Forward, Backward and Symmetric Solutions of Discrete ARMA-Representations

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Abstract

The main objective of this paper is to determine a closed formula for the forward, backward and symmetric solution of a general discrete time AutoRegressive Moving Average (ARMA) representation. The importance of the above formula is that it is easily implemented in a computer algorithm and gives rise to the solution of analysis, synthesis and design problems.

Key Words : ARMA-Representations, solutions, discrete time systems, forward, backward, symmetric.

1 Introduction

Consider a nonhomogeneous system of linear difference and algebraic equations described in matrix form by

$$A(\sigma)y_k = B(\sigma)u_k \tag{1}$$

where σ denotes the forward shift operator i.e. $\sigma^i y_k = y_{k+i}$,

$$A(\sigma) = A_0 + A_1\sigma + \ldots + A_q\sigma^q \in R[\sigma]^{r\times r},\ rank_{R(\sigma)}A(\sigma) = r$$
$$B(\sigma) = B_0 + B_1\sigma + \ldots + B_q\sigma^q \in R[\sigma]^{r\times m}$$

where at least one of A_q, B_q is nonzero, $y_k : \mathbb{Z}^+ \to \mathbb{R}^r$ is the input and $u_k : \mathbb{Z}^+ \to R^m$ is the input of the system. Following the terminology of [16] we call the set of equations (1) an ARMA representation of $\mathcal{B}$, where $\mathcal{B}$ is the solution space of the system de ned by

$$\mathcal{B} = \pi_y(\mathcal{B}_f)$$

with

$$\mathcal{B}_f := \{\begin{pmatrix} y_k & u_k \end{pmatrix} : \mathbb{Z}^+ \to R^r \times R^m \ |(1) \text{ is satis ed } \forall k \in Z^+\}$$
$$\text{and } \pi_y : R^r \times R^m \to R^r \text{ is given by } \pi_y \begin{pmatrix} y_k & u_k \end{pmatrix} = y_k$$

In case where $A(\sigma) = \sigma E - A \in R[\sigma]^{r\times r}$ and $B(\sigma) = B \in R^{r\times m}$ then the ARMA representation (1) is the known generalized state space representation, i.e.

$$Ex_{k+1} = Ax_k + Bu_k \tag{2}$$

while in case where $\det[E] \neq 0$, (2) is the known state space representation. For a survey of singular systems of the form (2) see [7].

ARMA representations of the form (1) nd numerous applications in analysis of circuits [12], neural networks [2], economics (the Leontieff model, see [9]) and power systems [14].

The solution of the ARMA-Representation (2) has been calculated by many different techniques ([1], [6], [10], [13], [15]), and among them we distinguish [8] and [11]. This teqhnique gives a solution of the singular system representation in terms of the fundamental matrix ϕ_k and the backward fundamental matrix τ_k of $(zE - A)^{-1}$. Following similar lines with [8], [11] we produce in section 3 a closed formula for the forward, backward and symmetric solution of the general ARMA-Representation (1) in terms now of the fundamental matrix H_k and the backward fundamental matrix V_k of $A(s)^{-1}$. A generalized Leverrier technique for computing the forward fundamental matrix is available [3], so that we may assume that this fundamental matrix is given. We shall show in section 2 that the backward fundamental matrix is the forward fundamental matrix of the dual polynomial matrix $\tilde{A}(\sigma) = A_0\sigma^q + A_1\sigma^{q-1} + \cdots + A_q$ of $A(s)$ and thus we may assume that V_k is also given. The whole theory is illustrated via an example in Section 4.

2 Preliminary Results

We are concerned with the discrete time ARMA-representation (1) where $y_k \in R^r$, $u_k \in R^m$, $k = 0, 1, ..., N - q$. We assume that $A(\sigma)$ is *regular* i.e. $\det A(\sigma) \neq 0$ for almost every s. Given regularity the Laurent series expansion about in nity of $A(s)^{-1}$ exists and is given by

$$A(\sigma)^{-1} = H_{\hat{q}_r}\sigma^{\hat{q}_r} + H_{\hat{q}_r-1}\sigma^{\hat{q}_r-1} + \cdots, \quad |\sigma| > \rho > 0 \tag{3}$$

where $\hat{q}_r$ is the greatest order of the zeros of $A(\sigma)$ at $\sigma = \infty$ and the sequence $\{H_k\}$ is known as the *forward fundamental matrix* [3]. The Laurent expansion about zero of $A(s)^{-1}$ exists and is given by :

$$A(\sigma)^{-1} = V_{-\ell}\sigma^{-\ell} + V_{-\ell+1}\sigma^{-\ell+1} + \cdots, \quad |\sigma| < \rho \tag{4}$$

where the sequence $\{V_k\}$ is known [8] as the *backward fundamental matrix*.

The Laurent expansion about zero of $A(s)^{-1}$ given in (4) is related with the Laurent expanion about in nity given in (3) of the inverse of the dual matrix $\tilde{A}(\sigma) = A_0\sigma^q + A_1\sigma^{q-1} + \cdots + A_q$ of $A(\sigma)$ as we can see in the following lemma.

Lemma 2.1. Let the Laurent expansion about in nity of $\tilde{A}(\sigma)^{-1}$ is

$$\tilde{A}(\sigma)^{-1} = \tilde{H}_f\sigma^f + \tilde{H}_{f-1}\sigma^{f-1} + \cdots + \tilde{H}_1\sigma + \tilde{H}_0 + \tilde{H}_{-1}\sigma^{-1} + \cdots \tag{5}$$

and (4) is the Laurent expansion about zero of $A(\sigma)^{-1}$. Then

$$q + f = \ell \text{ and } V_{-i} = \tilde{H}_{-\ell+f+i} \text{ for i}=\ell, \ell - 1, ..., 1, 0, -1, .. \tag{6}$$

Proof. We have that

$$
\begin{aligned}
A(\sigma) = \sigma^q \tilde{A}\left(\tfrac{1}{\sigma}\right) \Leftrightarrow A(\sigma)^{-1} = \sigma^{-q}\tilde{A}\left(\tfrac{1}{\sigma}\right)^{-1} \overset{(6)}{\Leftrightarrow} \\
A(\sigma)^{-1} = \sigma^{-q}\left[\tilde{H}_f\sigma^{-f} + \tilde{H}_{f-1}\sigma^{-f+1} + \cdots\right] \\
= \tilde{H}_f\sigma^{-q-f} + \tilde{H}_{f-1}\sigma^{-q-f+1} + \cdots \\
\equiv V_{-\ell}\sigma^{-\mu} + V_{-\ell+1}\sigma^{-\mu+1} + \cdots
\end{aligned}
$$

Equating the coefficients of the powers of σ we obtain the proof of Lemma 2.1. ■

A direct result from Lemma 2.1 is that the Leverrier algorithm in [3] may be used for the computation both of the forward and backward fundamental matrix.

An interesting result which connects the solutions of the ARMA-representation (1) and the ones of the dual discrete time ARMA-representation :

$$
A_q\tilde{y}_k + A_{q-1}\tilde{y}_{k+1} + \cdots + A_0\tilde{y}_{k+q} = B_q\tilde{u}_k + B_{q-1}\tilde{u}_{k+1} + \cdots + B_0\tilde{u}_{k+q} \quad (7)
$$

in the closed interval $[0, N]$ is given by the following :

Theorem 2.2

(a) If $\tilde{y}_k$ is a solution of (7) for the nonzero input $\tilde{u}_k$ then the sequence $y_k = \tilde{y}_{N-k}$ is a solution of the dual equation (1) for the nonzero input $u_k = \tilde{u}_{N-k}$.

(b) If y_k is a solution of (1) for the nonzero input u_k then the sequence $\tilde{y}_k = y_{N-k}$ is a solution of the dual equation (7) for the nonzero input $\tilde{u}_k = u_{N-k}$.

Proof. (a) Let $\tilde{y}_k$ be a solution of (7). This implies that (7) is satis ed. Now consider the equation (1). If we set $y_k = \tilde{y}_{N-k}$ and $u_k = \tilde{u}_{N-k}$ and take into account that $y_{k+j} = \tilde{y}_{N-(k+j)}, u_{k+j} = \tilde{u}_{N-(k+j)}$, $j = 0, 1, .., q$ we have

$$
A(\sigma)\tilde{y}_{N-k} = \sum_{i=0}^{q} A_i\tilde{y}_{N-k-i} \overset{(7)}{=} \sum_{i=0}^{q} B_i\tilde{u}_{N-k-i} \overset{u_k=\tilde{u}_{N-k}}{=} B(\sigma)\tilde{u}_{N-k} \quad (8)
$$

which veri es the rst part of the Theorem.

(b) Following the same way we can show the second part of the Theorem. ■

A direct result from the above theorem is that the backward solution of the ARMA-representation (1) comes directly from the forward solution of the dual ARMA-representation (7).

3 Solutions of ARMA-Representations

There are three different interpretations of the equation (1) [8] :

- We may consider that the initial conditions $\{y_0, y_1, ..., y_{q-1}\}$ are given and that is desired to determine y_k in a *forward* fashion from the input sequence and the previous values of the output.
- We may consider that the nal conditions $\{y_N, y_{N-1}, ..., y_{N-q+1}\}$ are given and that is desired to determine y_k in a *backward* fashion from the input sequence and the future values of the output.
- We may consider (1) as a relationship between the inputs and outputs i.e. economics, and thus no causality is assumed. It is desired to determine y_k for the values $k = q, q+1, ..., N-q$, in terms of the input sequence and the initial and nal conditions. We could call this the *symmetric* solution of (1).

3.1 The Forward Solution of ARMA-Representations

Consider the discrete time ARMA-representation (1) where $A(\sigma)$ is regular i.e. $\det A(\sigma) \neq 0$ and the Laurent series expansion about in nity for the resolvent matrix exists and is given by (3). Then we have

Theorem 3.1

The whole response of the system (1) will be :

$$y_k = \begin{bmatrix} H_{-k-q} & H_{-k-q+1} & \cdots & H_{-k-1} \end{bmatrix} \begin{bmatrix} A_q & 0 & \cdots & 0 \\ A_{q-1} & A_q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \cdots & A_q \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{q-1} \end{bmatrix} \quad (9)$$

$$+ \begin{bmatrix} H_{-k} & H_{-k+1} & \cdots & H_0 & \cdots & H_{\hat{q}_r} \end{bmatrix} \times$$

$$\times \begin{bmatrix} B_0 & B_1 & \cdots & B_q & 0 & \cdots & 0 \\ 0 & B_0 & B_1 & \cdots & B_q & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & B_0 & B_1 & \cdots & B_q \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{k+\hat{q}_r+q} \end{bmatrix}$$

or equivalently

$$y_k = \sum_{i=1}^{q}\sum_{j=1}^{q} H_{-k-i}A_j y_{j-i} + \sum_{i=0}^{k+\hat{q}_r}\sum_{j=0}^{q} H_{-k+i}B_j u_{i+j} \text{ with } k = q, q+1, .. \quad (10)$$

Proof. Equating the coefficients of the powers of σ in the relation $A(\sigma)\times A(\sigma)^{-1} = I_r$ we have that

$$\sum_{n=0}^{q} A_n H_{i-n} = \delta_i I_r \text{ or } \sum_{n=0}^{q} H_{i-n} A_n = \delta_i I_r \quad (11)$$

where $\delta_i = 0$ for $i \neq 0$ and $\delta_0 = 1$. Now substituting y_k from (10) in (1) we have that

$$A(\sigma)y_k = A(\sigma)\left[\sum_{i=1}^{q}\sum_{j=1}^{q} H_{-k-i}A_j y_{j-i} + \sum_{i=0}^{k+\hat{q}_r}\sum_{j=0}^{q} H_{-k+i}B_j u_{i+j}\right] =$$

$$= \sum_{n=0}^{q} A_n \sum_{i=1}^{q}\sum_{j=1}^{q} H_{-k-i}A_j y_{j-i} + \sum_{n=0}^{q} A_n \sum_{i=0}^{k+n+\hat{q}_r}\sum_{j=0}^{q} H_{-k-n+i}B_j u_{i+j} =$$

$$= \sum_{i=1}^{q}\sum_{j=1}^{q}\sum_{n=0}^{q} \left(A_n H_{-k-n-i}\right) A_j y_{j-i} +$$

$$+ \sum_{n=0}^{q} A_n H_{-k-n} \sum_{j=0}^{q} B_j u_j + \sum_{n=0}^{q} A_n H_{-k-n+1} \sum_{j=0}^{q} B_j u_{j+1} + \cdots + \sum_{n=0}^{q} A_n H_{-n} \sum_{j=0}^{q} B_j u_{j+k} +$$

$$+ \sum_{n=0}^{q} A_n H_{-n+1} \sum_{j=0}^{q} B_j u_{j+k+1} + \cdots + \sum_{n=0}^{q} A_n H_{-n+\hat{q}_r} \sum_{j=0}^{q} B_j u_{j+k+\hat{q}_r} + \cdots +$$

$$+ \sum_{n=0}^{q} A_n H_{-n+\hat{q}_r+1} \sum_{j=0}^{q} B_j u_{j+k+\hat{q}_r+1} + \cdots + A_q H_{\hat{q}_r} \sum_{j=0}^{q} B_j u_{j+k+\hat{q}_r+q} =$$

$$\stackrel{(11)}{=} \sum_{i=1}^{q}\sum_{j=1}^{q}\delta_{-k-i}A_j y_{j-i}+\delta_{-k}\sum_{j=0}^{q}B_j u_j+\delta_{-k+1}\sum_{j=0}^{q}B_j u_{j+1}+\cdots+\delta_0\sum_{j=0}^{q}B_j u_{j+k}=B(\sigma)\,u_k \tag{12}$$

which proves the Theorem. ■

It is important to note that the discrete time ARMA-representations does not always have a solution. A necessary and sufficient condition such that the ARMA-representation (1) has a solution is that the initial conditions $\{y_0, y_1, ..., y_{q-1}\}$ satis es the relation (1) for $k = 0, 1, .., q-1$. Therefore we de ne:

De nition 3.2 We de ne as

$$H_{iu} := \left\{ \begin{array}{c} y_i, u_i\ (i=0,1,..,q-1): \\ \tilde{A}_1 \begin{bmatrix} H_0 & H_1 & \cdots & H_{q-1} \\ H_{-1} & H_0 & \cdots & H_{q-2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{-q+1} & H_{-q+2} & \cdots & H_0 \end{bmatrix} \tilde{A}_1 \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{q-1} \end{bmatrix} = \\ = \tilde{A}_1 \begin{bmatrix} H_0 & \cdots & H_{\hat{q}_r} & 0 & \cdots & 0 \\ H_{-1} & \cdots & H_{\hat{q}_r-1} & H_{\hat{q}_r} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ H_{-q+1} & \cdots & H_{\hat{q}_r-q+1} & H_{\hat{q}_r-q+2} & \cdots & H_{\hat{q}_r} \end{bmatrix} \times \\ \times \begin{bmatrix} B_0 & B_1 & \cdots & B_q & 0 & \cdots & 0 \\ 0 & B_0 & B_1 & \cdots & B_q & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & B_0 & B_1 & \cdots & B_q \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{2q+\hat{q}_r-1} \end{bmatrix} \end{array} \right\} \tag{13}$$

where

$$\tilde{A}_1 = \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix}$$

the *admissible initial condition space* of (1) under nonzero inputs.

Proof. Consider the relation (1) for $k = 0, 1, ..., q-1$ and write this in the form

$$\begin{bmatrix} A_q & A_{q-1} & \cdots & A_0 & 0 & \cdots & 0 \\ 0 & A_q & A_{q-1} & \cdots & A_0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & A_q & A_{q-1} & \cdots & A_0 \end{bmatrix} \begin{bmatrix} y_{2q-1} \\ y_{2q-2} \\ \vdots \\ y_0 \end{bmatrix} =$$

$$= \begin{bmatrix} B_q & B_{q-1} & \cdots & B_0 & 0 & \cdots & 0 \\ 0 & B_q & B_{q-1} & \cdots & B_0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & B_q & B_{q-1} & \cdots & B_0 \end{bmatrix} \begin{bmatrix} u_{2q-1} \\ u_{2q-2} \\ \vdots \\ u_0 \end{bmatrix} \tag{14}$$

Substitution of the values $y_q, y_{q+1}, ..., y_{2q-1}$ with the respective formula of (9) and use of (11) give us that the initial conditions $\{y_0, y_1, ..., y_{q-1}\}$ satisfy the system if the relation (13) is satis ed. ■

As we can see in (9) the solution of (1) is determined in terms of the initial conditions $\{y_0, y_1, ..., y_{q-1}\}$ and the input sequence of the system. An obvious disadvantage is that for each succesive output y_k speci ed by $k = q, q+1, ..$, the coefficient matrices H_j comprising each speci c solution change. Therefore if the solution is required over a comparatively large range, say $y_q, y_{q+1}, .., y_{100}$ corresponding to $k = q, q+1, ..., 100$, we would require the coefficient matrices $H_{-101}, H_{-100}, ..., H_{\hat{q}_r}$. An equivalent forward solution is presented in what follows for the general solution y_k depends on the *previous* q outputs $\{y_{k-1}, y_{k-2}, ..., y_{k-q}\}$ and not on the q xed initial conditions $\{y_0, y_1, ..., y_{q-1}\}$. In this case the coefficient matrices required over a solution range is xed (i.e. independent of k), namely $H_{-q}, H_{-q+1}, ..., H_{\hat{q}_r}$.

Corollary 3.3 Equation (9) is equivalent to the following forward recursion :

$$y_k = -\begin{bmatrix} H_{-1} & H_{-2} & \cdots & H_{-q} \end{bmatrix} \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix} \begin{bmatrix} y_{k-1} \\ y_{k-2} \\ \vdots \\ y_{k-q} \end{bmatrix} + \tag{15}$$

$$+\left[\begin{array}{cccccc} H_{-q} & H_{-q+1} & \cdots & H_0 & \cdots & H_{\hat{q}_r}\end{array}\right]\left[\begin{array}{ccccccc} B_0 & B_1 & \cdots & B_q & 0 & \cdots & 0 \\ 0 & B_0 & B_1 & \cdots & B_q & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & B_0 & B_1 & \cdots & B_q \end{array}\right]\left[\begin{array}{c} u_{k-q} \\ u_{k-q+1} \\ \vdots \\ u_{k+\hat{q}_r+q}\end{array}\right] \tag{16}$$

or equivalently

$$y_k = -\sum_{i=1}^{q}\sum_{j=0}^{i-1} H_{-i}A_j y_{k-i+j} + \sum_{i=0}^{q+\hat{q}_r}\sum_{j=0}^{q} H_{-q+i}B_j u_{k-q+j+i} \tag{17}$$

Proof. It is easily seen that the state vector y_q will be connected with the previous vectors $\{y_0, y_1, ..., y_{q-1}\}$ according to (9) with the following relation :

$$y_q = \left[\begin{array}{cccc} H_{-2q} & H_{-2q+1} & \cdots & H_{-q-1}\end{array}\right]\left[\begin{array}{cccc} A_q & 0 & \cdots & 0 \\ A_{q-1} & A_q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \cdots & A_q \end{array}\right]\left[\begin{array}{c} y_0 \\ y_1 \\ \vdots \\ y_{q-1}\end{array}\right]+$$

$$+\left[\begin{array}{cccccc} H_{-q} & H_{-q+1} & \cdots & H_0 & \cdots & H_{\hat{q}_r}\end{array}\right]\left[\begin{array}{ccccccc} B_0 & B_1 & \cdots & B_q & 0 & \cdots & 0 \\ 0 & B_0 & B_1 & \cdots & B_q & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & B_0 & B_1 & \cdots & B_q \end{array}\right]\left[\begin{array}{c} u_0 \\ u_1 \\ \vdots \\ u_{2q+\hat{q}_r}\end{array}\right] \overset{(11)}{=}$$

$$= -\left[\begin{array}{cccc} H_{-1} & H_{-2} & \cdots & H_{-q}\end{array}\right]\left[\begin{array}{cccc} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{array}\right]\left[\begin{array}{c} y_{q-1} \\ y_{q-2} \\ \vdots \\ y_0\end{array}\right]+$$

$$+\left[\begin{array}{cccccc} H_{-q} & H_{-q+1} & \cdots & H_0 & \cdots & H_{\hat{q}_r}\end{array}\right]\left[\begin{array}{ccccccc} B_0 & B_1 & \cdots & B_q & 0 & \cdots & 0 \\ 0 & B_0 & B_1 & \cdots & B_q & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & B_0 & B_1 & \cdots & B_q \end{array}\right]\left[\begin{array}{c} u_0 \\ u_1 \\ \vdots \\ u_{2q+\hat{q}_r}\end{array}\right]$$

The system is time invariant and thus the same relation will connect the output y_k with the previous vectors q outputs $\{y_{k-1}, y_{k-2}, ..., y_{k-q}\}$. Thus if we replace $\{y_q, y_{q-1}, ..., y_0\}$ with $\{y_k, y_{k-1}, ..., y_{k-q}\}$ respectively and $\{u_0, u_1, ..., u_{2q+\hat{q}_r}\}$ with $\{u_{k-q}, u_{k-q+1}, ..., u_{k+q+\hat{q}_r}\}$ respectively we get the relation (15). ■

The advantage of the formula (15) is, as we have already mentioned, that it depends only on the $q+\hat{q}_r+1$ Laurent expansion terms $\{H_{-q}, H_{-q+1}, ..., H_0, ..., H_{\hat{q}_r}\}$. The above formula is very usefull when we need to determine y_k in the interval $k = q, q+1, q+2, ...,$ because we always have to start to compute from y_q, $y_{q+1}, ...$ in contrast to the solution formula (9) where only the q rst initial conditions are required for the determination of y_k. Another advantage of (15) is that the round-off errors for the determination of the $q+\hat{q}_r+1$ Laurent expansion terms $\{H_{-q}, H_{-q+1}, ..., H_0, ..., H_{\hat{q}_r}\}$ are less than the ones for the determination of $\{H_{-k}, ..., H_{\hat{q}_r}\}$ in (9).

3.2 The Backward Solution of ARMA-Representations

Consider the ARMA-Representation (1). The Laurent series expansion about zero for the resolvent matrix is given in (4). Then we have :

Theorem 3.4 The whole response of the system (1) will be :

$$\begin{aligned} y_k &= \begin{bmatrix} V_{N-k} & V_{N-k-1} & \cdots & V_{N-k-q+1} \end{bmatrix} \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix} \begin{bmatrix} y_N \\ y_{N-1} \\ \vdots \\ y_{N-q+1} \end{bmatrix} \\ &+ \begin{bmatrix} V_{N-k-q} & V_{N-k-q-1} & \cdots & V_{-\ell} \end{bmatrix} \times \\ &\times \begin{bmatrix} B_q & B_{q-1} & \cdots & B_0 & 0 & \cdots & 0 \\ 0 & B_q & B_{q-1} & \cdots & B_0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & B_q & B_{q-1} & \cdots & B_0 \end{bmatrix} \begin{bmatrix} u_N \\ u_{N-1} \\ \vdots \\ u_{k-\ell} \end{bmatrix} \end{aligned} \tag{18}$$

or equivalently

$$y_k = \sum_{i=0}^{q-1} \sum_{j=0}^{i} V_{N-k-i} A_j y_{N-i+j} + \sum_{i=0}^{q+k-N-\ell} \sum_{j=0}^{q} V_{N-k-q-i} B_j u_{N+j-i-q} \tag{19}$$

Proof. Consider the dual ARMA-Representation (8) of (1) :

$$\tilde{A}(\sigma)\tilde{y}_k = \tilde{B}(\sigma)\tilde{u}_k \tag{20}$$

where

$$\tilde{A}(\sigma) = A_0\sigma^q + \cdots + A_{q-1}\sigma + A_q \text{ and } \tilde{B}(\sigma) = B_0\sigma^q + \cdots + B_{q-1}\sigma + B_q$$

Consider also the Laurent expansion at $s = \infty$ from (6). Then from Theorem 3.1 the solution of (20) will be :

$$\begin{aligned}
\tilde{y}_k &= \begin{bmatrix} \tilde{H}_{-k-q} & \tilde{H}_{-k-q+1} & \cdots & \tilde{H}_{-k-1} \end{bmatrix} \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix} \begin{bmatrix} \tilde{y}_0 \\ \tilde{y}_1 \\ \vdots \\ \tilde{y}_{q-1} \end{bmatrix} + \\
&\quad + \begin{bmatrix} \tilde{H}_{-k} & \tilde{H}_{-k+1} & \cdots & \tilde{H}_0 & \cdots & \tilde{H}_f \end{bmatrix} \times \\
&\quad \times \begin{bmatrix} B_q & B_{q-1} & \cdots & B_0 & 0 & \cdots & 0 \\ 0 & B_q & B_{q-1} & \cdots & B_0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & B_q & B_{q-1} & \cdots & B_0 \end{bmatrix} \begin{bmatrix} \tilde{u}_0 \\ \tilde{u}_1 \\ \vdots \\ \tilde{u}_{k+f+q} \end{bmatrix} \underset{\substack{V_{-i}=\tilde{H}_{f-\ell+i} \\ q+f=\ell}}{\overset{(6)}{=}} \\
&= \begin{bmatrix} V_k & V_{k-1} & \cdots & V_{-q+k+1} \end{bmatrix} \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix} \begin{bmatrix} \tilde{y}_0 \\ \tilde{y}_1 \\ \vdots \\ \tilde{y}_{q-1} \end{bmatrix} + \\
&\quad + \begin{bmatrix} V_{-q+k} & V_{-q+k-1} & \cdots & V_{-\ell} \end{bmatrix} \times \\
&\quad \times \begin{bmatrix} B_q & B_{q-1} & \cdots & B_0 & 0 & \cdots & 0 \\ 0 & B_q & B_{q-1} & \cdots & B_0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & B_q & B_{q-1} & \cdots & B_0 \end{bmatrix} \begin{bmatrix} \tilde{u}_0 \\ \tilde{u}_1 \\ \vdots \\ \tilde{u}_{k+\hat{q}_r+q} \end{bmatrix}
\end{aligned}$$

From Theorem 2.2 we have that the solution y_k of (1) for an input u_k is given by the solution $\tilde{y}_{N-k}$ of (8) for an input $\tilde{u}_{N-k}$ and the converse. Thus we can replace the initial conditions $\tilde{y}_i, \tilde{u}_i$ of the system (8) with the nal conditions y_{N-i}, u_{N-i} of the system (1) as well as the solution $\tilde{y}_{N-k}$ of (8) with the solution y_k of (1), which proves the relation (18). ■

A necessary and sufficient condition such that the ARMA-Representation (1) has a solution is that the final conditions $\{y_N, y_{N-1}, ..., y_{N-q+1}\}$ satisfies (1) for $k = N, N-1, ..., N-q+1$. Therefore we define :

Definition 3.5. We define as

$$
\begin{aligned}
\bar{H}_{iu} := \{ & y_i,\, u_i \ (i = N, N-1, ..., N-q+1) : \\
& \tilde{A}_2 \begin{bmatrix} V_{-q} & \cdots & V_{-2q+1} \\ V_{-q+1} & \cdots & V_{-2q+2} \\ \vdots & \ddots & \vdots \\ V_{-1} & \cdots & V_{-q} \end{bmatrix} \tilde{A}_2 \begin{bmatrix} y_N \\ y_{N-1} \\ \vdots \\ y_{N-q+1} \end{bmatrix} = \\
= \tilde{A}_2 & \begin{bmatrix} V_{-q} & \cdots & V_{-\ell} & 0 & \cdots & 0 \\ V_{-q+1} & \cdots & V_{-\ell+1} & V_{-\ell} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ V_{-1} & \cdots & V_{-\ell+q-1} & V_{-\ell+q-2} & \cdots & V_{-\ell} \end{bmatrix} \\
& \begin{bmatrix} B_q & B_{q-1} & \cdots & B_0 & 0 & \cdots & 0 \\ 0 & B_q & B_{q-1} & \cdots & B_0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & B_q & B_{q-1} & \cdots & B_0 \end{bmatrix} \begin{bmatrix} u_N \\ u_{N-1} \\ \vdots \\ u_{N-q-\ell+1} \end{bmatrix} \}
\end{aligned} \tag{21}
$$

where

$$
\tilde{A}_2 = \begin{bmatrix} A_q & 0 & \cdots & 0 \\ A_{q-1} & A_q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \cdots & A_q \end{bmatrix}
$$

Proof. Consider the relation (1) for $k = N-q, N-q-1, ..., N-2q+1$ and write this in the form

$$
\begin{bmatrix} A_q & A_{q-1} & \cdots & A_0 & 0 & \cdots & 0 \\ 0 & A_q & A_{q-1} & \cdots & A_0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & A_q & A_{q-1} & \cdots & A_0 \end{bmatrix} \begin{bmatrix} y_N \\ y_{N-1} \\ \vdots \\ y_{N-2q+1} \end{bmatrix} =
$$

$$
= \begin{bmatrix} B_q & B_{q-1} & \cdots & B_0 & 0 & \cdots & 0 \\ 0 & B_q & B_{q-1} & \cdots & B_0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & B_q & B_{q-1} & \cdots & B_0 \end{bmatrix} \begin{bmatrix} u_N \\ u_{N-1} \\ \vdots \\ u_{N-2q+1} \end{bmatrix} \tag{22}
$$

Substitution of the values y_{N-q}, y_{N-q-1}, ..., y_{N-2q+1} with the respective formula of (18) and use of (11) give us that the nal conditions $\{y_N, y_{N-1}, ..., y_{N-q+1}\}$ satisfy the system iff the relation (21) is satis ed. ■

A backward solution formula in terms of the following q terms and the input sequence of the system is provided by the following :

Corollary 3.6. Equation (18) is equivalent to the backward recursion :

$$y_k = \begin{bmatrix} V_q & V_{q-1} & \cdots & V_1 \end{bmatrix} \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix} \begin{bmatrix} y_{k+q} \\ y_{k+q-1} \\ \vdots \\ y_{k+1} \end{bmatrix} + \quad (23)$$

$$+ \begin{bmatrix} V_0 & V_{-1} & \cdots & V_{-\ell} \end{bmatrix} \begin{bmatrix} B_q & B_{q-1} & \cdots & B_0 & 0 & \cdots & 0 \\ 0 & B_q & B_{q-1} & \cdots & B_0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & B_q & B_{q-1} & \cdots & B_0 \end{bmatrix} \begin{bmatrix} u_{k+q} \\ u_{k+q-1} \\ \vdots \\ u_{k-\ell} \end{bmatrix}$$

or equivalently

$$y_k = \sum_{i=0}^{q-1} \sum_{j=0}^{i} V_{q-i} A_j y_{k+q-i+j} + \sum_{i=0}^{-\ell} \sum_{j=0}^{q} V_{-i} B_j u_{k+j-i} \quad (24)$$

Proof. Following similar lines with the proof of Corollary 3.3 we obtain the result. ■

The advantage of the formula (23) is, that depends only from the $q + \ell + 1$ Laurent expansion terms $[V_q, V_{q-1}, ...V_{-\ell}]$ and thus we don t need the continuous computation of the Laurent expansion terms which gives rise to numerical errors.

3.3 The Symmetric Solution

In this section we consider (1) as a relation between the output y_k and the input u_k over an interval $k = 0, 1, ..., N$, where k not necessarily the time index. Such an interpretation is used in economics and elsewhere [7],[9]. Consdier the discrete time ARMA-representation (1) and the Laurent series expansion about in nity for its resolvent matrix in (3). Then

Lemma 3.7

(i) A right inverse of the matrix

$$A_N = \begin{bmatrix} A_q & A_{q-1} & \cdots & A_0 & 0 & \cdots & 0 \\ 0 & A_q & A_{q-1} & \cdots & A_0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & A_q & A_{q-1} & \cdots & A_0 \end{bmatrix} \in R^{(N-q+1)p\times(N+1)\ell}$$

is the following

$$A_N^r = \begin{bmatrix} H_{-q} & H_{-q-1} & \cdots & H_{-N} \\ H_{-q+1} & H_{-q} & \cdots & H_{-N+1} \\ \cdots & \cdots & \ddots & \cdots \\ H_{-q+N} & H_{-q+N-1} & \cdots & H_0 \end{bmatrix}$$

(ii) A left inverse of the matrix

$$T = \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_q & A_{q-1} & \cdots & A_0 \\ 0 & A_q & \cdots & A_1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_q \end{bmatrix} \in R^{(N-q+1)p\times(N-2q+1)\ell}$$

is the following

$$T^r = \begin{bmatrix} H_0 & H_{-1} & \cdots & H_{-N+q} \\ H_1 & H_0 & \cdots & H_{-N+q+1} \\ \cdots & \cdots & \ddots & \cdots \\ H_{N-2q} & H_{N-2q-1} & \cdots & H_{-q} \end{bmatrix}$$

Proof. Using the relation (11) we can easily show that $A_N \times A_N^r = I$ and $T^r \times T = I$ which proves the Lemma. ■

We can now show the following

Theorem 3.8 The solution of the ARMA-representation (1) in terms of the initial and nal conditions, $\{y_0, y_1, ..., y_{q-1}\}$ and $\{y_N, y_{N-1}, ..., y_{N-q+1}\}$ respectively, is given by the following formula :

$$y_k = \begin{bmatrix} H_{-k-1} & H_{-k-2} & \cdots & H_{-k-q} \end{bmatrix} \begin{bmatrix} A_q & A_{q-1} & \cdots & A_1 \\ 0 & A_q & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_q \end{bmatrix} \begin{bmatrix} y_{q-1} \\ y_{q-2} \\ \vdots \\ y_0 \end{bmatrix} + \tag{25}$$

$$+ \begin{bmatrix} H_{N-k} & H_{N-k-1} & \cdots & H_{N-k-q+1} \end{bmatrix} \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix} \begin{bmatrix} y_N \\ y_{N-1} \\ \vdots \\ y_{N-q+1} \end{bmatrix} +$$

$$+ \begin{bmatrix} H_{N-k-q} & H_{N-k-q-1} & \cdots & H_{-k} \end{bmatrix} \begin{bmatrix} B_q & B_{q-1} & \cdots & B_0 & 0 & \cdots & 0 \\ 0 & B_q & B_{q-1} & \cdots & B_0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & B_q & B_{q-1} & \cdots & B_0 \end{bmatrix} \begin{bmatrix} u_N \\ u_{N-1} \\ \vdots \\ u_0 \end{bmatrix}$$

or equivalently

$$y_k = \sum_{i=1}^{q} \sum_{j=1}^{q} H_{-k-i} A_j \, y_{j-i} + \sum_{i=0}^{q-1} \sum_{j=0}^{i} H_{N-k-i} A_j y_{N-i+j} + \sum_{i=0}^{N-q} \sum_{j=0}^{q} H_{N-k-q-i} B_j u_{N+j-i-q} \tag{26}$$

under the following restrictions between the initial conditions, nal conditions and input sequences :

$$\begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \begin{bmatrix} X_A y_{N-q+1,N} \\ X_{\tilde{A}} y_{0,q-1} \end{bmatrix} = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} B_N u_{0,N} \tag{27}$$

where

$$W_{11} = \begin{bmatrix} H_{-q} & H_{-q-1} & \cdots & H_{-2q+1} \\ H_{-q+1} & H_{-q} & \cdots & H_{-2q+2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{-1} & H_{-2} & \cdots & H_{-q} \end{bmatrix}$$

$$W_{12} = \begin{bmatrix} H_{-N+q-1} & H_{-N+q-2} & \cdots & H_{-N} \\ H_{-N+q} & H_{-N+q-1} & \cdots & H_{-N+1} \\ \vdots & \vdots & \ddots & \vdots \\ H_{-N+2q-2} & H_{-N+2q-3} & \cdots & H_{-N+q-1} \end{bmatrix}$$

$$W_{21} = \begin{bmatrix} H_{N-2q+1} & H_{N-2q} & \cdots & H_{N-3q+2} \\ H_{N-2q+2} & H_{N-2q+1} & \cdots & H_{N-3q+3} \\ \vdots & \vdots & \ddots & \vdots \\ H_{N-q} & H_{N-q-1} & \cdots & H_{N-2q+1} \end{bmatrix}$$

$$W_{22} = \begin{bmatrix} H_0 & H_{-1} & \cdots & H_{-q+1} \\ H_1 & H_0 & \cdots & H_{-q+2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{q-1} & H_{q-2} & \cdots & H_0 \end{bmatrix}$$

$$X_A = \begin{bmatrix} A_q & A_{q-1} & \cdots & A_1 \\ 0 & A_q & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_q \end{bmatrix} ; X_{\tilde{A}} = \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix} \tag{28}$$

$$y_{N-q+1,N} = \begin{bmatrix} y_N \\ y_{N-1} \\ \vdots \\ y_{N-q+1} \end{bmatrix} ; y_{0,q-1} = \begin{bmatrix} y_{q-1} \\ y_{q-2} \\ \vdots \\ y_0 \end{bmatrix} ; u_{0,N} = \begin{bmatrix} u_N \\ u_{N-1} \\ \vdots \\ u_0 \end{bmatrix}$$

$$
B_N = \begin{bmatrix} B_0 & B_1 & \cdots & B_q & 0 & \cdots & 0 \\ 0 & B_0 & \cdots & B_{q-1} & B_q & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & B_0 & B_1 & \cdots & B_q \end{bmatrix}
$$

$$
Z_1 = \begin{bmatrix} H_{-q} & H_{-q-1} & \cdots & H_{-N} \\ H_{-q+1} & H_{-q} & \cdots & H_{-N+1} \\ \vdots & \vdots & \ddots & \vdots \\ H_{-1} & H_{-2} & \cdots & H_{-N+q-1} \end{bmatrix} ; Z_2 = \begin{bmatrix} H_{N-2q+1} & H_{N-2q} & \cdots & H_{-q+1} \\ H_{N-2q+2} & H_{N-2q+1} & \cdots & H_{-q+2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{N-q} & H_{N-q-1} & \cdots & H_0 \end{bmatrix}
$$

We call the solution (25) the *symmetric solution* of (1) and the equations (27) *boundary mapping equations* of (1).

Proof. Rewritting (1) in the form

$$
\underbrace{\begin{bmatrix} A_q & \cdots & A_0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & A_q & A_{q-1} & \cdots & A_0 \end{bmatrix}}_{A_N} \underbrace{\begin{bmatrix} y_N \\ y_{N-1} \\ \vdots \\ y_0 \end{bmatrix}}_{y_{0,N}} =
$$

$$
= \underbrace{\begin{bmatrix} B_q & \cdots & B_0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & B_q & B_{q-1} & \cdots & B_0 \end{bmatrix}}_{B_N} \underbrace{\begin{bmatrix} u_N \\ u_{N-1} \\ \vdots \\ u_0 \end{bmatrix}}_{u_{0,N}} \Leftrightarrow
$$

$$
\begin{bmatrix} X_A y_{N-q+1,N} \\ 0 \\ X_{\tilde{A}} y_{0,q-1} \end{bmatrix} = \begin{bmatrix} -A_0 & \cdots & 0 & \\ \vdots & \ddots & \vdots & \\ -A_q & \ddots & -A_0 & \mid B_N \\ \vdots & \ddots & \vdots & \\ 0 & \cdots & -A_q & \end{bmatrix} \begin{bmatrix} y_{q,N-q} \\ u_{0,N} \end{bmatrix} \tag{29}
$$

where $y_{q,N-q} = [y_{N-q}^T, ..., y_q^T]^T$. Premultiply both sides of (29) by A_N^r we obtain from the rst q and the last q equations the relations (27), while from the middle $N-2q$ equations, after the use of Lemma 3.7 we obtain the formula (25). ■

A necessary and sufficient condition such that the ARMA-representation (1) has a solution is that the initial, nal conditions and input sequences satis es the relation (27). Therefore we de ne :

De nition 3.9 We de ne as

$$\begin{array}{c}\tilde{H}_{iu} := \{y_{0,q-1},\, y_{N-q+1,N} : \\ \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \begin{bmatrix} X_A y_{N-q+1,N} \\ X_{\tilde{A}} y_{0,q-1} \end{bmatrix} = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} B_N u_{0,N}\}\end{array} \tag{30}$$

the *symmetric boundary condition space* of (1) under nonzero inputs. ■

The boundary mapping equation (27) represents the restrictions that the system places on the boundary variables $y_{0,q-1}$, $y_{N-q+1,N}$ in order the system to be solvable. Addition restrictions on the variables can be applied to the system in the form of an auxiliary equation

$$W_{31} y_{N-q+1,N} + W_{32} y_{0,q-1} = C \tag{31}$$

The combined boundary equation formed from (27) and (31)

$$\begin{array}{c}\begin{bmatrix} W_{11}X_A & W_{12}X_{\tilde{A}} \\ W_{21}X_A & W_{22}X_{\tilde{A}} \\ W_{31} & W_{32} \end{bmatrix} \begin{bmatrix} y_{N-q+1,N} \\ y_{0,q-1} \end{bmatrix} = \begin{bmatrix} Z_1 u_{0,N} \\ Z_2 u_{0,N} \\ C \end{bmatrix} \Leftrightarrow \\ \Leftrightarrow ZY = \tilde{C}\end{array} \tag{32}$$

will subsequently de ne a unique solution iff $ZZ^+\tilde{C} = \tilde{C}$ and Z has full column rank, where Z^+ denotes the pseudoinverse of Z, i.e. $Y = Z^+\tilde{C}$.

Alternative forms of the solution formula (25) are given by the following

Corollary 3.10 The symmetric solution (25) can be written in the alternative forms

FORWARD - SYMMETRIC

$$y_k = -\begin{bmatrix} H_{-1} & H_{-2} & \cdots & H_{-q} \end{bmatrix} \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix} \begin{bmatrix} y_{k-1} \\ y_{k-2} \\ \vdots \\ y_{k-q} \end{bmatrix} + \tag{33}$$

$$+\begin{bmatrix} H_{N-k} & H_{N-k-1} & \cdots & H_{N-k-q+1} \end{bmatrix} \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix} \begin{bmatrix} y_N \\ y_{N-1} \\ \vdots \\ y_{N-q+1} \end{bmatrix} +$$

$$+\begin{bmatrix} H_{N-k-q} & H_{N-k-q-1} & \cdots & H_{-q} \end{bmatrix} \begin{bmatrix} B_q & B_{q-1} & \cdots & B_0 & 0 & \cdots & 0 \\ 0 & B_q & B_{q-1} & \cdots & B_0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & B_q & B_{q-1} & \cdots & B_0 \end{bmatrix} \begin{bmatrix} u_N \\ u_{N-1} \\ \vdots \\ u_0 \end{bmatrix}$$

or

$$y_k = -\sum_{i=1}^{q} \sum_{j=0}^{i-1} H_{-i} A_j \, y_{k-j-i} + + \sum_{i=0}^{q-1} \sum_{j=0}^{i} H_{N-k-i} A_j y_{N-i+j} + \sum_{i=0}^{N-k} \sum_{j=0}^{q} H_{N-k-q-i} B_j u_{N+j-i-q} \tag{34}$$

BACKWARD - SYMMETRIC

$$y_k = \begin{bmatrix} H_{-k-1} & H_{-k-2} & \cdots & H_{-k-q} \end{bmatrix} \begin{bmatrix} A_q & A_{q-1} & \cdots & A_1 \\ 0 & A_q & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_q \end{bmatrix} \begin{bmatrix} y_{q-1} \\ y_{q-2} \\ \vdots \\ y_0 \end{bmatrix} - \tag{35}$$

$$-\begin{bmatrix} H_0 & H_{-1} & \cdots & H_{-q+1} \end{bmatrix} \begin{bmatrix} A_q & A_{q-1} & \cdots & A_1 \\ 0 & A_q & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_q \end{bmatrix} \begin{bmatrix} y_{k+q} \\ y_{k+q-1} \\ \vdots \\ y_{k+1} \end{bmatrix} +$$

$$+\begin{bmatrix} H_0 & H_{-1} & \cdots & H_{-k} \end{bmatrix} \begin{bmatrix} B_q & B_{q-1} & \cdots & B_0 & 0 & \cdots & 0 \\ 0 & B_q & B_{q-1} & \cdots & B_0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & B_q & B_{q-1} & \cdots & B_0 \end{bmatrix} \begin{bmatrix} u_{k+q} \\ u_{k+q-1} \\ \vdots \\ u_0 \end{bmatrix}$$

or equivalently

$$y_k = \sum_{i=1}^{q}\sum_{j=i}^{q} H_{-k-i}A_j\; y_{j-i} - \sum_{i=0}^{q-1}\sum_{j=i+1}^{q} H_{-i}A_j y_{k+j-i} + \sum_{i=0}^{k}\sum_{j=0}^{q} H_{-i}B_j u_{k+j-i} \quad (36)$$

Proof. Taking the solution formula (25) and using the following three tasks

(i) Assume that $k = \nu q + \upsilon$ $(N-k = \nu q + \upsilon)$

(ii) Do the following replacement

$$\begin{bmatrix} H_{-s} & H_{-s+1} & \cdots & H_{-s+q-1} \end{bmatrix} \begin{bmatrix} A_q & A_{q-1} & \cdots & A_1 \\ 0 & A_q & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_q \end{bmatrix} \overset{\text{for } s\neq q}{=}$$

$$= -\begin{bmatrix} H_{-s+q} & H_{-s+q+1} & \cdots & H_{-s+2q-1} \end{bmatrix} \begin{bmatrix} A_0 & A_1 & \cdots & A_{q-1} \\ 0 & A_0 & \cdots & A_{q-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_0 \end{bmatrix} \quad (37)$$

$$\begin{bmatrix} H_{N-k-sq} & H_{N-k-sq-1} & \cdots & H_{N-k-(s+1)q+1} \end{bmatrix} \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix} \overset{\text{for } s\neq q}{=}$$

$$= -\begin{bmatrix} H_{N-k-(s+1)q} & H_{N-k-(s+1)q-1} & \cdots & H_{N-k-(s+2)q+1} \end{bmatrix} \begin{bmatrix} A_q & A_{q-1} & \cdots & A_1 \\ 0 & A_q & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_q \end{bmatrix} \quad (38)$$

which is based on (11).

(iii) Do the following replacement (using (1))

$$\begin{bmatrix} A_0 & A_1 & \cdots & A_{q-1} \\ 0 & A_0 & \cdots & A_{q-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_0 \end{bmatrix} \begin{bmatrix} y_{k+sq} \\ y_{k+sq+1} \\ \vdots \\ y_{k+(s+1)q-1} \end{bmatrix} = - \begin{bmatrix} A_q & 0 & \cdots & 0 \\ A_{q-1} & A_q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \cdots & A_q \end{bmatrix} \begin{bmatrix} y_{k+(s+1)q} \\ y_{k+(s+1)q+1} \\ \vdots \\ y_{k+(s+2)q-1} \end{bmatrix} +$$

$$+ \begin{bmatrix} B_0 & B_1 & \cdots & B_q & 0 & \cdots & 0 \\ 0 & B_0 & \cdots & B_{q-1} & B_q & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & B_0 & B_1 & \cdots & B_q \end{bmatrix} \begin{bmatrix} u_{k+sq} \\ u_{k+sq+1} \\ \vdots \\ u_{k+(s+2)q-1} \end{bmatrix} \qquad (39)$$

$$\begin{bmatrix} A_q & A_{q-1} & \cdots & A_1 \\ 0 & A_q & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_q \end{bmatrix} \begin{bmatrix} y_{N-sq} \\ y_{N-sq-1} \\ \vdots \\ y_{N-(s+1)q+1} \end{bmatrix} = - \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix} \begin{bmatrix} y_{N-(s+1)q} \\ y_{N-(s+1)q-1} \\ \vdots \\ y_{N-(s+2)q+1} \end{bmatrix} +$$

$$+ \begin{bmatrix} B_0 & B_1 & \cdots & B_q & 0 & \cdots & 0 \\ 0 & B_0 & \cdots & B_{q-1} & B_q & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & B_0 & B_1 & \cdots & B_q \end{bmatrix} \begin{bmatrix} u_{N-sq} \\ u_{N-sq-1} \\ \vdots \\ u_{N-(s+2)q+1} \end{bmatrix} \qquad (40)$$

which is based on (1) we get the solution formula (33) and (35). ■

In the *Forward-Symmetric* case we still solve within the region $[0, N]$ but now the solution depends on the q final conditions $\{y_N, y_{N-1},..., y_{N-q+1}\}$ and the *previous* q outputs $\{y_{k-1}, y_{k-2}, ..., y_{k-q}\}$ and no longer on the q fixed initial conditions $\{y_0, y_1, ..., y_{q-1}\}$. Therefore we solve *forwards* in the interval.

In the *Backward-Symmetric* case we again still solve within the region $[0, N]$ but now the solution depends on the q initial conditions $\{y_0, y_1, ..., y_{q-1}\}$ and the *future* q outputs $\{y_{k+1}, y_{k+2}, ..., y_{k+q}\}$ and no longer on the q fixed final conditions $\{y_N, y_{N-1},..., y_{N-q+1}\}$. Therefore we solve *backwards* in the interval.

4 Illustrative Example

Consider the following discrete time ARMA-representation :

$$\underbrace{\begin{bmatrix} \sigma^2+5\sigma+6 & \sigma+1 & 0 \\ 2\sigma-5 & 3\sigma+2 & 1 \\ 0 & -1 & 0 \end{bmatrix}}_{A(\sigma)=A_0+A_1\sigma+A_2\sigma^2} \underbrace{\begin{bmatrix} y_k^1 \\ y_k^2 \\ y_k^3 \end{bmatrix}}_{y_k} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B(\sigma)=B_0} u_k \tag{41}$$

Let also the Laurent expansion of $A(\sigma)^{-1}$ at $s=\infty$:

$$A(\sigma)^{-1} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}}_{H_1} \sigma + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}}_{H_0} + \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -2 & 0 & 13 \end{bmatrix}}_{H_{-1}} \sigma^{-1} +$$

$$+ \underbrace{\begin{bmatrix} 1 & 0 & -4 \\ 0 & 0 & 0 \\ 15 & 0 & -48 \end{bmatrix}}_{H_{-2}} \sigma^{-2} + \underbrace{\begin{bmatrix} -5 & 0 & 14 \\ 0 & 0 & 0 \\ -63 & 0 & 162 \end{bmatrix}}_{H_{-3}} \sigma^{-3} + \cdots$$

and the Laurent expansion at $s=0$ of $A(\sigma)^{-1}$:

$$A(\sigma)^{-1} = \underbrace{\begin{bmatrix} \frac{1}{6} & 0 & \frac{1}{6} \\ 0 & 0 & -1 \\ \frac{5}{6} & 1 & \frac{17}{6} \end{bmatrix}}_{V_0} + \underbrace{\begin{bmatrix} -\frac{5}{36} & 0 & \frac{1}{36} \\ 0 & 0 & 0 \\ -\frac{37}{36} & 0 & \frac{101}{36} \end{bmatrix}}_{V_1} \sigma + \underbrace{\begin{bmatrix} \frac{19}{216} & 0 & -\frac{11}{216} \\ 0 & 0 & 0 \\ \frac{155}{216} & 0 & -\frac{67}{216} \end{bmatrix}}_{V_1} \sigma^2 + \cdots$$

A forward recursive representation of (41) is given according to Corollary 3.3 by

$$y_k = -\begin{bmatrix} H_{-1} & H_{-2} \end{bmatrix} \begin{bmatrix} A_0 & 0 \\ A_1 & A_0 \end{bmatrix} \begin{bmatrix} y_{k-1} \\ y_{k-2} \end{bmatrix} +$$

$$+ \begin{bmatrix} H_{-2} & H_{-1} & H_0 & H_1 \end{bmatrix} \begin{bmatrix} B_0 & 0 & 0 & 0 \\ 0 & B_0 & 0 & 0 \\ 0 & 0 & B_0 & 0 \\ 0 & 0 & 0 & B_0 \end{bmatrix} \begin{bmatrix} u_{k-2} \\ u_{k-1} \\ u_k \\ u_{k+1} \end{bmatrix} =$$

$$= \begin{bmatrix} -5y_{k-1}^1 - 6y_{k-2}^1 - 5y_{k-2}^2 - 4u_{k-2} + u_{k-1} \\ -u_k \\ -63y_{k-1}^1 - 90y_{k-2}^1 - 63y_{k-2}^2 - 48u_{k-2} + 13u_{k-1} + 3u_{k+1} \end{bmatrix}$$

The admissible initial condition space H_{iu} of (41) under nonzero inputs is given from (13) as follows :

$$H_{iu} := \left\{ \begin{array}{c} y_i, u_i \ (i = 0, 1) : \\ \begin{bmatrix} A_0 & 0 \\ A_1 & A_0 \end{bmatrix} \begin{bmatrix} H_0 & H_1 \\ H_{-1} & H_0 \end{bmatrix} \begin{bmatrix} A_0 & 0 \\ A_1 & A_0 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} = \\ = \begin{bmatrix} A_0 & 0 \\ A_1 & A_0 \end{bmatrix} \begin{bmatrix} H_0 & H_1 & 0 \\ H_{-1} & H_0 & H_1 \end{bmatrix} \begin{bmatrix} B_0 & 0 & 0 \\ 0 & B_0 & 0 \\ 0 & 0 & B_0 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} \end{array} \right\}$$

or equivalently

$$H_{iu} := \left\{ \begin{array}{c} y_i, u_i \ (i = 0, 1) : \\ \begin{bmatrix} 0 & -4 & 0 & 0 & 1 & 0 \\ -5 & 2 & 1 & 2 & 3 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 & 1 & 0 \\ -12 & -10 & 0 & -15 & 2 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} y_0^1 \\ y_0^2 \\ y_0^3 \\ y_1^1 \\ y_1^2 \\ y_1^3 \end{bmatrix} = \begin{bmatrix} 4 & -1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 6 & -1 & 0 \\ 8 & -2 & 3 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} \end{array} \right\}$$

A backward recursive representation of (41) is given from Corollary 3.6 by :

$$y_k = \begin{bmatrix} V_2 & V_1 \end{bmatrix} \begin{bmatrix} A_0 & 0 \\ A_1 & A_0 \end{bmatrix} \begin{bmatrix} y_{k+2} \\ y_{k+1} \end{bmatrix} + V_0 B_0 u_k =$$

$$= \begin{bmatrix} -\frac{1}{6} y_{k+2}^1 - \frac{5}{6} y_{k+1}^1 - \frac{1}{6} y_{k+1}^2 + \frac{1}{6} u_k \\ -u_k \\ -\frac{5}{6} y_{k+2}^1 - \frac{37}{6} y_{k+1}^1 - \frac{25}{6} y_{k+1}^2 + \frac{17}{6} u_k \end{bmatrix}$$

The admissible nal condition space $\tilde{H}_{iu}$ of (41) under nonzero inputs is given by 21) as follows

$$\tilde{H}_{iu} := \left\{ \begin{array}{c} y_i, u_i \ (i = N, N-1) : \\ \begin{bmatrix} A_2 & 0 \\ A_1 & A_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_2 & 0 \\ A_1 & A_2 \end{bmatrix} \begin{bmatrix} y_N \\ y_{N-1} \end{bmatrix} = \\ = \begin{bmatrix} A_2 & 0 \\ A_1 & A_2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} B_0 u_N \end{array} \right\}$$

$$\equiv \{y_i, u_i \ (i = N, N-1) : \text{arbitrary}\}$$

In the same way we can use relation (25) to nd the symmetric solution of (41) under the restrictions between the nal and initial conditions described by (27). ■

5 Conclusions

In the case of regular discrete time ARMA-representations exact solutions where proposed in three different forms : a) forward solutions, b) backward solutions and c) symmetric solutions. It is easily seen that the proposed solutions are extensions of the ones proposed by [8] for discrete time generalized state space systems. the solution formula presented in this work has been implemented via MAPLE in a recent publication [4]. Certain controllability, reachability and observability criteria based on the proposed solutions are being studied and will be discussed in a future publication.

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