transformation (4), without changing the initial f.e.d. and i.e.d. of $A_1(s), A_2(s)$, by multiplying both of them by certain polynomial terms, as we can easily see in the following Lemma.

Lemma 13 *If $A_1(s)(s - s_0)^k \in \mathbb{R}^{m \times m}[s]$ has the same f.e.d. and i.e.d. as $A_2(s)(s - s_0)^k \in \mathbb{R}^{m \times m}[s]$, where $s_0 \neq 0$ is not a zero of either $A_1(s)$ or $A_2(s)$, then $A_1(s)$ and $A_2(s)$ have the same f.e.d. and i.e.d..*

Proof. See Appendix 4. ■

Now, using the above lemmata we are in a position to prove that the d.e. transformation leaves invariant both the finite and infinite elementary divisors of the polynomial matrices involved in the transformation.

Theorem 14 *$A_1(s), A_2(s) \in \mathbb{R}_c[s]$ are d.e. iff they have the same finite and infinite elementary divisors.*

Proof. Sufficiency. (a) According to condition (i) of d.e., $A_1(s)$ and $A_2(s)$ are also e.u.e. and thus have the same f.e.d..

(b) In the sequel we prove that the second condition of d.e. implies that the two polynomial matrices possess the same i.e.d.. In the following proof we distinguish two cases : i) $A_1(s)$ and $A_2(s) \in \mathbb{R}_c[s]$ have different dimensions and ii) $A_1(s)$ and $A_2(s) \in \mathbb{R}_c[s]$ have the same dimensions.

(i) Let $A_1(s)$ and $A_2(s) \in \mathbb{R}_c[s]$ with dimensions $m \times m$ and $(m+r) \times (m+r)$ respectively where $r \neq 0$. Then, from Lemma 12a the first two conditions of d.e. imply the following degree conditions $\deg M(s) \leq \deg A_2(s)$ and $\deg N(s) \leq \deg A_1(s)$. Let also $d_{MA_2} = \deg \begin{bmatrix} M(s) & A_2(s) \end{bmatrix}$ and $d_{A_1N} = \deg \begin{bmatrix} A_1(s)^T & -N(s)^T \end{bmatrix}^T$. Then, by setting $s = \frac{1}{w}$, (4) may be rewritten as

$$\begin{bmatrix} M(\frac{1}{w}) & A_2(\frac{1}{w}) \end{bmatrix} \begin{bmatrix} A_1(\frac{1}{w}) \\ -N(\frac{1}{w}) \end{bmatrix} = 0 \tag{8}$$

and then premultiplying and postmultiplying (8) by $w^{d_{MA_2}}$ and $w^{d_{A_1N}}$ respectively as

$$w^{d_{MA_2}} \begin{bmatrix} M(\frac{1}{w}) & A_2(\frac{1}{w}) \end{bmatrix} \begin{bmatrix} A_1(\frac{1}{w}) \\ -N(\frac{1}{w}) \end{bmatrix} w^{d_{A_1N}} = 0 \Longleftrightarrow$$

$$\widetilde{\begin{bmatrix} M(w) & A_2(w) \end{bmatrix}} \widetilde{\begin{bmatrix} A_1(w) \\ -N(w) \end{bmatrix}} = 0 \tag{9}$$

where with ~ we denote the dual matrix. Since $d_{MA_2} \leq \deg[A_2(s)]$ and $d_{A_1N} \leq \deg[A_1(s)]$ equation (9) may be rewritten as

$$\begin{bmatrix} M'(w) & \tilde{A}_2(w) \end{bmatrix} \begin{bmatrix} \tilde{A}_1(w) \\ -N'(w) \end{bmatrix} = 0 \tag{10}$$

The compound matrix $\begin{bmatrix} M(s) & A_2(s) \end{bmatrix}$ (respectively $\begin{bmatrix} A_1(s) \\ -N(s) \end{bmatrix}$) has no i.e.d. and therefore its dual $\begin{bmatrix} M'(w) & \tilde{A}_2(w) \end{bmatrix}$ (respectively $\begin{bmatrix} \tilde{A}_1(w) \\ -N'(w) \end{bmatrix}$) has no finite zeros at $w = 0$. Therefore relation (10) is a $\{0\}$-equivalence relation which preserves the f.e.d. of $\tilde{A}_1(w), \tilde{A}_2(w)$ at $w = 0$ or otherwise the i.e.d. of $A_1(s), A_2(s)$.

ii) Let $A_1(s), A_2(s) \in \mathbb{R}[s]^{m\times m}$ with the same degree d. Then according to Lemma 12b $d_t = \deg M(s) = \deg N(s)$. In case where $d_t \leq d$ then the proof is the same with the one presented in (i) above. Consider now the case where $d_t > d$. Let $s_0 \neq 0$ not a zero of either $A_1(s)$ or $A_2(s)$. Then

$$\left[\begin{array}{cc} M(s) & A_2(s)(s-s_0)^{d_t-d} \end{array}\right] \left[\begin{array}{c} A_1(s)(s-s_0)^{d_t-d} \\ -N(s) \end{array}\right] = 0 \tag{11}$$

is an e.u.e. relation which implies that $A_2(s)(s-s_0)^{d_t-d}$ and $A_1(s)(s-s_0)^{d_t-d}$ have the same f.e.d.. Therefore according to Lemma 13, $A_1(s)$ and $A_2(s)$ have the same f.e.d.. Taking the duals of (11) we have

$$\left[\begin{array}{cc} \widetilde{M}(s) & A_2(s)\widetilde{(s-s_0)}^{d_t-d} \end{array}\right] \left[\begin{array}{c} A_1(s)\widetilde{(s-s_0)}^{d_t-d} \\ \widetilde{-N(s)} \end{array}\right] = 0 \tag{12}$$

(12) is a $\{0\}$-equivalence relation, so

$$A_2(s)\widetilde{(s-s_0)}^{d_t-d} \text{ and } A_1(s)\widetilde{(s-s_0)}^{d_t-d}$$

have the same f.e.d. at 0 and thus $A_2(s)(s-s_0)^{d_t-d}$ and $A_1(s)(s-s_0)^{d_t-d}$ have the same i.e.d. i.e. $A_2(s)$ and $A_1(s)$ have the same i.e.d. (Lemma 13).

Necessity. Assume that $A_1(s), A_2(s) \in \mathbb{R}_c[s]$ have identical f.e.d. and i.e.d.. Then according to Lemma 11 $A_1(s)$ and $A_2(s)$ are d.e. to matrix pencils $sE_1 - A_1$ and $sE_2 - A_2$ of the form (7) and vice versa. The pencils $sE_1 - A_1$ and $sE_2 - A_2$ are also strictly equivalent to their respective Weierstrass forms denoted thereafter by $W(sE_1 - A_1)$ and $W(sE_2 - A_2)$ [1]. Since $sE_1 - A_1, sE_2 - A_2 \in \mathbb{R}_c[s]$ and share the same f.e.d. and i.e.d. they have the same Weierstrass form: $sE_w - A_w \equiv W(sE_1 - A_1) \equiv W(sE_2 - A_2)$. Repeated use of the transitivity property proved in Lemma 10 shows that $A_1(s)$ is divisor equivalent to $sE_2 - A_2$. This argument is summarized in Figure 1.

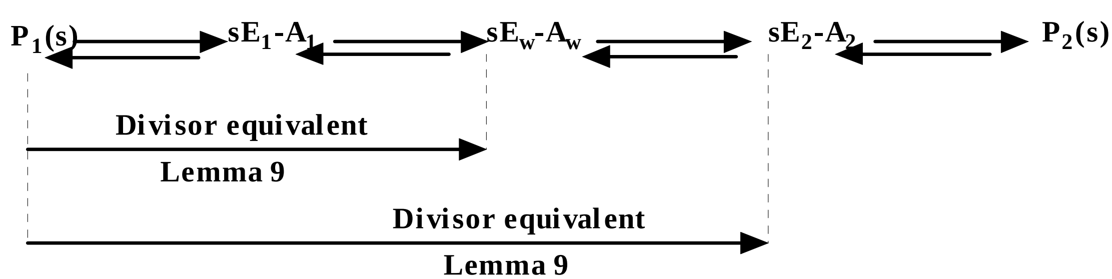

Figure 1.

Therefore there exist polynomial matrices $A(s), B(s)$ such that the following transformation :

$$A(s)(sE_2 - A_2) = A_1(s)B(s) \tag{13}$$

is a d.e. transformation. The polynomial matrices $sE_2 - A_2$ and $A_2(s)$ are also connected, according to Lemma 11 through the following divisor equivalent transformation

$$\begin{pmatrix} 0 \\ sI-\Lambda_0 \end{pmatrix} A_2(s) = (sE_2 - A_2) \begin{pmatrix} sI-\Lambda_0 \\ (sI-\Lambda_0)\, s \\ \vdots \\ (sI-\Lambda_0)\, s^{q_2-1} \end{pmatrix} \tag{14}$$

where $q_2 = \deg[A_2(s)]$ and $sI - \Lambda_0$ as defined in Lemma 11. Premultiplying (14) by $A(s)$ and then using (13) we get the following transformation:

$$\left\{ A(s) \begin{pmatrix} 0 \\ (sI-\Lambda_0) \end{pmatrix} \right\} A_2(s) = A_1(s) \left\{ B(s) \begin{pmatrix} sI-\Lambda_0 \\ (sI-\Lambda_0)\, s \\ \vdots \\ (sI-\Lambda_0)\, s^{q_2-1} \end{pmatrix} \right\} \tag{15}$$

We divide both sides with $(s-s_0)$ and thus we have the following transformation:

$$\left\{ A(s) \begin{pmatrix} 0 \\ I \end{pmatrix} \right\} A_2(s) = A_1(s) \left\{ B(s) \begin{pmatrix} I \\ sI \\ \vdots \\ s^{q_2-1} I \end{pmatrix} \right\} \Longleftrightarrow \tag{16}$$

$$\begin{bmatrix} A(s) \begin{pmatrix} 0 \\ I \end{pmatrix} & A_1(s) \end{bmatrix} \begin{bmatrix} A_2(s) \\ -B(s) \begin{pmatrix} I \\ sI \\ \vdots \\ s^{q_2-1} I \end{pmatrix} \end{bmatrix} = 0$$

We shall show in the sequel that (16) is a d.e. transformation. In order to prove the absence of finite and infinite elementary divisors of the compound matrices

$$\begin{bmatrix} A(s) \begin{pmatrix} 0 \\ I \end{pmatrix} & A_1(s) \end{bmatrix} \text{ and } \begin{bmatrix} A_2(s) \\ -B(s) \begin{pmatrix} I \\ sI \\ \vdots \\ s^{q_2-1} I \end{pmatrix} \end{bmatrix} \tag{17}$$

it is enough to prove that (17) possess the same finite and infinite elementary divisors respectively with

$$\begin{bmatrix} A(s) & A_1(s) \end{bmatrix} \text{ and } \begin{bmatrix} sE_2 - A_2 \\ -B(s) \end{bmatrix} \tag{18}$$

since according to the d.e. transformation (13), the matrices in (18) have neither finite nor infinite elementary divisors.

(i) Our first goal is to prove that the matrices

$$\begin{bmatrix} sE_2 - A_2 \\ -B(s) \end{bmatrix} \text{ and } \begin{bmatrix} A_2(s) \\ -B(s)\begin{pmatrix} I \\ sI \\ \vdots \\ s^{q_2-1}I \end{pmatrix} \end{bmatrix} \tag{19}$$

possess the same finite and infinite elementary divisors.

(i-a) - *Same finite elementary divisors*

Consider the transformation

$$\begin{bmatrix} \begin{pmatrix} 0 \\ I \end{pmatrix} & 0 & sE_2 - A_2 \\ 0 & I & -B(s) \end{bmatrix} \begin{bmatrix} A_2(s) \\ -B(s)\begin{pmatrix} I \\ sI \\ \vdots \\ s^{q_2-1}I \end{pmatrix} \\ -\begin{pmatrix} I \\ sI \\ \vdots \\ s^{q_2-1}I \end{pmatrix} \end{bmatrix} = 0$$

It is easily seen (see also (7)) that both compound matrices of the above transformation include the unit matrix and therefore do not possess finite elementary divisors. Therefore the matrices (19) are e.u.e. and thus have the same finite elementary divisors.

(i-b) - *Same infinite elementary divisors*

Consider now the transformation

$$\begin{bmatrix} \begin{pmatrix} 0 \\ (sI - \Lambda_0) \end{pmatrix} & 0 & sE_2 - A_2 \\ 0 & (sI - \Lambda_0) & -B(s) \end{bmatrix} \begin{bmatrix} A_2(s) \\ -B(s)\begin{pmatrix} I \\ sI \\ \vdots \\ s^{q_2-1}I \end{pmatrix} \\ -\begin{pmatrix} I \\ sI \\ \vdots \\ s^{q_2-1}I \end{pmatrix}(sI - \Lambda_0) \end{bmatrix} = 0 \tag{20}$$

The highest coefficient matrices of the above compound matrices are

$$\begin{bmatrix} 0 & 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & I & 0 \\ I & 0 & 0 & 0 & \cdots & 0 & A_{q_2} \\ 0 & I & & & -B_1 & & \end{bmatrix} \text{ and } \begin{bmatrix} A_{2,q_2} \\ -B_1 \begin{pmatrix} 0 \\ 0 \\ \vdots \\ I \end{pmatrix} \\ -\begin{pmatrix} 0 \\ 0 \\ \vdots \\ I \end{pmatrix} \end{bmatrix}$$

where $B(s) = B_0 + B_1 s$ and $A_2(s) = A_{2,0} + A_{2,1}s + \cdots + A_{2,q_2}s^{q_2}$ with $A_{2,q_2} \neq 0$. The above highest coefficient matrices have full row and column rank respectively and therefore the compound matrices of (20) have no i.e.d.. Consequently the matrices in (19) are d.e. and thus possess the same i.e.d. .

(ii) Our second goal is to prove that the matrices

$$\begin{bmatrix} A(s)\begin{pmatrix} 0 \\ I \end{pmatrix} & A_1(s) \end{bmatrix} \text{ and } \begin{bmatrix} A(s) & A_1(s) \end{bmatrix} \tag{21}$$

possess the same finite and infinite elementary divisors.

(ii-a) *Same finite elementary divisors*

Consider the transformation

$$I \begin{bmatrix} A(s)\begin{pmatrix} 0 \\ I \end{pmatrix} & A_1(s) \end{bmatrix} = \begin{bmatrix} A(s) & A_1(s) \end{bmatrix} \begin{bmatrix} \begin{pmatrix} 0 \\ I \end{pmatrix} & 0 \\ 0 & I \end{bmatrix}$$

We observe that both compound matrices of the above transformation include the unit matrix and therefore does not possess finite elementary divisors. Therefore the matrices (21) are e.u.e and thus have the same finite elementary divisors.

(ii-b) *Same infinite elementary divisors*

Consider the transformation

$$[(sI - \Lambda_0)^{q_1}] \begin{bmatrix} A(s)\begin{pmatrix} 0 \\ I \end{pmatrix} & A_1(s) \end{bmatrix} = \begin{bmatrix} A(s) & A_1(s) \end{bmatrix} \begin{bmatrix} \begin{pmatrix} 0 \\ (sI - \Lambda_0)^{q_1} \end{pmatrix} & 0 \\ 0 & (sI - \Lambda_0)^{q_1} \end{bmatrix} \Longleftrightarrow \tag{22}$$

$$\begin{bmatrix} (sI - \Lambda_0)^{q_1} & A(s) & A_1(s) \end{bmatrix} \begin{bmatrix} A(s)\begin{pmatrix} 0 \\ I \end{pmatrix} & A_1(s) \\ -\begin{pmatrix} 0 \\ (sI - \Lambda_0)^{q_1} \end{pmatrix} & 0 \\ 0 & -(sI - \Lambda_0)^{q_1} \end{bmatrix} = 0$$

where $A(s) = A_0 + A_1 s + \cdots + A_{q_1} s^{q_1}$ and $A_1(s) = A_{1,0} + A_{1,1} s + \cdots + A_{1,q_1} s^{q_1}$ with $A_{1,q_1} \neq 0$. The highest coefficient matrices of the above compound matrices are :

$$\begin{bmatrix} I & A_{q_1} & A_{1,q_1} \end{bmatrix} \text{ and } \begin{bmatrix} A_{q_1}\begin{pmatrix} 0 \\ I \end{pmatrix} & A_{1,q_1} \\ -\begin{pmatrix} 0 \\ I \end{pmatrix} & 0 \\ 0 & -I \end{bmatrix}$$

The above highest coefficient matrices have full row and column rank respectively and therefore the compound matrices in (20) have no i.e.d.. Therefore the matrices in (21) are d.e. and thus possess the same i.e.d..

N.B. In the proof of the necessity of the above Theorem 14, it is possible to start with the matrix $A_2(s)$ and follow identical arguments to yield a transformation of divisor equivalence between $A_2(s)$ and $A_1(s)$ i.e. $M(s)A_2(s) = A_1(s)N(s)$. ∎

Remark 15 *Examining closely the constructive proof of Theorem 14, we can conclude that if $A_1(s), A_2(s) \in \mathbb{R}_c[s]$ have the same f.e.d. and i.e.d. then we can find :*

$$M(s) := A(s)\begin{pmatrix} 0 \\ I \end{pmatrix} \quad ; \quad N(s) := B(s)\begin{pmatrix} I \\ sI \\ \vdots \\ s^{q_2-1}I \end{pmatrix}$$

such that $M(s)A_1(s) = A_2(s)N(s)$ is a d.e. transformation with $\deg[M(s)] \leq \deg[A_2(s)]$ *and* $\deg[N(s)] \leq \deg[A_1(s)]$.

Remark 16 *It is easily seen from the proof of the above theorem that condition (i) of the d.e. guarantees the invariance of the f.e.d., while condition (ii) guarantees the invariance of the i.e.d. of the involved matrices. Therefore in order to prove that two polynomial matrices $A_1(s), A_2(s) \in \mathbb{R}_c[s]$ possess the same finite (resp. infinite) elementary divisors it is enough to prove that there exist polynomial matrices $M(s), N(s)$ of appropriate dimensions that satisfy $M(s)A_1(s) = A_2(s)N(s)$ and condition (i) (resp. (ii)).*

Corollary 17 *The matrix pencil $sE - A$ defined in (11) is d.e. to the square polynomial matrix $A(s)$ defined in (2) and thus according to Theorem 14, $A(s)$ and $sE - A$ possess the same f.e.d. and i.e.d.. This has already been proved in a different way in [29] and [2].*

Example 18 *Consider the transformation*

$$\underbrace{\begin{bmatrix} s+1 & 0 \\ s^2 & 0 \\ 0 & -1 \end{bmatrix}}_{M(s)}\underbrace{\begin{bmatrix} s^2 & 1 \\ 0 & s^3 \end{bmatrix}}_{A_1(s)} = \underbrace{\begin{bmatrix} s^2 & 1 & 0 \\ 0 & s & 1 \\ 0 & 0 & s^2 \end{bmatrix}}_{A_2(s)}\underbrace{\begin{bmatrix} 1 & 0 \\ s^3 & s+1 \\ 0 & -s \end{bmatrix}}_{N(s)}$$

Note that $A_1(s), A_2(s) \in \mathbb{R}_6[s]$. The Smith forms of the compound matrix $\begin{bmatrix} M(s) & A_2(s) \end{bmatrix}$ and of its dual are

$$S^{\mathbb{C}}_{\begin{bmatrix} M(s) & A_2(s) \end{bmatrix}}(s) = \begin{bmatrix} I_4 & 0 \end{bmatrix}$$

and

$$M_1(s) := \widetilde{\begin{bmatrix} M(s) & A_2(s) \end{bmatrix}} = \begin{bmatrix} s+s^2 & 0 & 1 & s^2 & 0 \\ 1 & 0 & 0 & s & s^2 \\ 0 & -s^2 & 0 & 0 & 1 \end{bmatrix} \text{ with } S^{0}_{M_1(s)}(s) = \begin{bmatrix} I_3 & 0 \end{bmatrix}$$

and thus the compound matrix $\begin{bmatrix} M(s) & A_2(s) \end{bmatrix}$ has no f.e.d. nor i.e.d.. Also since

$$S^{\mathbb{C}}_{\begin{bmatrix} A_1(s) \\ -N(s) \end{bmatrix}}(s) = \begin{bmatrix} I_2 \\ 0 \end{bmatrix}$$

and

$$N_1(s) := \widetilde{\begin{bmatrix} A_1(s) \\ -N(s) \end{bmatrix}} = \begin{bmatrix} s & s^3 \\ 0 & 1 \\ -s^3 & 0 \\ -1 & -s^2 - s^3 \\ 0 & s^2 \end{bmatrix} \text{ with } S^{0}_{N_1(s)}(s) = \begin{bmatrix} I_2 \\ 0 \end{bmatrix}$$

the compound matrix $\begin{bmatrix} A_1(s)^T & -N(s)^T \end{bmatrix}^T$ possesses no f.e.d. nor i.e.d.. Therefore $A_1(s)$ and $A_2(s)$ are divisor equivalent and thus according to Theorem 14 they should have the same f.e.d. and i.e.d. Checking their Smith forms

$$S^{\mathbb{C}}_{A_1(s)}(s) = \begin{bmatrix} 1 & 0 \\ 0 & s^5 \end{bmatrix} ;\ S^{\mathbb{C}}_{A_2(s)}(s) = \begin{bmatrix} I_3 & 0 \\ 0 & s^5 \end{bmatrix}$$

and

$$S^{0}_{\tilde{A}_1(s)}(s) = \begin{bmatrix} 1 & 0 \\ 0 & s \end{bmatrix} ;\ S^{0}_{\tilde{A}_2(s)}(s) = \begin{bmatrix} I_3 & 0 \\ 0 & s \end{bmatrix}$$

this is indeed confirmed.

As it shown in the following theorem regarding the special case of matrix pencils with the same dimension, strict equivalence and d.e. define the same equivalence class.

Theorem 19 *Let $sE_1 - A_1, sE_2 - A_2 \in \mathbb{R}[s]^{m\times m}$ with $\det[sE_i - A_i] \neq 0$, $i = 1, 2$. Then $sE_1 - A_1, sE_2 - A_2$ are strictly equivalent iff they are d.e..*

Proof. Sufficiency. If $sE_1 - A_1, sE_2 - A_2$ are strictly equivalent then there exists nonsingular matrices $M, N \in \mathbb{R}^{m\times m}$ such that :

$$M[sE_1 - A_1]N = [sE_2 - A_2] \Longrightarrow M[sE_1 - A_1] = [sE_2 - A_2]N^{-1}$$

Select s_0 such that $\{\det[s_0 E_i - A_i] \neq 0, i = 1, 2\}$ and construct the transformation :

$$[(s - s_0) M] [sE_1 - A_1] = [sE_2 - A_2] \left[(s - s_0) N^{-1}\right]$$

It is easily seen that the above transformation is a d.e. transformation.

Necessity. Suppose that $sE_1 - A_1, sE_2 - A_2$ are d.e.. Then according to Theorem 14, $sE_1 - A_1$ and $sE_2 - A_2$ possess the same f.e.d. and i.e.d.. Therefore [1] the pencils $sE_1 - A_1$ and $sE_2 - A_2$ are strictly equivalent. ■

Strict equivalence defines an equivalence class on the set of square and nonsingular pencils, and thus according to the above theorem defines the same equivalence class with d.e.. However d.e. defines an equivalence class in the more general set of polynomial matrices that belong to $\mathbb{R}_c[s]$ as it can be seen in the following Theorem.

Theorem 20 *Divisor equivalence is an equivalence relation on* $\mathbb{R}_c[s]$.

Proof. (i) *Reflexivity*

Let $A(s) \in \mathbb{R}_c[s]$ and consider the following relation

$$\begin{bmatrix} (s - s_0)^{\deg[A]} I & A(s) \end{bmatrix} \begin{bmatrix} A(s) \\ -(s - s_0)^{\deg[A]} I \end{bmatrix} = 0$$

where s_0 is not a zero of $A(s)$. It is easily proved that the above transformation is a d.e. transformation.

(ii) *Symmetry*

Let $A_1(s), A_2(s) \in \mathbb{R}_c[s]$ be related by a d.e. transformation of the form

$$M(s)A_1(s) = A_2(s)N(s) \tag{23}$$

Then from Theorem 14, $A_1(s)$ and $A_2(s)$ have identical f.e.d. and i.e.d.. Hence from the converse of the Theorem 14 (see N.B. in the proof) there exists a d.e. relation between $A_2(s)$ and $A_1(s)$ of the form (4).

(iii) *Transitivity*

Suppose that $A_1(s), A_2(s) \in \mathbb{R}_c[s]$ are d.e. and that $A_2(s), A_3(s) \in \mathbb{R}_c[s]$ are also d.e.. Then from Theorem 14 $A_1(s)$ and $A_3(s)$ have identical f.e.d and i.e.d.. Hence, from the converse of the Theorem 14, $A_1(s)$ and $A_3(s)$ are divisor equivalent. ■

4 On the connection of d.e. and strict equivalence

In this section we make clear the connection between the proposed transformation of d.e. and the transformations of *strict equivalence* and *fundamental equivalence* defined in [2].

Definition 21 *[2]* $A_1(s)$,$A_2(s) \in \mathbb{R}_c[s]$ *are called* ***strictly equivalent*** *iff their equivalent matrix pencils* $sE_1 - A_1 \in \mathbb{R}^{c\times c}$ *and* $sE_2 - A_2 \in \mathbb{R}^{c\times c}$ *proposed in (7), are strictly equivalent according to Definition 5.*

The coincidence of divisor equivalence and strict equivalence (Definition 21) is proved in the following Theorem.

Theorem 22 *Strict equivalence (Definition 21) belongs to the same equivalence class with d.e..*

Proof. Necessity. Suppose that $A_1(s)$, $A_2(s)$ are d.e.. Then

$$sE_1 - A_1 \overset{d.e}{\sim} A_1(s) \overset{d.e}{\sim} A_2(s) \overset{d.e}{\sim} sE_2 - A_2$$

Therefore from the transitivity property of d.e. $sE_1 - A_1 \overset{d.e}{\sim} sE_2 - A_2$. However d.e. coincides with strict equivalence in pencils as has already been proved in Theorem 19 and thus the two pencils are also strictly equivalent.
Sufficiency. Suppose that $A_1(s)$ and $A_2(s)$ are strictly equivalent according to Definition 21. Then

$$A_1(s) \overset{d.e}{\sim} sE_1 - A_1 \underset{s.e.}{\overset{d.e}{\sim}} sE_2 - A_2 \overset{d.e}{\sim} A_2(s)$$

Then from the transitivity property of d.e. $A_1(s) \overset{d.e}{\sim} A_2(s)$. ∎

The geometrical meaning of strict equivalence, presented in [2], is given in the sequel.

Definition 23 *[2] Two discrete time AR-representations of the form*

$$A_i\,(\sigma)\,\beta_i(k) = 0, k = 0, 1, 2, ..., N - \deg[A_i(s)] \geq 0, i = 1, 2$$

where σ is the shift operator, $A_i(\sigma) \in \mathbb{R}[\sigma]^{r_i \times r_i}$, $\det[A_i(s)] \neq 0, i = 1, 2$ will be called ***fundamentally equivalent*** *(f.e.) over the finite time interval $k = 0, 1, 2, ..., N \geq \max\{\deg[A_i(s)]\}$ iff there exists a bijective polynomial map between their respective behaviors $\mathcal{B}_{A_1(\sigma)}, \mathcal{B}_{A_2(\sigma)}$.*

Note that fundamental equivalence, constitutes an extension of the behavior homomorphism presented in [23], [30] and [31], for the case of non-proper discrete time AR-representations, when studied in a closed time interval. Fundamental equivalence (Definition 23) is the geometric interpretation of strict equivalence (Definition 21) as can be easily seen in the following Theorem.

Theorem 24 *[2] Two discrete time AR-representations of the form (1) are strict equivalent iff they are fundamentally equivalent.*

Based on the above theorem, we can now extend the results presented in Theorem 22.

Theorem 25 *Two discrete time AR-representations of the form (1) are d.e. iff they are fundamentally equivalent.*

Proof. The theorem can be proved using Theorem 22 and Theorem 24. ∎

Therefore, strict equivalence, fundamental equivalence and d.e. define the same equivalence classes. In the following remark we give a geometrical meaning of the right transforming matrix $N(s)$ involved in the d.e. transformation (4).

Remark 26 *Suppose that $A_1(s), A_2(s) \in \mathbb{R}_c[s]$ are d.e. Then there exists polynomial matrices $M(s), N(s)$ such that*

$$M(\sigma)A_1(\sigma) = A_2(\sigma)N(\sigma) \tag{24}$$

By multiplying (24) on the right by $\beta_1(k)$ we get

$$\begin{aligned} M(\sigma)A_1(\sigma)\beta_1(k) = A_2(\sigma)N(\sigma)\beta_1(k) \Longrightarrow 0 = A_2(\sigma)N(\sigma)\beta_1(k) \Longrightarrow \\ \exists\beta_2(k) \in \mathcal{B}_{A_2(\sigma)} \text{ such that } \beta_2(k) = N(\sigma)\beta_1(k) \end{aligned} \tag{25}$$

According to the conditions of d.e., $\begin{bmatrix} A_1(\sigma)^T & -N(\sigma)^T \end{bmatrix}^T$ has full rank and no f.e.d. or i.e.d.. This implies [32] that $\beta_1(k) = 0$. Therefore the map defined by the polynomial matrix $N(s) : \mathcal{B}_{A_1(s)} \to \mathcal{B}_{A_2(s)} \mid \beta_1(k) \mapsto \beta_2(k)$ is injective. Using the symmetry property of d.e. we can find polynomial matrices $\hat{M}(\sigma), \hat{N}(\sigma)$ such that $\hat{M}(\sigma)A_2(\sigma) = A_1(\sigma)\hat{N}(\sigma)$ is a d.e. relation. Then in a similar manner we get that the map defined by the polynomial matrix $\hat{N}(\sigma) : \mathcal{B}_{A_2(s)} \to \mathcal{B}_{A_1(s)} \mid \beta_2(k) \mapsto \beta_1(k)$ is also injective. Therefore both maps are bijections between $\mathcal{B}_{A_1(\sigma)}, \mathcal{B}_{A_2(\sigma)}$.

Note, that the results of the above remark, concerning the right transforming matrix $N(s)$, comes in full accordance with the ones presented by Fuhrmann ([23], Theorem 4.3). The only difference is that here we are referring both to the purely causal and purely anticausal part of the behavior of a non-proper discrete time AR-representations over a closed time interval, while the result in [23] refers only to the causal part of the behavior, but to a more general class of discrete time AR-representations (where the polynomial matrices are not necessary square and nonsingular) and with time axis equal to $\mathbb{Z}^+$ (and not a closed time interval). However, since the study of the causal part of the behavior over the closed time interval comes in accordance with the study of the causal part of the behavior according to Fuhrmann on $\mathbb{Z}^+$, half of our results coincide with the ones of [23] i.e. full rank and coprimeness of the compound matrices in divisor equivalence, ensures the isomorphism between the causal parts of the behaviors. The extra condition arising in divisor equivalence i.e. the absence of infinite elementary divisors, ensures that an isomorphism exists also, between the anti-causal parts of the behaviors of the equivalent systems.

5 Conclusions

It is known [9] that linear homogeneous matrix difference equations of the form (1) exhibit a forward in time behavior which is due to the finite elementary divisors of $A(\sigma)$ and a backward in time behavior which is due to the infinite elementary divisors of $A(\sigma)$. On the other hand continuous time systems of the respective form i.e. $A(\rho)\beta(t) = 0$ with $\rho := d/dt$, exhibit smooth and impulsive behavior [24] due respectively to the finite and infinite zero structure of $A(\sigma)$. Since the behavior of continuous and discrete time systems depends on different structural invariants of the polynomial matrix describing such systems, a new transformation, between square and nonsingular polynomial matrices of diferent degrees and dimensions, termed *divisor equivalence*, has been introduced. It is shown that *divisor equivalence* is an equivalence relation on a specific set of polynomial matrices and has the special property of preserving both the finite and the infinite elementary divisors of the polynomial matrices involved in the transformation. For the special case of matrix pencils with the same dimension, *divisor equivalence* gives rise to the same equivalence class defined through the transformation of *strict equivalence* and extends it in the case of general polynomial matrices. Furthermore, it is shown that *divisor equivalence* constitutes a closed form to test fundamental equivalence of discrete time AR-Representations.

Although a study of the backward behavior of regular discrete time systems of the form (1) has been carried out in [9], there are still no results concerning the connection of the backward in time behavior of regular discrete time polynomial matrix descriptions driven by some nonzero inputs (ARMA models) with the structural invariants of the system. Further research will establish the connection between properties of discrete time polynomial matrix descriptions and the structural invariants of the systems related with both finite and infinite elementary divisors of certain polynomial matrices. Divisor equivalence will then play a key role in the study of equivalence between system matrices as *full equivalence* [8] has played in the study of equivalence between continuous time polynomial matrix descriptions.

A new approach of the past decade has been the so-called "behavioral approach" introduced by Willems, where the behavior of a linear, time invariant, discrete time dynamical system is defined as the set of solutions of a system of the form (1) where now the polynomial matrix $A(\sigma)$ is nonregular i.e. $A(\sigma) \in \mathbb{R}[\sigma]^{r\times m}$ with $r \neq m$ or $A(\sigma) \in \mathbb{R}[\sigma]^{r\times r}$ with $rank_{\mathbb{R}(\sigma)}A(\sigma) < r$. In that case, not only the finite and infinite elementary divisors but also the right minimal indices of $A(\sigma)$ play a crucial role in both the forward and the backward behavior of the AR-representation, while the left minimal indices of $A(\sigma)$ influence the existence of a solution of the AR-representation under specific initial conditions [33]. In order to treat equivalence between nonregular polynomial matrices in a manner similar with that of regular polynomial matrices, an extension of the notion of divisor equivalence is needed. The authors have presented in [34] a new equivalence transformation between nonregular polynomial matrices by using a twofold approach : (a) the "homogeneous polynomial matrix approach" where in place of polynomial matrices we have studied their homogeneous polynomial matrix forms and use 2-D equivalence transformations in order to preserve their infinite elementary divisor structure, and (b) the "polynomial matrix approach" where an additional condition has been added on divisor equivalence in order to treat nonregular polynomial matrices. However, in contrast to divisor equivalence presented in this work, both equivalence transformations presented in [34] give necessary (and not sufficient) conditions in order for two nonregular polynomial matrices to possess the same finite and infinite elementary divisor structure. Further research will be carried out on finding sufficient conditions for two nonregular polynomial matrices to possess the same finite and infinite elementary divisor structure. Since the results concerning the fundamental equivalence of AR-representations presented in [2] were applied only to regular polynomial matrices, an extension of fundamental equivalence to nonregular polynomial matrices and its connection with divisor equivalence has to be established. Additional invariants, appart the finite and infinite elementary divisors, may also be included in the extension of divisor equivalence since the right and left null structure of nonregular polynomial matrices play a key role in the behavior of nonregular AR-representations. Another application of divisor equivalence could be on the study of the equivalence of AR-representations according to the "behavioral approach" of Willems. According to Willems, two AR-representations are equivalent if and only if they share the same behavior. In the particular case where only the forward behavior of the AR-representations is of interest, then "unimodular equivalence" between the corresponding polynomial matrices is the transformation that we are looking for. However, in case where both forward and backward behavior is under research, an extension of unimodular equivalence is needed, following the lines of divisor equivalence. Similar approaches can be applied to the wider class of Normalized AutoRegressive Moving Average (NARMA) representations presented in [23].

References

[1] F. Gantmacher, *The Theory of Matrices.* Chelsea Pub.Co., 1959.

[2] A. Vardulakis and E. Antoniou, "Fundamental equivalence of discrete-time AR-representations," *International Journal of Control*, vol. 76, pp. 1078–1088, 2003.

[3] H. H. Rosenbrock, *State-space and multivariable theory.* John Wiley & Sons, Inc. [Wiley Interscience Division], New York, 1970.

[4] A. C. Pugh and A. K. Shelton, "On a new definition of strict system equivalence," *Internat. J. Control*, vol. 27, no. 5, pp. 657–672, 1978.

[5] A. I. G. Vardulakis and N. Karcanias, "Relations between strict equivalence invariants and structure at infinity of matrix pencils," *IEEE Trans. Automat. Control*, vol. 28, no. 4, pp. 514–516, 1983.

[6] A. C. Pugh, G. E. Hayton, and P. Fretwell, "Transformations of matrix pencils and implications in linear systems theory," *Internat. J. Control*, vol. 45, no. 2, pp. 529–548, 1987.

[7] G. E. Hayton, A. C. Pugh, and P. Fretwell, "Infinite elementary divisors of a matrix polynomial and implications," *Internat. J. Control*, vol. 47, no. 1, pp. 53–64, 1988.

[8] G. E. Hayton, A. B. Walker, and A. C. Pugh, "Infinite frequency structure-preserving transformations for general polynomial system matrices," *Internat. J. Control*, vol. 52, no. 1, pp. 1–14, 1990.

[9] E. N. Antoniou, A. I. G. Vardulakis, and N. P. Karampetakis, "A spectral characterization of the behavior of discrete time AR-representations over a finite time interval," *Kybernetika (Prague)*, vol. 34, no. 5, pp. 555–564, 1998.

[10] F. Lewis, "Descriptor systems : Decomposition into forward and backward subsystems," *IEEE Trans. Autom. Control*, vol. 29, no. 2, pp. 167–170, 1984.

[11] N. Karampetakis, "On the construction of the forward and backward solution space of a discrete time AR-representation." 15th IFAC World Congress 2002, 2002, 15th IFAC World Congress.

[12] J. Willems, "From time series to linear systems : Part i. finite dimensional linear time invariant systems." *Automatica*, vol. 22, no. 5, pp. 561–580, 1986.

[13] ——, *Models for Dynamics, in : U. Kirchgraber, H.O. Walther (Eds.), Dynamics Reported*. Wiley-Teubner, 1989, vol. 2.

[14] ——, "Paradigms and puzzles in the theory of dynamical systems," *IEEE Trans. Automat. Control*, vol. AC-36, pp. 259–294, 1991.

[15] D. Luenberger, "Dynamic equations in descriptor form," *IEEE Trans. Autom. Control*, vol. 22, no. 3, pp. 312–321, 1977.

[16] ——, "Boundary recursion for descriptor variable systems," *IEEE Trans. Autom. Control*, vol. 34, no. 3, pp. 287–292, 1989.

[17] F. Lewis and B. G. Mertzios, "On the analysis of discrete linear time-invariant singular systems," *IEEE Trans. Autom. Control*, vol. 35, no. 4, pp. 506–511, 1990.

[18] F. Lewis, "A survey of linear singular systems," *Circuit Systems and Signal Processing*, vol. 5, no. 1, pp. 3–36, 1986.

[19] R. Nikoukhah, A. Wilski, and B. Levy, "Boundary-value descriptor systems : Well posedness, reachability and observability," *Int. J. Control*, vol. 46, pp. 1715–1737, 1987.

[20] N. P. K. S. N. Antoniou and A. I. G. Vardulakis, "A classification of the solution of non-regular, discrete time descriptor systems." 36th IEEE Conference on Decision and Control, 1997.

[21] I. Gohberg, P. Lancaster, and L. Rodman, *Matrix polynomials*. New York: Academic Press Inc. [Harcourt Brace Jovanovich Publishers], 1982, computer Science and Applied Mathematics.

[22] N. P. Karampetakis, "On the solution space of discrete time AR-representations over a finite time horizon," *Linear Algebra and its Applications*, 2004.

[23] P. A. Fuhrmann, "A study of behaviors," *Linear Algebra and its Applications*, vol. 351-352, pp. 303–380, 2002.

[24] A. I. G. Vardulakis, *Linear multivariable control*. Chichester: John Wiley & Sons Ltd., 1991, algebraic analysis and synthesis methods.

[25] H. Rosenbrock, *State Space and Multivariable Theory*. New York: Nelson-Wiley, 1970.

[26] P. Fuhrmann, "Algebraic system theory: An analyst's point of view," *J. Franklin Institute*, vol. 301, pp. 521–540, 1976.

[27] ——, "On strict system equivalence and similarity," *Int. J. Control*, vol. 25, no. 1, pp. 5–10, 1977.

[28] N. P. Karampetakis, A. C. Pugh, and A. I. Vardulakis, "Equivalence transformations of rational matrices and applications," *Internat. J. Control*, vol. 59, no. 4, pp. 1001–1020, 1994.

[29] C. Praagman, "Invariants of polynomial matrices." 1rst European Control Conference, 1991, pp. 1274–1277.

[30] P. Fuhrmann, "On behavior homomorphisms and system equivalence," *System and Control Letters*, vol. 44, pp. 127–134, 2001.

[31] P. A. Fuhrmann, "A note on continuous behavior homomorphisms," *System and Control Letters*, vol. to appear, 2003.

[32] N. Karampetakis, "On the determination of the dimension of the solution space of discrete time AR-representations." Barcelona, Spain: 15th IFAC World Congress, 2002, 15th IFAC World Congress.

[33] N. P. Karampetakis, "On the solution space of discrete time AR-representations over a finite time horizon," *Linear Algebra and its Application*, vol. to appear, 2004.

[34] N. P. Karampetakis and S. Vologiannidis, "Infinite elementary divisor structure-preserving transformations for polynomial matrices," *Int. J. Appl. Math. Comput. Sci.*, vol. 13, no. 4, pp. 493–503, 2003.

Appendix 1

Proof of Lemma 10

Let that $A_1(s), A_2(s) \in \mathbb{R}_c[s]$ are related by the following divisor equivalence transformation :

$$M_1(s)A_1(s) = A_2(s)N_1(s) \tag{26}$$

while $A_2(s), A_3(s) \in \mathbb{R}_c[s]$ are related by the following strict equivalence transformation :

$$M_2A_2(s) = A_3(s)N_2 \tag{27}$$

where M_2, N_2 are constant and nonsingular matrices (therefore $A_2(s)$ and $A_3(s)$ have the same dimensions). Premultiplying relation (26) by M_2 and using relation (27) we get the following transformation:

$$\begin{aligned} &[M_2M_1(s)]\, A_1(s) = A_3(s)\, [N_2N_1(s)] \Longleftrightarrow \\ &\begin{bmatrix} M_2M_1(s) & A_3(s) \end{bmatrix} \begin{bmatrix} A_1(s) \\ -N_2N_1(s) \end{bmatrix} = 0 \end{aligned} \tag{28}$$

Our main goal is to prove that the compound matrices involved in (28) satisfy the properties of divisor equivalence. We observe that

$$\begin{bmatrix} I & 0 \\ 0 & N_2 \end{bmatrix} \begin{bmatrix} A_1(s) \\ -N_1(s) \end{bmatrix} = \begin{bmatrix} A_1(s) \\ -N_2N_1(s) \end{bmatrix} I$$

and

$$M_2 \begin{bmatrix} M_1(s) & A_2(s) \end{bmatrix} = \begin{bmatrix} M_2M_1(s) & A_3(s) \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & N_2 \end{bmatrix}$$

are strict equivalent transformations and thus the compound matrices

$$\begin{bmatrix} M_2M_1(s) & A_3(s) \end{bmatrix} \quad ; \quad \begin{bmatrix} A_1(s) \\ -N_2N_1(s) \end{bmatrix} \tag{29}$$

possess the same f.e.d. and i.e.d. with the respective compound matrices

$$\begin{bmatrix} M_1(s) & A_2(s) \end{bmatrix} \quad ; \quad \begin{bmatrix} A_1(s) \\ -N_1(s) \end{bmatrix} \tag{30}$$

However, (26) is a divisor equivalence transformation and thus the compound matrices (30) or equivalently (29) does not possess any f.e.d. or i.e.d.. Therefore, (28) is a divisor equivalence transformation.

Appendix 2

Proof of Lemma 11

Let $s_0 \in \mathbb{R}$ such that $det[A(s_0)] \neq 0, \Lambda_0 := diag\,[s_0, s_0, ..., s_0] \in \mathbb{R}^{r\times r}$ so that

$sI_r - \Lambda_0, A(s)$ are coprime. Then consider the identity

$$\underbrace{\begin{bmatrix} 0_{(q-1)r,r} \\ sI_r - \Lambda_0 \end{bmatrix}}_{M(s)} A(s) = \underbrace{\begin{bmatrix} sI_r & -I_r & 0 & \cdots & 0 & 0 \\ 0 & sI_r & -I_r & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & sI_r & -I_r \\ A_0 & A_1 & A_2 & \cdots & A_{q-2} & sA_q + A_{q-1} \end{bmatrix}}_{sE-A} \underbrace{\begin{bmatrix} I_r \\ sI_r \\ \vdots \\ s^{q-1} I_r \end{bmatrix} (sI_r - \Lambda_0)}_{N(s)} \tag{31}$$

In the compound matrix

$$\begin{bmatrix} M(s) & sE - A \end{bmatrix} = \begin{bmatrix} 0 & sI_r & -I_r & 0 & \cdots & 0 & 0 \\ 0 & 0 & sI_r & -I_r & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & sI_r & -I_r \\ (sI_r - \Lambda_0) & A_0 & A_1 & A_2 & \cdots & A_{q-2} & sA_q + A_{q-1} \end{bmatrix}$$

firstly the rq-order minors $\det[sE - A]$ and $\det[L(s)]$ where

$$L(s) := \begin{bmatrix} 0 & -I_r & 0 & \cdots & 0 & 0 \\ 0 & sI_r & -I_r & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & sI_r & -I_r \\ (sI_r - \Lambda_0) & A_1 & A_2 & \cdots & A_{q-2} & sA_q + A_{q-1} \end{bmatrix}$$

are coprime since $A(s)$ and $sI_r - \Lambda_0$ are coprime.

Secondly it is easily seen that the rank of the highest degree coefficient matrix of $\begin{bmatrix} M(s) & sE - A \end{bmatrix}$ is

$$rank_{\mathbb{R}} \begin{bmatrix} \tilde{M}(0) & E \end{bmatrix} = rank_{\mathbb{R}} \begin{bmatrix} 0 & I_r & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_r & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & I_r & 0 \\ I_r & 0 & 0 & 0 & \cdots & 0 & A_q \end{bmatrix} = rq$$

and therefore the compound matrix $\begin{bmatrix} M(s) & sE - A \end{bmatrix}$ has no i.e.d..

Now consider the second compound matrix

$$\begin{bmatrix} A(s) \\ -N(s) \end{bmatrix} = \begin{bmatrix} A(s) \\ (sI_r - \Lambda_0) \\ (sI_r - \Lambda_0)\, s \\ \vdots \\ (sI_r - \Lambda_0)\, s^{q-1} \end{bmatrix}$$

Firstly we can easily find two greatest order minors i.e.

$$Q_1(s) = A(s) \text{ and } Q_2(s) = sI_r - \Lambda_0$$

with

$$det\,[Q_1(s)] = det\,[A(s)] \text{ and } det\,[Q_2(s)] = det\,[sI_r - \Lambda_0]$$

However Λ_0 is selected such that the greatest common divisor of $det\,[Q_1(s)]$ and $det\,[Q_2(s)]$ is 1. Therefore the compound matrix $\begin{bmatrix} A(s) \\ -N(s) \end{bmatrix}$ has no f.e.d..

Secondly the highest degree coefficient matrix of $\begin{bmatrix} A(s) \\ -N(s) \end{bmatrix}$ i.e. $\begin{bmatrix} A_q^T & 0 & \cdots & I_r \end{bmatrix}^T$ has full column rank and thus the compound matrix $\begin{bmatrix} A(s) \\ -N(s) \end{bmatrix}$ has no i.e.d. .

Thus, we have proved that the compound matrices $\begin{bmatrix} M(s) & sE - A \end{bmatrix}$ and $\begin{bmatrix} A(s) \\ -N(s) \end{bmatrix}$ involved in (31), satisfy the conditions of d.e., and therefore $A(s)$ and $sE - A$ are d.e..

Similarly we can show that the following identity between $sE - A$ and $A(s)$ is a d.e. transformation :

$$\underbrace{\begin{bmatrix} -(sI_r - \Lambda_0)\, s^{q-2} E_0(s) & -(sI_r - \Lambda_0)\, s^{q-3} E_1(s) & \cdots & -(sI_r - \Lambda_0)\, E_{q-2}(s) & (sI_r - \Lambda_0)\, s^{q-1} \end{bmatrix}}_{M(s)} \times$$

$$\times \underbrace{\begin{bmatrix} sI_r & -I_r & 0 & \cdots & 0 & 0 \\ 0 & sI_r & -I_r & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & sI_r & -I_r \\ A_0 & A_1 & A_2 & \cdots & A_{q-2} & sA_q + A_{q-1} \end{bmatrix}}_{sE-A} = A(s) \underbrace{\begin{bmatrix} 0_{r,(q-1)r} & (sI_r - \Lambda_0) \end{bmatrix}}_{N(s)} \tag{32}$$

where $E_i(s) = E_{i-1}(s) + A_i s, i = 0, 1, ..., q-2$ with $E_0 = A_0$ and Λ_0 as defined above.

Appendix 3.

Proof of Lemma 12a.

Assume that there exists $M(s)$ with $\deg M(s) > \deg A_2(s)$. In order for $\begin{bmatrix} M(s) & A_2(s) \end{bmatrix}$ to have no i.e.d its highest coefficient matrix must be of full row rank i.e. $rank \begin{bmatrix} M_{hc} & 0 \end{bmatrix} = m + r$. This is impossible because of the dimension of the matrix $M_{hc} \in \mathbb{R}^{(m+r)\times m}$. So $\deg M(s) <= \deg A_2(s)$.

Let $\deg N(s) > \deg A_1(s)$. Then there exists $d \neq 0$ such that $\deg A_1(s) + d = \deg N(s)$. Condition (i) of d.e. implies that the matrices $A_1(s)$ and $A_2(s)$ are e.u.e. so they have the same f.e.d. (and of course the same number of f.e.d. i.e. $S_{\mathbb{R}}\left(A_1(s)\right) = S_{\mathbb{R}}\left(A_2(s)\right)$ where $S_{\mathbb{R}}(A(s))$ denotes the total number of f.e.d. of $A(s)$ (order accounted for)). Taking the duals of the compound matrices in (4) we get

$$\begin{bmatrix} M'(w) & \widetilde{A}_2(w) \end{bmatrix} \begin{bmatrix} w^d \widetilde{A}_1(w) \\ -\widetilde{N}(w) \end{bmatrix} = 0 \tag{33}$$

Since the compound matrices in (5) have no i.e.d., the compound matrices in (33) will have no f.e.d. at $s = 0$ and thus (33) is a $\{0\}$-equivalence relation. Therefore, $\widetilde{A}_2(w)$ and $w^d\widetilde{A}_1(w)$ have the same f.e.d. at 0. Denoting by $S_l(A(w))$ the total number of f.e.d. (order accounted for) at l of $A(w)$ we have

$$S_0(\widetilde{A}_2(w)) = S_0(w^d\widetilde{A}_1(w)) > S_0(\widetilde{A}_1(w)) \Longrightarrow$$
$$S_0(\widetilde{A}_2(w)) > S_0(\widetilde{A}_1(w)) \Longrightarrow S_\infty(A_2(s)) > S_\infty(A_1(s)) \tag{34}$$

According to our assumption $A_1(s), A_2(s) \in \mathbb{R}_c[s]$. Thus $c = S_\infty(A_2(s)) + S_\mathbb{R}(A_2(s)) = S_\infty(A_1(s)) + S_\mathbb{R}(A_1(s))$ or equivalently since $S_\mathbb{R}(A_2(s)) = S_\mathbb{R}(A_1(s))$ we have that $S_\infty(A_2(s)) = S_\infty(A_1(s))$ which contradicts with (34).

Proof of Lemma 12b.

Firstly we shall prove that if one of the chosen transforming matrices has degree more than d then $\deg M(s) = \deg N(s)$. Let

$$d_M = \deg M,\ d_N = \deg N$$
$$D_L = \deg \begin{bmatrix} M(s) & A_2(s) \end{bmatrix} \ ; \ D_R = \deg \begin{bmatrix} A_1(s) \\ -N(s) \end{bmatrix}$$
$$d = \deg A_1(s) = \deg A_2(s)$$

Then

$$\widetilde{\begin{bmatrix} M(s) & A_2(s) \end{bmatrix}} = \begin{bmatrix} w^{-d_M+D_L}\tilde{M}(w) & w^{-d+D_L}\widetilde{A}_2(w) \end{bmatrix}$$
$$\widetilde{\begin{bmatrix} A_1(s) \\ -N(s) \end{bmatrix}} = \begin{bmatrix} w^{-d+D_R}\widetilde{A}_1(w) \\ -w^{-d_N+D_R}\widetilde{N}(w) \end{bmatrix}$$

>From (4) taking the duals gives

$$\begin{bmatrix} w^{-d_M+D_L}\tilde{M}(w) & w^{-d+D_l}\widetilde{A}_2(w) \end{bmatrix} \begin{bmatrix} w^{-d+D_R}\widetilde{A}_1(w) \\ -w^{-d_N+D_R}\widetilde{N}(w) \end{bmatrix} = 0 \tag{35}$$

It is easily seen for the reasons explained in the proof of the previous Lemma that (35) is a $\{0\}$-equivalence relation either between $w^{-d+D_l}\widetilde{A}_2(w)$ and $w^{-d+D_R}\widetilde{A}_1(w)$ or between $w^{-d_M+D_L}\tilde{M}(w)$ and $w^{-d_N+D_R}\widetilde{N}(w)$ and thus

$$\left\{ \begin{array}{c} S_0(w^{-d_M+D_L}\tilde{M}(w)) = S_0(w^{-d_N+D_R}\widetilde{N}(w)) \\ S_0(w^{-d+D_L}\widetilde{A}_2(w)) = S_0(w^{-d+D_R}\widetilde{A}_1(w)) \end{array} \right\} \Rightarrow$$
$$S_\infty M(s) + (-d_M + D_L)m = S_\infty N(s) + (-d_N + D_R)m \tag{36}$$
$$S_\infty A_2(s) + (-d + D_L)m = S_\infty A_1(s) + (-d + D_R)m \tag{37}$$

Also (4) is an e.u.e. relation and

$$S_\mathbb{R} M(s) = S_\mathbb{R} N(s) \ ; \ S_\mathbb{R} A_2(s) = S_\mathbb{R} A_1(s) \tag{38}$$

But since $A_1(s)$ and $A_2(s) \in \mathbb{R}_c[s]$ we have

$$S_{\mathbb{R}}A_1(s) + S_\infty A_1(s) = S_{\mathbb{R}}A_2(s) + S_\infty A_2(s) \overset{(37)}{\Longrightarrow} \tag{39}$$

$$S_{\mathbb{R}}A_1(s) + S_\infty A_2(s) + (-d + D_L)m-$$

$$-(-d + D_R)m = S_\infty A_2(s) + S_{\mathbb{R}}A_2(s) \overset{(38)}{\Longrightarrow}$$

$$D_L = D_R \tag{40}$$

Assume that $d_M > d$. Then

$$d_M = D_L \overset{(40)}{=} D_R \tag{41}$$

i.e. the degree of the compound matrix is equal to the maximum degree of $M(s)$ and $A_2(s)$ which is d_M. From (40) and (41) we have that

$$D_R = d_M > d$$

The above inequality tell us that the degree of the right compound matrix in (4) (the maximum degree of $N(s)$ and $A_1(s)$) is more than $\deg(A_1(s))$ and thus $D_R = d_N$. Therefore we conclude that $d_M = D_L = D_R = d_N$. Similar proofs are also apply in case where we assume that $d_N > d$.

Secondly we shall show that if one of the chosen transforming matrices has degree less than d then $\deg M(s) = \deg N(s)$. Suppose that $N(s)$ has degree d_N such that $d_N(s) < d_M(s) \leq d$ i.e. note that from the first part of the proof if one of the matrices has degree less than d then the other one cannot have degree more than d. Then by equating the coefficient matrices of the highest degrees of s in (4) we get that

$$M^{hc}A_1^{hc} = 0 \tag{42}$$

where A^{hc} denotes the highest degree coefficient matrix of the polynomial matrix $A(s)$. Since $d_N(s) < d$ we have that A_1^{hc} has full column rank and therefore $\dim(Ker(M^{hc})) = m$. Thus $rank(M^{hc}) = 0$ and therefore $\deg M(s) < d_M$ which contradicts with our second assumption.

Appendix 4.

Proof of Lemma 13.

The proof is trivial having in mind that a) the f.e.d. of $A_1(s)(s - s_0)^k$ are the f.e.d. of $A_1(s)$ plus m divisors of the form $(s - s_0)^k$ and b) the i.e.d. of $A_1(s)(s - s_0)^k \in \mathbb{R}^{m\times m}[s]$, where $s_0 \neq 0$ is not a zero of $A_1(s)$, are exactly the i.e.d. of $A_1(s)$.