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On the computation and parametrization of proper denominator assigning compensators for strictly proper plants

E. N. ANTONIOU† AND A. I. G. VARDULAKIS‡
Department of Mathematics, Aristotle University of Thessaloniki, 54006 - Thessaloniki, Greece

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Given a right coprime MFD of a strictly proper plant $P(s) = N_R(s)D_R(s)^{-1}$ with $D_R(s)$ column proper a simple numerical algorithm is derived for the computation of all polynomial solutions $[X_L(s), Y_L(s)]$ of the polynomial matrix Diophantine equation $X_L(s)D_R(s) + Y_L(s)N_R(s) = D_C(s)$ which give rise to the class $\Phi(P, D_C)$ of proper compensators $C(s) := X_L(s)^{-1}Y_L(s)$ that when employed in a unity feedback loop, result in closed-loop systems $S(P, C)$ with a desired denominator $D_C(s)$. The parametrization of the proper compensators $C(s) \in \Phi(P, D_C)$ is obtained and the number of independent parameters in the parametrization is given.

Keywords: linear multivariable control; coprime factorizations; diophantine euqations.

1. Introduction

We consider linear, time invariant, multivariable systems which are assumed to be free of unstable hidden modes, whose input–output relation is described by a strictly proper transfer function matrix $P(s)$ (the plant). In this note we describe a numerically efficient algorithm for the computation of the class of proper compensators $C(s)$ which, when employed in the unity feedback loop of Fig. 1, gives rise to a closed-loop system $S(P, C)$ with a specific closed-loop denominator $D_C(s)$ (Rosenbrock & Hayton, 1978; Kucera, 1970). In particular, given a right coprime MFD of a strictly proper plant $P(s) = N_R(s)D_R(s)^{-1}$ with $D_R(s)$ column proper (column reduced) and an appropriately defined polynomial matrix $D_C(s)$ with desired zeros, we extend the (Wolovich, 1974) resultant theorem and a theorem by Callier & Desoer (1982), Callier (2001) and Kucera & Zagalak (1999) in order to obtain an algorithm for the computation of all polynomial solutions $[X_L(s), Y_L(s)]$ of the polynomial matrix Diophantine equation

$$X_L(s)D_R(s) + Y_L(s)N_R(s) = D_C(s) \tag{1}$$

which give rise to the class $\Phi(P, D_C)$ of proper compensators $C(s) := X_L(s)^{-1}Y_L(s)$ that result in closed-loop systems $S(P, C)$ with $D_C(s)$ as their closed-loop denominator. The issues of the parametrization of the proper compensators $C(s) \in \Phi(P, D_C)$ and the number of independent parameters in the parametrization are also resolved. This is done by investigating the properties of a generalized version of Wolovich's resultant to obtain a series of new results regarding its algebraic structure. Despite the fact that similar results for Sylvester-type resultants have been presented in Bitmead *et al.* (1978), the Wolovich resultant has not received the expected attention, except perhaps by Wolovich (1974) and Hayton (1980) where Wolovich's resultant is used as a tool for testing the coprimeness of polynomial matrices.

†Corresponding author. Email: antoniou@math.auth.gr
‡Email: avardula@auth.gr

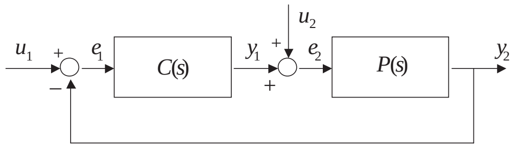

FIG. 1. The unity feedback system $S(P, C)$.

The method presented here can be compared to the one in Antsaklis & Gao (1993), where Wolovich's resultant is employed as a tool for the construction of the interpolation matrix. However, our method requires only knowledge of the coefficients of the polynomial matrices $D_R(s)$, $N_R(s)$ and provides a parametrization of all proper denominator assigning controllers, unifying in this way the 'resultant' approach with the approaches in Callier & Desoer (1982), Callier (2001) and Kucera & Zagalak (1999). The proposed approach can be viewed as a generalization of the method presented in Antsaklis & Michel (1997, Theorem 2.13, p. 547) where the solution of a degree-specific Diophantine equation is obtained using Wolovich's resultant. Furthermore, through the investigation of the rank of the generalized Wolovich resultant, we establish the lower bound for the (McMillan) degree of an arbitrary closed-loop denominator, a fact which has been used throughout the constructions in Callier & Desoer (1982), Callier (2001) and Kucera & Zagalak (1999), but not justified via some theoretic argument.

2. Preliminaries

In the following $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}(s)$, $\mathbb{R}[s]$, $\mathbb{R}_{pr}(s)$, $\mathbb{R}_{po}(s)$ are respectively the fields of *real numbers, complex numbers, real rational functions*, the rings of *polynomials*, *proper rational* and *strictly proper rational* functions all with coefficients in $\mathbb{R}$ and indeterminate s. For a set $\mathbb{F}$, $\mathbb{F}^{p\times m}$ denotes the set of $p \times m$ matrices with entries in $\mathbb{F}$. $\mathbb{N}^+$ is the set of positive integers. If $m \in \mathbb{N}^+$ then $\mathbf{m}$ denotes the set $\{1, 2, \ldots, m\}$. Finally $\delta_M[\cdot]$ denotes the McMillan degree of $[\cdot]$

Let $N_R(s) \in \mathbb{R}[s]^{p\times m}$, $D_R(s) \in \mathbb{R}[s]^{m\times m}$ be a pair of polynomial matrices with $D_R(s)$ invertible for almost every $s \in \mathbb{C}$ and define the compound matrix $F(s) := \left[D_R^\top(s), N_R^\top(s)\right]^\top$. Respectively let $D_L(s) \in \mathbb{R}[s]^{p\times p}$, $N_L(s) \in \mathbb{R}[s]^{p\times m}$ (with $D_L(s)$ invertible for a.e. $s \in \mathbb{C}$) and $E(s) := [-N_L(s), D_L(s)]$ such that

$$E(s)F(s) = 0. \tag{2}$$

The pair of matrices $N_R(s)$, $D_R(s)$ (resp. $N_L(s)$, $D_L(s)$) will be called right (resp. left) coprime iff $F(s)$ has full column rank (resp. $E(s)$ has full row rank) for every $s \in \mathbb{C}$. It is known that if $N_R(s)$, $D_R(s)$ are right coprime and $N_L(s)$, $D_L(s)$ left coprime, then

$$\deg|D_R(s)| = \deg|D_L(s)|. \tag{3}$$

A polynomial matrix $X(s) \in \mathbb{R}[s]^{p\times m}$ $(m \leqslant p)$ is called column proper or column reduced iff its highest column coefficient matrix denoted X^{hc}, which is formed by the coefficients of the highest degrees of s in each column of $X(s)$, has full column rank. The column degrees of $X(s)$ are usually denoted by $\deg_{ci} X(s)$, $i \in \mathbf{m}$. Respectively $Y(s) \in \mathbb{R}[s]^{p\times m}$ $(p \leqslant m)$ is called row proper or row reduced iff $Y^T(s)$ is column proper and the row degrees of $Y(s)$ are denoted by $\deg_{ri} Y(s)$, $i \in \mathbf{p}$. Furthermore, a square polynomial matrix $X(s) \in \mathbb{R}[s]^{m\times m}$ is called row-column reduced (Callier & Desoer, 1982) with row powers r_i and column powers c_i, $i \in \mathbf{m}$ iff the matrix $\operatorname{diag}_{i\in\mathbf{m}}\{s^{-r_i}\}X(s)\operatorname{diag}_{i\in\mathbf{m}}\{s^{-c_i}\}$ is biproper (i.e. it is proper and its inverse exists and it is proper as well).

LEMMA 1 (Vardulakis, 1991; Corollary 3.100, p. 144) If $X(s) \in \mathbb{R}[s]^{p\times m}$ $(p \geqslant m)$ is column proper then $X(s)$ has no zeros at infinity and its (ordered) column degrees are the orders of its poles at infinity i.e. if

$$S^{\infty}_{X(s)} = \begin{bmatrix} \operatorname{diag}\{s^{q_1}, s^{q_2}, \ldots, s^{q_m}\} \\ 0_{p-m,m} \end{bmatrix}$$

is the Smith–McMillan form of $X(s)$ at infinity, with $q_1 \geqslant q_2 \geqslant \cdots \geqslant q_m \geqslant 0$, then $q_i = \deg_{ci} X(s)$, $i \in \mathbf{m}$. Furthermore, since $X(s)$ (as polynomial matrix) has no finite poles and due to s^{q_i} has (possibly) only poles at infinity, $\delta_M X(s) = \sum_{i=1}^{m} \deg_{ci} X(s)$.

Obviously a similar result holds for row proper matrices.

When (2) is satisfied and $E(s)$ is row proper with $D_L(s)$, $N_L(s)$ left coprime, $E(s)$ is a minimal polynomial basis of the (rational) vector space spanning the left kernel of $F(s)$ and the row degrees $\deg_{ri} E(s) =: \mu_i$, $i \in \mathbf{p}$ of $E(s)$ are the invariant row minimal (dual) dynamical indices of

$$P(s) = N_R(s)D_R^{-1}(s) = D_L^{-1}(s)N_L(s). \tag{4}$$

In such a case it is known (Forney, 1975) that $E(s)$ has the following properties:

1. if $p(s)^\top \in \mathbb{R}[s]^{1\times(p+m)}$ is a polynomial vector such that $p(s)^\top F(s) = 0$ then there exists a polynomial vector $w(s)^\top = [w_1(s), w_2(s), \ldots, w_p(s)] \in \mathbb{R}[s]^{1\times p}$ such that

$$p(s)^\top = w(s)^\top E(s); \tag{5}$$

2. if $p(s)^\top = w(s)^\top E(s)$ then

$$\deg p(s)^\top = \max_{i\in\mathbf{p}}\{\deg w_i(s) + \mu_i\}. \tag{6}$$

The following result establishes a relation between the McMillan degrees of $P(s)$ and $E(s)$ (or $F(s)$).

LEMMA 2 (Vardulakis, 1991; p. 140) If $E(s)$ has no zeros in $\mathbb{C} \cup \{\infty\}$ (equiv. $D_L(s)$, $N_L(s)$ are left coprime in $\mathbb{C} \cup \{\infty\}$) then

$$\delta_M P(s) = \delta_M E(s). \tag{7}$$

When $E(s)$ is a minimal polynomial basis of the left kernel of $F(s)$, i.e. $E(s)$ has no zeros in $\mathbb{C}$ and is row proper, by Lemma 1 $E(s)$ will have no zeros in $\mathbb{C} \cup \{\infty\}$ and thus, from the last statement of Lemma 1,

$$\delta_M P(s) = \delta_M E(s) = \sum_{i=1}^{p} \deg_{ri} E(s). \tag{8}$$

Furthermore, if also $D_R(s), N_R(s)$ are right coprime and $F(s)$ is column proper then again, from Lemmata 1 and 2,

$$\delta_M P(s) = \delta_M F(s) = \sum_{i=1}^{m} \deg_{ci} F(s), \tag{9}$$

thus in such a case we get the well known result (Forney, 1975) that

$$\sum_{i=1}^{p} \deg_{ri} E(s) = \sum_{i=1}^{m} \deg_{ci} F(s). \tag{10}$$

3. Generalized Wolovich resultant

Let $k_i = \deg_{ci} F(s), i \in \mathbf{m}$ be the column degrees of $F(s)$ and similarly to Wolovich (1974, p. 242) for $k \geqslant 1$ define the $(m+p)k \times m$ polynomial matrix $X_k(s)$ via

$$X_k(s) := S_k(s) \begin{bmatrix} D_R(s) \\ N_R(s) \end{bmatrix} = \underbrace{\begin{bmatrix} I_{m+p} \\ sI_{m+p} \\ \vdots \\ s^{k-1} I_{m+p} \end{bmatrix}}_{(m+p)k\times(m+p)} \underbrace{\begin{bmatrix} D_R(s) \\ N_R(s) \end{bmatrix}}_{(m+p)\times m} = \underbrace{\begin{bmatrix} F(s) \\ sF(s) \\ \vdots \\ s^{k-1} F(s) \end{bmatrix}}_{(m+p)k\times m} \tag{11}$$

and notice that $X_k(s)$ can be written as

$$X_k(s) = M_{ek} \underbrace{\left[\underset{i\in\mathbf{m}}{block\ \mathrm{diag}} \left\{ \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{k_i+k-1} \end{bmatrix} \right\} \right]}_{\left(\sum_{i=1}^{m} k_i + mk\right)\times m} =: M_{ek} S_{ek}(s) \tag{12}$$

where $M_{ek} \in \mathbb{R}^{(m+p)k\times(mk+\sum_{i=1}^{m} k_i)}$. Notice that M_{ek} does not coincide with the one in Wolovich (1974) since Wolovich assumes that $D_R(s)$ is column proper and $P(s) = N_R(s)D_R^{-1}(s)$ is proper. Apart from that, essentially the two matrices differ only up to row permutations.

One of our goals is to describe the left null space (kernel) of M_{ek} which in what follows is denoted by

$$Ker\, M_{ek}^{\top} = \{x^{\top} \in \mathbb{R}^{1\times(m+p)k} : x^{\top} M_{ek} = 0\}. \tag{13}$$

The following theorem determines the dimension of $\ker M_{ek}^{\top}$.

THEOREM 3 Let $N_R(s) \in \mathbb{R}[s]^{p\times m}, D_R(s) \in \mathbb{R}[s]^{m\times m}$ be a pair of polynomial matrices with $\mathrm{rank}_{\mathbb{R}(s)} \left[D_R^{\top}(s), N_R^{\top}(s)\right]^{\top} = m$. Let also $P(s) = N_R(s)D_R^{-1}(s) \in \mathbb{R}(s)^{p\times m}$, $\mu_i, i \in \mathbf{p}$ be the invariant row minimal dynamical indices of $P(s)$ and $M_{ek} \in \mathbb{R}^{(m+p)k\times(mk+\sum_{i=1}^{m} k_i)}$ as defined in (12). Then

$$\dim_{\mathbb{R}} \ker M_{ek}^{\top} = \sum_{i:\mu_i \leqslant k} (k - \mu_i). \tag{14}$$

Proof. The proof is identical to that of Theorem 1 in Bitmead *et al.* (1978). □

It is interesting to notice that the dimension of the kernel obtained here is identical to the one given in Theorem 1 in Bitmead *et al.* (1978), despite the fact that the generalized Sylvester resultant S_k in Bitmead *et al.* (1978) does not coincide in general with M_{ek}. Notice also that the above result does not require $D_R(s)$, $N_R(s)$ to be right coprime nor $D_R(s)$ to be column proper. We give now a generalization of the result that appears in Antsaklis & Gao (1993, Lemma 3.2), in the sense that we relax $P(s) = N_R(s)D_R^{-1}(s)$ from the properness requirement as well as from the assumption that $D_R(s)$ is column proper.

COROLLARY 4 Under the assumptions of Theorem 3, we have

$$\text{rank} M_{ek} = (p+m)k - \sum_{i:\mu_i \leqslant k} (k - \mu_i). \tag{15}$$

Furthermore, if k is chosen s.t. $k \geqslant \mu$, where $\mu = \max_{i \in \mathbf{p}}\{\mu_i\}$ then

$$\text{rank} M_{ek} = mk + \delta_M P(s). \tag{16}$$

Proof. Equation (15) follows simply from the fact that $\text{rank} M_{ek} = (p+m)k - \dim_{\mathbb{R}} \ker M_{ek}^{\top}$ and (14). Now for $k \geqslant \mu$, (15) becomes $\text{rank} M_{ek} = (p+m)k - \sum_{i=1}^{p}(k - \mu_i)$ or equivalently $\text{rank} M_{ek} = mk + \sum_{i=1}^{p} \mu_i$, thus (16) follows from the facts $\sum_{i=1}^{p} \mu_i = \delta_M E(s) = \delta_M P(s)$ in Lemmata 1 and 2. □

Notice that in the case that $D_R(s)$ is column proper and $P(s) := N_R(s)D_R^{-1}(s)$ is proper, $\delta_M P(s) = \{\#$ of poles of $P(s)$ in $\mathbb{C}\} = \deg|D_R(s)|$. Therefore, for $k \geqslant \mu$ the above result coincides with the result of Lemma 3.2 in Antsaklis & Gao (1993). The following corollary provides a generalization of the corresponding result in Wolovich (1974, p. 242).

COROLLARY 5 Let $N_R(s) \in \mathbb{R}[s]^{p\times m}$, $D_R(s) \in \mathbb{R}[s]^{m\times m}$ be a pair of polynomial matrices with $F(s) = \left[D_R^{\top}(s), N_R^{\top}(s)\right]^{\top}$ column proper with column degrees $\deg_{ci} F(s) = k_i$, $i \in \mathbf{m}$. Then $N_R(s), D_R(s)$ are right coprime in $\mathbb{C}$ iff M_{ek} has full column rank for $k \geqslant \mu$, or equivalently if $F(s) = \left[D_R^{\top}(s), N_R^{\top}(s)\right]^{\top}$ is column proper then $N_R(s), D_R(s)$ are right corpime in $\mathbb{C}$ iff for $k \geqslant \mu$, $\text{rank} M_{ek} = mk + \delta_M F(s)$.

Proof. First notice that from (12) the number of columns in M_{ek} is $mk + \sum_{i=1}^{m} k_i$. Since $F(s)$ is column proper it has no zeros at infinity and from Lemma 1 $\sum_{i=1}^{m} k_i = \delta_M F(s)$. Hence the number of columns in M_{ek} is $mk + \delta_M F(s)$.

($\Rightarrow$) Let $N_R(s), D_R(s)$ be right coprime in $\mathbb{C}$. Then from Corollary 4 for $k \geqslant \mu$, $\text{rank} M_{ek} = km + \delta_M P(s)$. Since $F(s)$ is column proper from Lemma 1 it has no zeros at infinity, thus $N_R(s), D_R(s)$ are right coprime at $s = \infty$. Hence $N_R(s), D_R(s)$ are right coprime in $\mathbb{C} \cup \{\infty\}$, thus from Lemma 2 $\delta_M P(s) = \delta_M F(s)$ and $\text{rank} M_{ek} = km + \delta_M F(s)$.

($\Leftarrow$) Assume that $N_R(s), D_R(s)$ are not right coprime in $\mathbb{C}$. Then there exists $0 \neq x \in \mathbb{R}^{m\times 1}$ and $s_0 \in \mathbb{C}$ such that $F(s_0)x = 0$. In view of (12)

$$X_k(s_0)x = M_{ek} S_{ek}(s_0)x = 0$$

hence M_{ek} does not have full column rank. □

The following remark establishes the fact that M_{ek} can have full column rank only for $k \geqslant \mu$.

REMARK 6 Let $D_R(s) \in \mathbb{R}[s]^{m\times m}$, $N_R(s) \in \mathbb{R}[s]^{p\times m}$ such that $D_R(s)$, $N_R(s)$ be right coprime in $\mathbb{C}$ and $F(s) = \left[D_R^\top(s),\, N_R^\top(s)\right]^\top$ be column proper with column degrees $\deg_{ci} F(s) = k_i$, $i \in \mathbf{m}$. Let also μ_i, $i \in \mathbf{p}$ be the left minimal indices of $F(s)$ and define $\mu = \max\limits_{i\in\mathbf{p}}\{\mu_i\}$. Then for $k < \mu$

$$\mathrm{rank} M_{ek} < mk + \sum_{i=1}^{m} k_i \tag{17}$$

i.e. M_{ek} cannot have full column rank for $k < \mu$.

Proof. Assume $k < \mu$ and let a be the number of μ_i satisfying $\mu_i > k$. It is easy to see that

$$ka < \sum_{i:\mu_i>k} \mu_i. \tag{18}$$

Using the fact that $\sum_{i=1}^{p} \mu_i = \sum_{i:\mu_i>k} \mu_i + \sum_{i:\mu_i\leqslant k} \mu_i$ we can write (18) as $ka + \sum_{i:\mu_i\leqslant k} \mu_i < \sum_{i=1}^{p} \mu_i$ or equivalently as

$$pk - k(p-a) + \sum_{i:\mu_i\leqslant k} \mu_i < \sum_{i=1}^{p} \mu_i. \tag{19}$$

Notice that the number of terms in $\sum_{i:\mu_i\leqslant k} \mu_i$ is exactly $p - a$, thus we can write (19) as

$$pk - \sum_{i:\mu_i\leqslant k} (k-\mu_i) < \sum_{i=1}^{p} \mu_i. \tag{20}$$

Adding mk on both sides of (20) we get $(m+p)k - \sum_{i:\mu_i\leqslant k}(k-\mu_i) < mk + \sum_{i=1}^{p} \mu_i$ where obviously the left-hand side is $\mathrm{rank} M_{ek}$ and $\sum_{i=1}^{p} \mu_i = \sum_{i=1}^{m} k_i$ due to the assumptions of coprimeness (of $D_R(s)$, $N_R(s)$ in $\mathbb{C}$) and the column properness of $F(s)$ (see 10). Thus (17) follows.

The above result has a direct implication on the choice of the row degrees of $D_C(s)$ in (1) which will be discussed in the following section.

4. Application to matrix Diophantine equations

Consider a strictly proper linear multivariable plant, $P(s) \in \mathbb{R}_{po}(s)^{p\times m}$ with m inputs and p outputs and let

$$P(s) = N_R(s)D_R(s)^{-1} = D_L(s)^{-1}N_L(s) \tag{21}$$

be respectively right and left coprime MFDs of $P(s)$ with $N_R(s) \in \mathbb{R}[s]^{p\times m}$ and $D_R(s) \in \mathbb{R}[s]^{m\times m}$ and *column proper* with column degrees $\deg D_{Rci}(s) = k_i$, $i \in \mathbf{m}$, $N_L(s) \in \mathbb{R}[s]^{p\times m}$ and $D_L(s) \in \mathbb{R}[s]^{p\times p}$ and *row proper* with row degrees $\deg D_{Lri}(s) = \mu_i$, $i \in \mathbf{p}$. Define $\mu := \max\limits_{i\in\mathbf{p}}\{\mu_i\}$ (the observability index of $P(s)$).

The problem of assigning the denominator of the closed-loop system using unity feedback and a dynamic precompensator $C(s) \in \mathbb{R}(s)^{m\times p}$ can be reduced to the solution of the polynomial matrix Diophantine equation of the form

$$X_L(s)D_R(s) + Y_L(s)N_R(s) = D_C(s) \tag{22}$$

where $D_C(s) \in \mathbb{R}[s]^{m\times m}$ is the desired closed-loop denominator matrix and $X_L(s) \in \mathbb{R}[s]^{m\times m}$, $Y_L(s) \in \mathbb{R}[s]^{m\times p}$ is a left (not necessarily coprime) MFD of $C(s)$, i.e.

$$C(s) = X_L(s)^{-1}Y_L(s) \in \mathbb{R}(s)^{m\times p}. \tag{23}$$

It is well known that (22) has a solution for arbitrary $D_C(s)$ iff $D_R(s)$, $N_R(s)$ are right coprime. Furthermore, if $\overline{X}_L(s), \overline{Y}_L(s)$ is a particular solution of (22) then every pair of the form $X_L(s) = \overline{X}_L(s) + T(s)N_L(s)$, $Y_L(s) = \overline{Y}_L(s) - T(s)D_L(s)$ is also a solution of (22) for any arbitrary polynomial matrix $T(s) \in \mathbb{R}[s]^{m\times p}$.

However, the question usually posed is under what conditions (22) can have solutions that give rise to a *proper* compensator $C(s) \in \mathbb{R}_{pr}(s)^{m\times p}$. For a particular type of closed-loop denominator this problem has been studied and solved by several authors (see Rosenbrock & Hayton, 1978; Emre, 1980; Callier & Desoer, 1982; Kucera, 1970) and a parametrization of all possible proper denominator assigning compensators has been given (see Kucera & Zagalak, 1999; Callier, 2001). According to this approach the desired denominator is chosen to be row-column reduced with particular row and column powers in order to be able to apply degree control on the numerator and denominator of $C(s)$.

The contribution of the present paper is to provide a numerical algorithm which employs Wolovich's resultant proposed in the previous section to obtain a parametrization of all denominator assigning proper compensators. Let $X_L(s), Y_L(s)$ be a solution of (22) for a particular choice of $D_C(s)$ and let $k-1$ be the maximum degree of s occurring amongst the elements of the matrix $\Omega(s) := [\,X_L(s), Y_L(s)] \in \mathbb{R}[s]^{m\times(m+p)}$. Then $\Omega(s)$ can be written as

$$\Omega(s) = \overline{\Omega}_k S_k(s) \tag{24}$$

where $\overline{\Omega} \in \mathbb{R}^{m\times k(p+m)}$ and $S_k(s)$ is as defined in (11). Then (22) can be written as

$$\overline{\Omega}_k M_{ek} S_{ek}(s) = D_C(s) \tag{25}$$

with $S_{ek}(s)$ defined if (12). Comparing the degrees of s in both sides of (25) it is easily seen that $\deg_{ci} D_C(s) \leqslant k_i + k - 1, i \in \mathbf{m}$ thus $D_C(s)$ can be written as $D_C(s) = \overline{D}_k S_{ek}(s)$, $\overline{D}_k \in \mathbb{R}^{m\times(\sum_{i=1}^{m} k_i + mk)}$ and (25) becomes

$$\overline{\Omega}_k M_{ek} S_{ek}(s) = \overline{D}_k S_{ek}(s) \tag{26}$$

or equivalently

$$\overline{\Omega}_k M_{ek} = \overline{D}_k \tag{27}$$

since (26) must hold for every $s \in \mathbb{C}$. Thus every solution of (22) can be determined from a set of numerical equations of the form (27) given the maximum degree of $\Omega(s)$ and selecting the appropriate k.

The following lemma can be found in Kucera & Zagalak (1999) stated for the dual of (22), i.e. for a left MFD of $P(s)$. For our purposes we shall state the corresponding assumptions and the result for a right MFD of $P(s)$.

LEMMA 7 (Kucera & Zagalak, 1999; Lemma 2) Consider (22) under the following assumptions:

1. $D_R(s)$ is column proper with column degrees $k_i = \deg_{ci} D_R(s), i \in \mathbf{m}$,
2. $D_R(s)$, $N_R(s)$ are right coprime,
3. $P(s) = D_R^{-1}(s)N_R(s) = N_L(s)D_L^{-1}(s)$ is strictly proper,
4. $N_L(s)$, $D_L(s)$ are left coprime,
5. $D_L(s)$ is row proper with row degrees $\mu_i = \deg_{ri} D_L(s), i \in \mathbf{p}$ and define $\mu = \max_{i\in\mathbf{p}}\{\mu_i\}$,
6. $D_C(s)$ is both row and column reduced with $\deg_{ci} D_C(s) = \deg_{ri} D_C(s) = k_i + \xi_i, i \in \mathbf{m}$ where ξ_i are integers s.t. $\xi_i \geqslant \mu - 1, i \in \mathbf{m}$.

If $X_L(s), Y_L(s)$ is a solution of (22) and $C(s) = X_L^{-1}(s)Y(s) \in \mathbb{R}_{pr}(s)$ then $X_L(s)$ is row proper with row degrees $\deg_{ri} X_L(s) = \xi_i, i \in \mathbf{m}$.

Notice that if $X_L(s)^{-1}Y(s) \in \mathbb{R}_{pr}(s)^{m\times p}$ then the row degrees of $Y_L(s)$ cannot exceed ξ_i, i.e. $\deg_{ri} Y_L(s) \leqslant \xi_i, i \in \mathbf{m}$ (Kailath, 1980; Callier & Desoer, 1982), thus the maximum degree of the ith row of $\Omega(s) = [\, X_L(s), Y_L(s)]$ will be ξ_i. Denote the rows of $\Omega(s)$ by $\omega_i^\top(s) \in \mathbb{R}[s]^{1\times(m+p)}, i \in \mathbf{m}$. Write

$$\omega_i^\top(s) = \sum_{j=0}^{\xi_i} \omega_{ij}^\top s^j, \quad \omega_{ij}^\top \in \mathbb{R}^{1\times(m+p)}, i \in \mathbf{m} \tag{28}$$

and define the row vectors $\overline{\omega}_i^\top = [\omega_{i0}^\top, \omega_{i1}^\top, \ldots, \omega_{i\xi_i}^\top] \in \mathbb{R}^{1\times(p+m)(\xi_i+1)}, i \in \mathbf{m}$.

Now let $d_i^\top(s), i \in \mathbf{m}$ be the rows of $D_C(s)$ and using assumption 6 of Lemma 7 define $\overline{d_i}^\top \in \mathbb{R}^{1\times m(\xi_i+1)+\sum_{i=1}^m k_i}, i \in \mathbf{m}$ from the relation

$$d_i^\top(s) = \overline{d_i}^\top S_{e(\xi_i+1)}(s), i \in \mathbf{m} \tag{29}$$

where $S_{e(\xi_i+1)}$ is the $(m(\xi_i+1) + \sum_{i=1}^m k_i) \times m$ matrix defined in (12).

THEOREM 8 Let the assumptions 1–6 of Lemma 7 hold. Then every solution pair $X_L(s), Y_L(s)$ of (22) such that $C(s) = X_L(s)^{-1}Y_L(s) \in \mathbb{R}_{pr}^{m\times p}(s)$ can be obtained from the solutions of the numerical equations

$$\overline{\omega}_i^\top M_{e(\xi_{i+1})} = \overline{d_i}^\top, \ i \in \mathbf{m} \tag{30}$$

and vice versa: that is, every solution $\overline{\omega}_i^\top$ of (30) gives rises via (28) to a $\Omega(s) = [\, X_L(s), Y_L(s)]$, s.t. $C(s) = X_L(s)^{-1}Y_L(s) \in \mathbb{R}_{pr}^{m\times p}(s)$.

Proof. First notice that (30) are always solvable for arbitrary $\overline{d_i}^T$ since $\xi_i + 1 \geqslant \mu$ and thus from Lemma 5 in conjunction with assumptions 1–2 of Lemma 7 $M_{e(\xi_i+1)}$ has full column rank.

If $X_L(s), Y_L(s)$ is a solution of (22) and $X_L^{-1}(s)Y_L(s)$ is proper according to Lemma 7 the row degrees of $\Omega(s)$ will be ξ_i and thus we can write $\omega_i^\top(s)$ as in (28). It is easy to see that the corresponding $\overline{\omega}_i^\top$ will satisfy (30).

Conversely, if $\overline{\omega}_i^\top$ satisfy (30) then post-multiplying (30) by $S_{e(\xi_i+1)}(s)$ gives $\overline{\omega}_i^\top M_{e(\xi_{i+1})} S_{e(\xi_i+1)}(s) = \overline{d_i}^\top S_{e(\xi_i+1)}(s), i \in \mathbf{m}$ or equivalently from (12)

$$\overline{\omega}_i^\top S_{(\xi_i+1)}(s) \begin{bmatrix} D_R(s) \\ N_R(s) \end{bmatrix} = d_i^\top(s), i \in \mathbf{m}$$

which equivalently gives

$$\omega_i^\top(s)\begin{bmatrix} D_R(s) \\ N_R(s) \end{bmatrix} = d_i^\top(s), i \in \mathbf{m}. \tag{31}$$

Obviously $\Omega(s) = \left[\omega_1^\top(s), \omega_2^\top(s), \ldots, \omega_m^\top(s)\right]^\top$ satisfies (22) and $\deg_{ri} \Omega(s) \leqslant \xi_i, i \in \mathbf{m}$. Hence $\deg_{ri} X(s) \leqslant \xi_i$ and $\deg_{ri} Y(s) \leqslant \xi_i, i \in \mathbf{m}$.

Now let $\Lambda_k(s) = \mathrm{diag}\{s^{k_1}, s^{k_2}, \ldots, s^{k_m}\}$, $\Lambda_\xi(s) = \mathrm{diag}\{s^{\xi_1}, s^{\xi_2}, \ldots, s^{\xi_m}\}$ and pre- and post-multiply (22) respectively by $\Lambda_\xi(s)^{-1}$ and $\Lambda_k(s)^{-1}$ to get

$$\Lambda_\xi(s)^{-1}X_L(s)D_R(s)\Lambda_k(s)^{-1} + \Lambda_\xi(s)^{-1}Y_L(s)N_R(s)\Lambda_k(s)^{-1} = \Lambda_\xi(s)^{-1}D_C(s)\Lambda_k(s)^{-1}. \tag{32}$$

Since $D_R(s)$ is column proper with column degrees k_i, $D_R(s)\Lambda_k(s)^{-1}$ is biproper. Similarly, since $D_C(s)$ is row and column reduced with row powers ξ_i and column powers k_i, $\Lambda_\xi(s)^{-1}D_C(s)\Lambda_k(s)^{-1}$ is also biproper. Using the fact that $P(s)$ is strictly proper, $\Lambda_k(s)$ is column proper and $\deg_{ci} N_R(s) < k_i, i \in \mathbf{m}$ it follows that $N_R(s)\Lambda_k(s)^{-1}$ is strictly proper. Finally, since $\deg_{ri} X(s) \leqslant \xi_i$ and $\deg_{ri} Y(s) \leqslant \xi_i, i \in \mathbf{m}$, $\Lambda_\xi(s)^{-1}X_L(s)$ and $\Lambda_\xi(s)^{-1}Y_L(s)$ are proper in general. Thus taking limits for $s \to \infty$ on both sides of (32) we obtain the equation

$$X_L^{hr} D_R^{hc} = D_C^{hrc}$$

where X_L^{hr} is the highest row degree coefficient matrix of $X_L(s)$, D_R^{hc} is the highest column degree coefficient matrix of $D_R(s)$ and D_C^{hrc} is the highest row–column power coefficient matrix of $D_C(s)$. Obviously X_L^{hr} is invertible since D_R^{hc}, D_C^{hrc} are invertible. Hence, $X_L(s)$ is row proper with row degrees ξ_i and since $\deg_{ri} Y(s) \leqslant \xi_i, i \in \mathbf{m}$, $X_L(s)^{-1}Y_L(s) \in \mathbb{R}_{pr}(s)$ is proper. □

The above result allows us to determine the number of independent parameters in the parametrization of all proper denominator assigning compensators for a strictly proper plant, in terms the McMillan degree of the plant, the number of inputs and outputs and the particular choice of ξ_i.

COROLLARY 9 Let assumptions 1–6 of Lemma 7 hold. Then the number of independent parameters in the parametrization of all denominator assigning proper compensators is

$$v = m(p - \delta_M P(s)) + p\sum_{i=1}^m \xi_i. \tag{33}$$

Proof. Using the result of Theorem 8, the degrees of freedom in the choice of $\omega_i^T(s)$ is essentially equal to the dimension of the left kernel of $M_{e(\xi_i+1)}$. Thus the total number of independent parameters will be $v = \sum_{i=1}^m \dim_{\mathbb{R}} \ker M_{e(\xi_i+1)}$. Now since $\xi_i + 1 \geqslant \mu$, $\dim_{\mathbb{R}} \ker M_{e(\xi_i+1)} = \sum_{j=1}^p (\xi_i + 1 - \mu_j)$. Thus

$$v = \sum_{i=1}^m \sum_{j=1}^p (\xi_i + 1 - \mu_j) = mp + p\sum_{i=1}^m \xi_i - m\sum_{j=1}^p \mu_j$$

which using the fact that $\delta_M P(s) = \sum_{j=1}^p \mu_j$ gives (33).

Notice that in case we choose $\xi_1 = \xi_2 = \cdots = \xi_m := \xi$, we do not need to solve (30) independently for each row, but we can use one resultant, namely $M_{e(\xi+1)}$, to determine all rows $\omega_i^\top(s)$. In such a case the number of independent parameters in the parametrization will be $v = m(p(\xi + 1) - \delta_M P(s))$.

Although Theorem 8 provides a way to reduce the computation of proper compensators to the solution of a set of numerical equations of the form (30), we can go a step further and propose a method that reduces the problem to a single numerical equation. This can be done by exploiting the shift-invariant form of the generalized Wolovich resultant and using Gaussian elimination.

Let $i_1, i_2, \ldots, i_m$ be indices such that $\xi_{i_1} \leqslant \xi_{i_2} \leqslant \cdots \leqslant \xi_{i_m}$. Let also $\xi := \xi_{i_m} = \max_{i\in\mathbf{m}}\{\xi_i\}$. In order to solve (30) for $i = i_1$ we can apply Gaussian elimination on the columns of $M_{e(\xi_{i_1}+1)}$ to obtain the reduced column echelon form $R_{e(\xi_{i_1}+1)}$. Due to the shift-invariant form of the resultant, the columns of $M_{e(\xi_{i_1}+1)}$ appear in the first $(p+m)\xi_{i_1}$ rows of $M_{e(\xi_{i_2}+1)}$ (together with m zero columns). Since $M_{e(\xi_{i_1}+1)}$ has full column rank, the reduced column echelon form of $M_{e(\xi_{i_2}+1)}$ will have the block triangular form

$$R_{e(\xi_{i_2}+1)} = \begin{bmatrix} R_{e(\xi_{i_1}+1)} & 0 \\ Q_{11} & Q_{12} \end{bmatrix}.$$

Proceeding inductively it is easy to see that $R_{e(\xi_{i_{j+1}}+1)}$ will also have a similar block triangular form

$$R_{e(\xi_{i_{j+1}}+1)} = \begin{bmatrix} R_{e(\xi_{i_j}+1)} & 0 \\ Q_{j1} & Q_{j2} \end{bmatrix}$$

for $j = 1, 2, \ldots, m-1$. Thus, reducing $M_{e(\xi+1)}$ into column echelon form essentially provides a solution to all (30) since $R_{e(\xi+1)}$ consists of blocks that give successively $R_{e(\xi_{i_j}+1)},\ j \in \mathbf{m}$.

In the light of the above analysis we provide the following algorithm:

- Step 1. Obtain a right coprime MFD $N_R(s) \in \mathbb{R}[s]^{p\times m}$, $D_R(s) \in \mathbb{R}[s]^{m\times m}$ of $P(s)$ with $D_R(s)$ column proper with column degrees $\deg_{ci} D_R(s) = k_i$, $i \in \mathbf{m}$.
- Step 2. Determine the minimum k for which M_{ek} has full column rank. Then $\mu = k$ and choose $\xi_i \geqslant \mu - 1$, $i \in \mathbf{m}$.
- Step 3. Using (12) construct the generalized Wolovich resultant $M_{e(\xi+1)}$ where $\xi = \max_{i\in\mathbf{m}}\{\xi_i\}$.
- Step 4. Choose $D_C(s) \in \mathbb{R}[s]^{m\times m}$ to be row and column reduced with column powers k_i, and row powers ξ_i and construct $\overline{D}_{(\xi+1)}$ by decomposing $D_C(s) = \overline{D}_{(\xi+1)} S_{e(\xi+1)}(s)$ as in (26).
- Step 5. Construct the compound matrix $\overline{M}_{e(\xi+1)} = \begin{bmatrix} M_{e(\xi+1)} \\ \overline{D}_{(\xi+1)} \end{bmatrix}$.
- Step 6. Reduce $\overline{M}_{e(\xi+1)}$ into column echelon form to obtain $\overline{R}_{e(\xi+1)} = \begin{bmatrix} R_{e(\xi+1)} \\ \Delta_{(\xi+1)} \end{bmatrix}$.
- Step 7. Compute the (general) solution for each row $\overline{\omega}_i^\top$ for $i = 1, 2, \ldots, m$, using the first $(\xi_i + 1)(p+m)$ rows of $\overline{R}_{e(\xi+1)}$ and the ith row of $Z_{(\xi+1)}$ (discarding the last $(\xi - \xi_i)m$ columns on both matrices because they contain only zeros).
- Step 8. Using (28) calculate $\omega_i^\top(s)$ of $\Omega(s)$ from $\overline{\omega}_i^\top$ for $i = 1, 2, \ldots, m$.

Notice that the above method does not require calculation of a left coprime MFD of $P(s)$ for the parametrization of solutions as in Kucera & Zagalak (1999) or Callier (2001) nor the computation of a Y-minimal particular solution as in Kucera & Zagalak (1999). The only information that affects the choice of the closed-loop denominator is the observability index μ of $P(s)$ which can be determined using rank tests on M_{ek} for successive choices of $k = 1, 2, 3, \ldots$, since due to Remark 6, μ is equal to the minimum k for which M_{ek} has full column rank. This fact justifies the choice of the lower bound for the row powers ξ_i of the desired closed-loop denominator. In the previous section we showed that $k \geqslant \mu$ is a necessary and sufficient condition (provided that $D_R(s), N_R(s)$ are coprime and $D_R(s)$ is column proper) in order for M_{ek} to have full column rank, imposing this way the lower bound for the choice of ξ_i that make (30) solvable for arbitrary choice of the right-hand side matrix. This lower bound on the choice of ξ_i has been used in the past but has not been justified via theoretic argument. With $\xi_i = \mu - 1, i \in \mathbf{m}$, the McMillan degree of the controller $C(s) = X_L(s)^{-1}Y_L(s)$, is genericaly $\delta_M C(s) = m(\mu - 1)$. However, there might be cases when $X_L(s), Y_L(s)$ turn out to have a left (non-unimodular) common divisor, giving rise to a $C(s)$ with McMillan degree $\delta_M C(s) < m(\mu - 1)$.

We should also notice that the Gaussian elimination method has been chosen here only for simplicity of presentation. The above algorithm can be applied equally well using unitary Householder transformations to reduce $M_{e(\xi+1)}$ to a lower (block) triangular form, which performs better from a numerical point of view.

We demonstrate the above procedure via the following example (the plant and MFDs appear in the example in Callier (2001) but the desired closed-loop denominator has been changed in order to illustrate the method for $\xi_1 \neq \xi_2$).

EXAMPLE 10 Let

$$P(s) = \begin{bmatrix} \frac{s+1}{s(s-2)} & 0 \\ \frac{1}{s(s-1)} & \frac{1}{s-1} \end{bmatrix}$$

with

$$D_R(s) = \begin{bmatrix} s^2 - 2s & 0 \\ 1 & s-1 \end{bmatrix}, \; N_R(s) = \begin{bmatrix} s+1 & 0 \\ 1 & 1 \end{bmatrix}$$

so that $k_1 = 2, k_2 = 1$ and $\delta_M P(s) = k_1 + k_2 = 3$. The observability index of $P(s)$ is $\mu = 2$, since it can be easily seen that M_{e1} does not have full column rank while M_{e2} does. Let the desired closed-loop denominator polynomial matrix be

$$D_C(s) = \operatorname{diag}\{s^3 + 8s^2 + 24s + 32, s^3 + 15s^2 + 62s + 48\}$$

with $\xi_1 = 1, \xi_2 = 2, \xi = \max\{\xi_i\} = 2$. We should expect the parametrization of all proper compensators giving rise to $D_C(s)$ to have $m(p - \delta_M P(s)) + p\sum_{i=1}^{m} \xi_i = 4$ independent parameters. Create the generalized Wolovich resultant for $k = \xi + 1 = 3$:

$$M_{e3} = \begin{bmatrix} 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \in \mathbb{R}^{12\times 9}.$$

Write $D_C(s)$ in terms of its coefficients as follows:

$$D_C(s) = \overline{D}_3 S_{e3}(s)$$

$$= \begin{bmatrix} 32 & 24 & 8 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 48 & 62 & 15 & 1 \end{bmatrix} S_{e3}(s).$$

Now define the compound matrix $\overline{M}_{e3} = \begin{bmatrix} M_{e3} \\ \overline{D}_3 \end{bmatrix}$ and apply Gaussian elimination on the columns of $\overline{M}_{e3}$ to obtain the column echelon form which is

$$\overline{R}_{e3} = \left[\begin{array}{ccccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & -2 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 1 & 1 & 0 & 0 \\ \hline 6 & 0 & 32 & 0 & 1 & 0 & 4 & 0 & 0 \\ -63 & 78 & -204 & 126 & 0 & 16 & 62 & 0 & 1 \end{array}\right] = \begin{bmatrix} R_{e3} \\ \Delta_3 \end{bmatrix}$$

where $R_{e3} \in \mathbb{R}^{12\times 9}$, $\Delta_3 \in \mathbb{R}^{2\times 9}$. To determine $\omega_1^T(s)$ take the first $(p+m)(\xi_1+1) = 8$ rows of $\overline{R}_{e3}$ as well as the first row of Δ_3 discarding the last two columns on both matrices. This corresponds to the reduced echelon form of (30) for $i = 1$, i.e.

$$\overline{\omega}_1^\top \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & -2 & 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 32 & 0 & 1 & 0 & 4 \end{bmatrix}$$

whose general solution is

$$\overline{\omega}_1^\top = \begin{bmatrix} 6 & 0 & 32 & 0 & 1 & 0 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 2 & -1 & 0 & 0 & -1 & 1 \end{bmatrix} t_1$$

where $t_1 \in \mathbb{R}$. Thus, from (28),

$$\omega_1^\top(s) = \begin{bmatrix} s+6+t_1 & -t_1 & (4-t_1)s+32+2t_1 & t_1 s - t_1 \end{bmatrix}.$$

Accordingly, to determine $\omega_2^\top(s)$ take the first $(p+m)(\xi_2+1) = 12$ rows of $\overline{R}_{e3}$ as well as the second row of Δ_3. This corresponds to the reduced echelon form of (30) for $i = 2$, i.e.

$$\overline{\omega}_2^\top \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & -2 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -63 & 78 & -204 & 126 & 0 & 16 & 62 & 0 & 1 \end{bmatrix}$$

whose general solution is

$$\begin{aligned} \overline{\omega}_2^\top = & \begin{bmatrix} -63 & 78 & -204 & 126 & 0 & 16 & 62 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \\ & + \begin{bmatrix} t_2 & t_3 & t_4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & -2 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

where $t_2, t_3, t_4 \in \mathbb{R}$. Thus, from (28),

$$\begin{aligned} \omega_2^\top(s) = \big[& -63 + t_2 - t_3 - st_3 \quad 78 - t_2 - t_4 + s(16 - t_4) + s^2 \\ & -204 + 2t_2 + 2t_4 + s(62 - t_2 - 2t_3 - t_4) + s^2 t_3 \quad 126 - t_2 - t_4 + st_2 + s^2 t_4 \big]. \end{aligned}$$

Now $\Omega(s) = [X_L(s), Y_L(s)] = \frac{\omega_1^\top(s)}{\omega_2^\top(s)}$, thus the parametrization of all proper compensators is

$$X_L(s) = \begin{bmatrix} s + 6 + t_1 & -t_1 \\ -63 + t_2 - t_3 - st_3 & 78 - t_2 - t_4 + s(16 - t_4) + s^2 \end{bmatrix}$$

$$Y_L(s) = \begin{bmatrix} (4 - t_1)s + 32 + 2t_1 & t_1 s - t_1 \\ -204 + 2t_2 + 2t_4 + s(62 - t_2 - 2t_3 - t_4) + s^2 t_3 & 126 - t_2 - t_4 + st_2 + s^2 t_4 \end{bmatrix}$$

for free parameters $t_1, t_2, t_3, t_4 \in \mathbb{R}$. Notice that the number of parameters is the expected one, i.e. 4. Obviously $X_L(s)$ is row proper with row degrees 1, 2 while the corresponding row degrees of $Y_L(s)$ do not exceed 1, 2. Thus $X_L^{-1}(s) Y_L(s)$ is proper. □

5. Conclusions

In this paper we have investigated the problem of the determination of a proper denominator assigning compensator for the class of strictly proper linear multivariable plants. Our approach focuses on the numerical computation of the coefficients of the polynomial matrices that describe the dynamic compensator and a parametrization of all such compensators corresponding to the one in Kucera & Zagalak (1999) and Callier (2001) has been provided.

The suggested method utilizes a generalized version of the resultant attributed to Wolovich (see Wolovich, 1974) whose structural properties surprisingly have not been studied in detail. In the light of the results presented in Section 3 the generalized Wolovich resultant is proved to be the ideal tool for handling polynomial matrix Diophantine equations when degree control of the solution is required. The entire procedure is reduced to the computation of a solution of a set of numerical equations and the determination of the left kernel of the generalized Wolovich resultant. Furthermore, our analysis shows that the number of independent parameters in the parametrization of all proper compensators can be calculated beforehand in terms of the row powers of the closed-loop denominator and the McMillan degree of the plant.

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