# Plato's Hidden Theorem on the Distribution of Primes

ANTONIS VARDULAKIS AND CLIVE PUGH

One of us (Vardulakis) met the late Andreas Zachariou, a professor of Mathematics at the University of Athens, very briefly some years ago during a dinner party and a follow-up mini-symposium ($\sigma\upsilon\mu\pi o\sigma\iota o\nu =$ (wine) "drinking gathering"). The symposium was held in honour of a colleague and friend, on the occasion of his election to the position of Full Professor in the Department of Mathematics of the Aristotelian University of Thessaloniki in Greece. Zachariou was visiting the Department of Mathematics as an external referee in these election proceedings. Vardulakis remembers that over a glass of red wine, Zachariou told him a mathematical story that impressed him deeply.

In early autumn of 2003, after many attempts, Vardulakis contacted him again in order to share with him some new developments concerning his old story.

What Zachariou had told Vardulakis during their earlier meeting was that a passage in Plato's "Laws" [1] concealed a theorem regarding the arrangement of prime numbers, which Zachariou together with his wife had conjectured and subsequently had proved. The passage is in Book 5, 737e, 738 of Plato's "Laws"; it is stated there that the number of citizens of an ideal city state should be 5040 because this number is divisible by a total of 59 numbers and in particular by all integers from 1 to 10.

Here is an English translation of the part of book 5 of Plato's Laws where the number 5040 is mentioned. As will become clear in the sequel, it is interesting that number 5040 appears in Plato's text exactly 7! times.[1]

> *[737e] Let us assume that there are—as a suitable number—5040 men, to be land-holders and to defend their plots; and let the land and houses be likewise divided into the same number of parts—the man and his allotment forming together one division. First, let the whole number be divided into two; next into three; then follow in natural order four and five, and so on up to ten. Regarding numbers, every man who is making laws must understand at least thus much,—[738a] what number and what kind of number will be most useful for all States. Let us choose that which contains the most numerous and most consecutive sub-divisions. Number as a whole comprises every division for all purposes; whereas the number 5040, for purposes of war, and in peace for all purposes connected with contributions and distributions, will admit of division [738b] into no more than 59 sections, these being consecutive from one up to ten. These facts about numbers must be grasped firmly and with deliberate attention by those who are appointed by law to grasp them: they are exactly as we have stated them, and the reason for stating them when founding a State is this:—in respect of gods, and shrines, and the temples which have to be set up for the various gods in the State, and the gods and daemons they are to be named after, no man of sense,—whether he be framing a new State or reforming an old one that has been corrupted,—will attempt to alter [738c] the advice from Delphi or Dodona or Ammon, or that of ancient sayings, whatever form they take—whether derived from visions or from some reported inspiration from heaven. By this advice they instituted sacrifices combined with rites, either of native origin or imported from Tuscany or Cyprus or elsewhere; and by means of such sayings they sanctified oracles and statues and altars and temples, and marked off for each of them sacred glebes. Nothing of all these [738d] should the lawgiver alter in the slightest degree; to each section he should assign a god or daemon, or at the least a hero; and in the distribution of the land he should assign first*

[1]The reader can interpret the exclamation mark according to his taste. Nowadays one can check this claim easily by google-ing "Plato Laws 5040" and counting the repetitions of the 5040s using "find 5040" in any text that comes along.

*to these divinities choice domains with all that pertains to them, so that, when assemblies of each of the sections take place at the appointed times, they may provide an ample supply of things requisite, and the people may fraternize with one another at the sacrifices and gain knowledge and intimacy, [738e] since nothing is of more benefit to the State than this mutual acquaintance; for where men conceal their ways one from another in darkness rather than light, there no man will ever rightly gain either his due honor or office, or the justice that is befitting. Wherefore every man in every State must above all things endeavor to show himself always true and sincere towards everyone, and no humbug, and also to allow himself to be imposed upon by no such person.*

Zachariou observed that

- $5040 = 7!$
- $10 = 11 - 1$
- 7 and 11 are successive primes

and finally that

- according to Plato " . . . *the number 5040,* for purposes of war, and in peace for all purposes connected with contributions and distributions, *will admit of division* [738b] *into no more than 59 sections, these being consecutive from one up to ten*."

This led him to believe that in this passage of the Laws, Plato is in fact stating (in a cryptic way) a theorem, which can be formulated as follows:

**THEOREM 1** *(Plato's hidden Theorem) Let $3 < P < Q$, where $P$ and $Q$ are successive primes. Then each integer $r < Q$, divides $P!$*

**REMARK** The theorem is not true when $P = 3$, so that $Q = 5$, because $3! = 6$ is not a multiple of 4.

## Again in Crete almost 2500 Years Later

On June 25, 2003, we met together with some fellow mathematicians and engineers during a three-day informal meeting on control theory that Vardulakis had organized in his home village of Anidri, located in the south tip of the western part of the island of Crete. According to Plato, it was in Crete where almost 2500 years earlier "the Athenian" (probably Plato himself) of the "Laws," talking to a local Cretan and a Spartan, mentioned for the first time in recorded history the number 5040 and its remarkable divisibility property. It was at this "Anidri meeting" that Vardulakis mentioned to the rest of the participants Zachariou's conjecture, asking everybody to try to prove it.

Although the conjecture had been tested to be true for very big successive primes, to our knowledge, up to the summer of 2003, no proof of the theorem was available. Zachariou had failed to indicate where a proof could be located. (He mentioned the references [2] [3].) The first proof we know of was given after the Anidri meeting by Peter Shiu of the Mathematics Department at Loughborough University after the story was mentioned to him by one of us (Clive Pugh). Another shorter proof was given by Mr. Georgios Velisaris, who in 2007 was a first-year student in the Department of Medicine of the Aristotle University of Thessaloniki. Using the idea of the proof by George Velisaris, the referee gave us a short proof of a slightly stronger result:

**THEOREM 3** *If $n$ is a positive integer $> 5$, then $n!$ is divisible by all the integers $1, 2, \ldots, n$ and by all the composite numbers among the integers $n + 1$, $n + 2, \ldots 2n$.*

Although all proofs are available from the authors, we hope that readers would enjoy seeking themselves proofs for both theorems.

According to Wikipedia, the number 5040 is "highly composite", "superior highly composite", and "colossally abundant" (see Wikipedia for the definition of these terms). It is also stated there[2] that if $\sigma(n) = \sum_{d|n}$, $d$ is the *divisor function*, and $\gamma$ is the *Euler–Mascheroni constant* defined

[2]http://en.wikipedia.org/wiki/5040_(number).

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as the limiting difference between the harmonic series and the natural logarithm:

$$\gamma = \lim_{n \to \infty} \left[ \left( \sum_{k=1}^{n} \frac{1}{k} \right) - \ln\,(n) \right] = 0.57721566\ldots,$$

then 5040 is the largest *known* number for which the inequality $\sigma\,(n) \geq e^{\gamma} n\,(\ln \ln\, n)$ holds. From this, we might conjecture that $\sigma\,(n) < e^{\gamma} n\,(\ln \ln\, n)$, $\forall n > 5040$. In 1984 Guy Robin[3] showed that, in fact, this statement is true if and only if the Riemann hypothesis is true. This is Robin's theorem. We think that it is delightful that the property of this particular integer is mentioned for the first time in Plato's "Laws."


## REFERENCES

[1] Plato, Laws (4th century BC).

[2] Andreas Zachariou and Eleni Zachariou. Abstract of papers presented to the American Mathematical Society, February 1982, Issue 16, Volume 3, Number 2, pp. 145–220.

[3] Andreas Zachariou and Eleni Zachariou. H Glossa ton Epigrafon tou Kafiziou tis Aglantzias kai ta Kypriaka Toponymia, Praktika tou Tritou Diethnous Kyprologikou Synedriou (Lefkosia, 16–20 Apriliou, 1996) Tomos A, ΑΡΧΑΙΟΝ ΤΜΗΜΑ, Ekdosi Georgiou Ioannidi kai Steliou Xatjistyli, Horigia Idrimatos A. G. Leventi, Lefkosia 2000, pp. 63–136.

[4] Andreas Zachariou and Eleni Zachariou. Logos kai Texni, Aristotle University of Thessaloniki, Polytechnic School, Department of Mathematics, Volume dedicated to Professor Emeritus Ioannis D. Mittas, Thessaloniki, 1999–2000. ←AU1


[3]http://blogs.msdn.com/devdev/archive/2007/07/16/robin-s-theorem.aspx http://mathworld.wolfram.com/RobinsTheorem.html.

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