Infinite elementary divisor structure-preserving transformations for polynomial matrices

N. P. Karampetakis and S. Vologiannidis

Aristotle University of Thessaloniki, Department of Mathematics,
Thessaloniki 54006, GREECE, karampet@auth.gr

Abstract. The main purpose of this work is to propose new notions of equivalence between polynomial matrices, that preserves both the finite and infinite elementary divisor structure. The approach we use is twofold : a) the "homogeneous polynomial matrix approach" where in place of the polynomial matrices, we study their homogeneous polynomial matrix forms and use 2-D equivalence transformations in order to preserve its elementary divisor structure, and b) the "polynomial matrix approach" where certain conditions between the 1-D polynomial matrices and their transforming matrices are proposed.

1 Introduction

Consider a linear homogeneous matrix difference equation of the form

$$A(\sigma)\beta(k) = 0,\ k \in [0, N] \tag{1}$$

$$A(\sigma) = A_q\sigma^q + A_{q-1}\sigma^{q-1} + \ldots + A_0 \in R[s]^{r\times r} \tag{2}$$

where σ denotes the forward shift operator. It is known from [1] that (1) exhibits forward behavior due to the finite elementary divisors of $A(\sigma)$ and backward behavior due to the infinite elementary divisors of $A(\sigma)$ (and not due to its infinite zeros, as in the continuous time case). Actually, if $A(s)$ is nonsquare or square with zero determinant, then additionally, the right minimal indices play a crucial role both in the forward and backward behavior of the AR-representation [2]. Therefore, it seems quite natural to search for transformations that preserve both the finite and infinite elementary divisor structure of polynomial matrices. [3] proposed the extended unimodular equivalence transformation (e.u.e.) which has the nice property of preserving only the finite elementary divisors. However, e.u.e. preserves the i.e.d., only if additional conditions are added.

In the present work we use two different ways to approach and solve this problem of polynomial matrix equivalence. Specifically, in the third section we notice that the finite elementary divisor structure of the homogeneous polynomial matrix form $A_q\sigma^q + A_{q-1}\sigma^{q-1}w + \ldots + A_0w^q$ corresponding to (2) gives us complete information for both finite and infinite elementary divisor structure of (2). Based on this line of thought, we reduce the problem of equivalence between 1-D polynomial matrices to a transformation between 2-D polynomial matrices. A more direct and transparent approach is given in the fourth section, where we propose additional conditions to e.u.e. . It is shown, that both transformations provide necessary conditions in order for two polynomial matrices to possess the same elementary divisor structure. However, in the special set of square and nonsingular polynomial matrices: a) the providing conditions are necessary and sufficient, and b) the proposed transformations are equivalent transformations and define the same equivalence class.

2 Discrete time autoregressive representations and elementary divisor structure

In what follows $\mathbb{R}, \mathbb{C}$ denote respectively the fields of real and complex numbers and $\mathbb{Z}, \mathbb{Z}^+$ denote respectively the integers and non negative integers. By $\mathbb{R}[s]$ and $\mathbb{R}[s]^{p\times m}$ we denote the sets of polynomials and $p \times m$ polynomial matrices respectively with real coefficients and indeterminate $s \in \mathbb{C}$. Consider a polynomial matrix

$$A(s) = A_qs^q + A_{q-1}s^{q-1} + .. + A_0 \in \mathbb{R}[s]^{p\times m} \tag{3}$$

$$A_j \in \mathbb{R}^{p\times m}, j = 0, 1, ..., q \geq 1, A_q \neq 0$$

Definition 1 *Let* $A(s) \in \mathbb{R}[s]^{p\times m}$ *with* $rank_{R(s)}A(s) = r \leq \min(p,m)$ *. The values* $\lambda_i \in \mathbb{C}$ *that satisfy the condition* $rank_{\mathbb{C}}A(\lambda_i) < r$ *are called finite zeros of* $A(s)$*. Assume that* $A(s)$ *has* l *distinct zeros* $\lambda_1, \lambda_2, \ldots, \lambda_l \in \mathbb{C}$*, and let*

$$S^{\lambda_i}_{A(s)}(s) = \begin{bmatrix} Q & 0_{r,m-r} \\ 0_{p-r,r} & 0_{p-r,m-r} \end{bmatrix}$$
$$Q = diag\{(s-\lambda_i)^{m_{i1}}, ..., (s-\lambda_i)^{m_{ir}}\}$$

be the local Smith form $S^{\lambda_i}_{A(s)}(s)$ *of* $A(s)$ *at* $s = \lambda_i, i = 1,2,...,l$ *where* $m_{ij} \in \mathbb{Z}^+$ *and* $0 \leq m_{i1} \leq m_{i2} \leq ... \leq m_{ir}$*. The terms* $(s-\lambda_i)^{m_{ij}}$ *are called the finite elementary divisors (f.e.d.) of* $A(s)$ *at* $s = \lambda_i$*. Define also as* $n := \sum_{i=1}^{l}\sum_{j=1}^{r} m_{ij}$*.*

Definition 2 *If* $A_0 \neq 0$*, the dual matrix* $\tilde{A}(s)$ *of* $A(s)$ *is defined as* $\tilde{A}(s) := A_0 s^q + A_1 s^{q-1} + \ldots + A_q$*. Since* $rank\tilde{A}(0) = rankA_q$ *the dual matrix* $\tilde{A}(s)$ *of* $A(s)$ *has zeros at* $s = 0$ *iff* $rankA_q < r$*. Let* $rankA_q < r$ *and let*

$$S^0_{\tilde{A}(s)}(s) = \begin{bmatrix} diag\{s^{\mu_1}, ..., s^{\mu_r}\} & 0_{r,m-r} \\ 0_{p-r,r} & 0_{p-r,m-r} \end{bmatrix} \quad (4)$$

be the local Smith form of $\tilde{A}(s)$ *at* $s = 0$ *where* $\mu_j \in \mathbb{Z}^+$ *and* $0 \leq \mu_1 \leq \mu_2 \leq ... \leq \mu_r$*. The infinite elementary divisors (i.e.d.) of* $A(s)$ *are defined as the finite elementary divisors* s^{μ_j} *of its dual* $\tilde{A}(s)$ *at* $s = 0$*. Define also as* $\mu = \sum_{i=1}^{r} \mu_i$*.*

An interesting consequence of the above definition is that in order to prove that the polynomial matrix (3) has no infinite elementary divisors it is enough to prove that $rankA_q = r$. It is easily seen also that the finite elementary divisors of $A(s)$ describe the finite zero structure of the matrix polynomial. In contrast, the infinite elementary divisors give a complete description of the total structure at infinity (pole and zero structure) and not simply that associated with the zeros [4],[5].

The structured indices of a polynomial matrix, (finite-infinite elementary divisors and right-left minimal indices) are connected with the rank and the degree of the matrix, as follows.

Proposition 1 *[6],[1]*

(a) If $A(s) = A_0 + A_1 s + \cdots + A_q s^q \in R[s]^{r\times r}$ *and* $\det|A(s)| \neq 0$*, then the total number of elementary divisors (finite and infinite ones and multiplicities accounted for) is equal to the product* rq *i.e.* $n + \mu = rq$*.*

(b) If $A(s) = A_0 + A_1 s + \cdots + A_q s^q$ *is nonsquare or square with determinant equal to zero then the total number of elementary divisors plus the left and right minimal indices of* $A(s)$ *(order accounted for) is equal to* rq *where now* r *denotes the rank of the polynomial matrix* $A(s)$*.*

Example 1 *Consider the polynomial matrix*

$$A(s) = \begin{bmatrix} 1 & s^2 \\ 0 & s+1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{A_0} +$$
$$+ \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{A_1} s + \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A_2} s^2$$

and its dual

$$\tilde{A}(s) = \begin{bmatrix} s^2 & 1 \\ 0 & s+s^2 \end{bmatrix} = A_2 + A_1 s + A_0 s^2$$

Then

$$S^C_{A(s)}(s) = \begin{bmatrix} 1 & 0 \\ 0 & s+1 \end{bmatrix}; S^0_{\tilde{A}(s)}(s) = \begin{bmatrix} 1 & 0 \\ 0 & s^3 \end{bmatrix}$$

Therefore $A(s)$ *has one finite elementary divisor* $(s+1)$ *and one infinite elementary divisor* s^3 *i.e.* $n + \mu = 1 + 3 = 2 \times 2 = r \times q$*.* ∎

The elementary divisor structure of a polynomial matrix plays a crucial role for the study of the behaviour of discrete time AR-Representations over a closed time interval. Consider for example the q-th order discrete time AutoRegressive representation :

$$A_q\xi_{k+q} + A_{q-1}\xi_{k+q-1} + \cdots + A_0\xi_k = 0 \quad (5)$$

If σ denotes the forward shift operator $\sigma^i\xi_k = \xi_{k+i}$ then (5) can be written as

$$A(\sigma)\xi_k = 0, k = 0,1,...,N-q \quad (6)$$

where $A(\sigma)$ as in (3) and $\xi_k \in R^r, k = 0,1,...,N$ is a vector sequence. The solution space or behavior $B^N_{A(\sigma)}$ of the AR-representation (6) over the finite time interval $[0,N]$ is defined as

$$B^N_{A(\sigma)} := \left\{ \begin{array}{c} (\xi_k)_{k=0,1,..,N} \subseteq R^r | \\ \xi_k \text{ satisfies (6) for } k \in [0,N] \end{array} \right\} \subseteq (R^r)^{N+1} \quad (7)$$

and we have

Theorem 2 *[7],[8]*

(a) If $A(s) \in R[s]^{r\times r}, \det|A(s)| \neq 0$ *then the behavior of (6) over the finite time interval* $k = 0,1,...,N \geq q$ *is given by*

$$\dim B^N_{A(\sigma)} = rq = n + \mu$$

(b) If $A(s)$ *is nonsquare or square with zero determinant then* $B^N_{A(\sigma)}$ *is separated into an equivalence class space i.e.* $\hat{B}^N_{A(\sigma)}$*, where each equivalence class corresponds to the family of solutions that corresponds to certain boundary conditions, and the dimension of* $\hat{B}^N_{A(\sigma)}$ *is* $n + \mu + 2\varepsilon$ *where* ε *denotes the total number of right minimal indices (since the left minimal indices play no role in the construction of the right solution space).*

It is easily seen from [7],[8], and [9] that the solution space $B^N_{A(\sigma)}$ is the maximal forward decomposition of (5). Actually, it consists of two subspaces, the one corresponding to the finite elementary divisors of $A(\sigma)$, which gives rise to solutions moving in the forward direction of time and the other corresponding to the infinite elementary divisors of $A(\sigma)$, which gives rise to solutions moving in the backward direction of time.

Example 2 *Consider the AR-Representation*

$$\underbrace{\left[\begin{array}{cc} 1 & \sigma^2 \\ 0 & \sigma+1 \end{array}\right]}_{A(\sigma)}\underbrace{\left[\begin{array}{c} \xi_k^1 \\ \xi_k^2 \end{array}\right]}_{\xi_k} = 0_{2\times 1}$$

Then

$$S^C_{A(s)}(s) = \left[\begin{array}{cc} 1 & 0 \\ 0 & s+1 \end{array}\right]; S^0_{\tilde{A}(s)}(s) = \left[\begin{array}{cc} 1 & 0 \\ 0 & s^3 \end{array}\right]$$

$$B^N_{A(\sigma)} = \left\langle \begin{array}{c} \underbrace{\left(\begin{array}{c} 1 \\ -1 \end{array}\right)(-1)^k}_{\text{due to } S^C_{A(s)}(s)}, \\ \underbrace{\left(\begin{array}{c} \delta_{N-k} \\ 0 \end{array}\right), \left(\begin{array}{c} \delta_{N-k+1} \\ 0 \end{array}\right), \left(\begin{array}{c} \delta_{N-k+2} \\ -\delta_{N-k} \end{array}\right)}_{\text{due to } S^0_{\tilde{A}(s)}(s)} \end{array} \right\rangle$$

and $\dim B^N_{A(\sigma)} = rq = 2\times 2 = 1+3 = n+\mu.$

Example 3 *Consider the polynomial matrix description*

$$\begin{array}{rcl} (\sigma^2)\,\xi_k & = & -u_k \\ y_k & = & (\sigma+1)\,\xi_k \end{array}$$

In order to find the state-input pair which gives rise to zero output (output zeroing problem) we have to solve the following system of difference equations

$$\underbrace{\left[\begin{array}{cc} \sigma^2 & 1 \\ \sigma+1 & 0 \end{array}\right]}_{P(\sigma)}\underbrace{\left[\begin{array}{c} \xi_k \\ u_k \end{array}\right]}_{x_k} = 0_{2\times 1}$$

It is easily seen that the above discrete time AR-Representation is the one we have already studied in the previous example and therefore the state-input pair which gives rise to zero output is given by a simple transformation of the space $B^N_{A(\sigma)}$ defined in the previous example i.e.

$$\left(\begin{array}{c} \xi_k \\ u_k \end{array}\right) = \left(\begin{array}{c} -l_1(-1)^k - l_4\delta_{N-k} \\ \\ l_1(-1)^k + l_2\delta_{N-k} + \\ +l_3\delta_{N-k+1} + l_4\delta_{N-k+2} \end{array}\right)$$

Since the elementary divisor structure of a polynomial matrix plays a crucial role in the study of discrete time AR-representations and/or polynomial matrix descriptions over a closed time interval, we are interested in finding transformations that leave invariant the elementary divisor structure of polynomial matrices.

3 The homogeneous polynomial matrix approach

We present in this section two different approaches in order to study the infinite elementary divisor structure of a polynomial matrix. The first approach is to apply a suitably chosen conformal mapping to bring the infinity point to some finite point, while the second approach is to use homogeneous polynomials to study the infinity point.

Let now $P(m,l)$ be the class of $(r+m)\times(r+l)$ polynomial matrices where l and m are fixed integers and r ranges over all integers which are greater than $\max(-m,-l)$.

Definition 3 *[3] $A_1(s), A_2(s) \in P(m,l)$ are said to be **extended unimodular equivalent** (e.u.e.) if there exist polynomial matrices $M(s), N(s)$ such that*

$$\left[\begin{array}{cc} M(s) & A_2(s) \end{array}\right]\left[\begin{array}{c} A_1(s) \\ -N(s) \end{array}\right] = 0 \qquad (8)$$

where the compound matrices

$$\left[\begin{array}{cc} M(s) & A_2(s) \end{array}\right] \quad ; \quad \left[\begin{array}{c} A_1(s) \\ -N(s) \end{array}\right] \qquad (9)$$

have full rank $\forall s \in C$.

E.u.e. relates matrices of different dimensions and preserves the f.e.d. of the polynomial matrices involved. In the case that we are interested to preserve only the elementary divisors at a specific point s_0 then we introduce $\{s_0\}$-equivalence transformation.

Definition 4 *[10] $A_1(s), A_2(s) \in P(m,l)$ are said to be **$\{s_0\}$-equivalent** if there exist rational matrices $M(s), N(s)$, having no poles at $s = s_0$, such that (8) is satisfied and where the compound matrices in (9) have full rank at $s = s_0$.*

$\{s_0\}$-equivalence preserves only the f.e.d. of $A_1(s), A_2(s) \in P(m,l)$ of the form $(s-s_0)^i, i > 0$.

Based on the above two polynomial matrix transformations we can easily define the following polynomial matrix transformation :

Definition 5 *$A_1(s), A_2(s) \in P(m,l)$ are said to be **strongly equivalent** if there exists :*

(i) polynomial matrices $M_1(s), N_1(s)$, such that (8) is satisfied and where the compound matrices in (9) have full rank $\forall s \in C$.

(ii) rational matrices $M_2(s), N_2(s)$, having no poles at $s = 0$, such that (8) between the dual polynomial matrices $\tilde{A}_1(s), \tilde{A}_2(s)$ is satisfied and where the respective compound matrices in (9) have full rank at $s = 0$.

Some nice properties of the above transformation are given by the following Theorem.

Theorem 3 *(a) Strong equivalence is an equivalence relation on $P(m,l)$.*

(b) $A_1(s), A_2(s) \in P(m,l)$ are strongly equivalent iff $S^C_{A_1(s)}(s)$ is a trivial expansion of $S^C_{A_2(s)}(s)$ and $S^0_{\tilde{A}_1(s)}(s)$ is a trivial expansion of $S^0_{\tilde{A}_2(s)}(s)$ i.e. s.e. leaves invariant the finite and infinite elementary divisors.

Proof. (a) E.u.e. and $\{s_0\}-$equivalence are equivalent relations on $P(m,l)$ and thus strong equivalence is an equivalence relation, since it consists of the above two equivalence relations.

(b) Strong equivalence is consisted of two transformations, the e.u.e. and the $\{s_0\}-$equivalence. However, $A_1(s), A_2(s) \in P(m,l)$ are : a) e.u.e. iff $S^C_{A_1(s)}(s)$ is a trivial expansion of $S^C_{A_2(s)}(s)$, and b) $\{s_0\}-$equivalent iff $S^0_{\tilde{A}_1(s)}(s)$ is a trivial expansion of $S^0_{\tilde{A}_2(s)}(s)$. ■

Based on the properties of e.u.e. and $\{s_0\}$-equivalent of preserving respective the f.e.d. and the i.e.d. at $s = s_0$, we can easily observe that the above transformation has the nice property of preserving both the finite and infinite elementary divisors of $A_i(s)$.

Example 4 *Consider the polynomial matrices*

$$A_1(s) = \begin{bmatrix} 1 & s^2 \\ 0 & s+1 \end{bmatrix} \; ; \; A_2(s) = \begin{bmatrix} s & 0 & -1 & 0 \\ 0 & s & 0 & -1 \\ 1 & 0 & 0 & s \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

and their dual polynomial matrices

$$\tilde{A}_1(s) = \begin{bmatrix} s^2 & 1 \\ 0 & s+s^2 \end{bmatrix} \; ; \; \tilde{A}_2(s) = \begin{bmatrix} 1 & 0 & -s & 0 \\ 0 & 1 & 0 & -s \\ s & 0 & 0 & 1 \\ 0 & s & 0 & s \end{bmatrix}$$

Then we can find polynomial matrices $M(s), N(s)$ and $\tilde{M}(s), \tilde{N}(s)$ such that

$$\underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{M(s)} \underbrace{\begin{bmatrix} 1 & s^2 \\ 0 & s+1 \end{bmatrix}}_{A_1(s)} = \underbrace{\begin{bmatrix} s & 0 & -1 & 0 \\ 0 & s & 0 & -1 \\ 1 & 0 & 0 & s \\ 0 & 1 & 0 & 1 \end{bmatrix}}_{A_2(s)} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ s & 0 \\ 0 & s \end{bmatrix}}_{N(s)}$$

is an e.u.e. and

$$\underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\tilde{M}(s)} \underbrace{\begin{bmatrix} s^2 & 1 \\ 0 & s+s^2 \end{bmatrix}}_{\tilde{A}_1(s)} = \underbrace{\begin{bmatrix} 1 & 0 & -s & 0 \\ 0 & 1 & 0 & -s \\ s & 0 & 0 & 1 \\ 0 & s & 0 & s \end{bmatrix}}_{\tilde{A}_2(s)} \underbrace{\begin{bmatrix} s & 0 \\ 0 & s \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\tilde{N}(s)}$$

is a $\{0\}-$equivalence. Therefore $A_1(s), A_2(s)$ are s.e. and thus according to Theorem 3 possess the same finite and infinite elementary divisors i.e.

$$S^C_{A(s)}(s) = \begin{bmatrix} 1 & 0 \\ 0 & s+1 \end{bmatrix} \; ; \; S^C_{A_2(s)}(s) = \begin{bmatrix} I_3 & 0 \\ 0 & s+1 \end{bmatrix}$$

$$S^0_{\tilde{A}_1(s)}(s) = \begin{bmatrix} 1 & 0 \\ 0 & s^3 \end{bmatrix} \; ; \; S^0_{\tilde{A}_2(s)}(s) = \begin{bmatrix} I_3 & 0 \\ 0 & s^3 \end{bmatrix}$$

Strong equivalence transformation has the disadvantage that it consists of two separate transformations. In order to overcome this difficulty we use the homogeneous variable to represent infinity.

Define as the *homogeneous form* of $A(s)$ the matrix

$$A^H(s,r) = A_0 s^n + A_1 s^{n-1} r + \cdots + A_n r^n \quad (10)$$

It is easily seen that the finite elementary divisors of $A(s)$ are actually the finite elementary divisors of $A^H(s,1)$. Similarly to the previous definition we can easily see that the infinite elementary divisors of $A(s)$ are actually the finite elementary divisors of $A^H(1,r)$ at $r = 0$. An alternative definition of the finite and infinite elementary divisors in terms of the homogeneous polynomial matrix (10) is given below.

Definition 6 *[6] Let D_i be the greatest common divisor of the $i \times i$ minors of A^H, and define $D_0 = 1$. Then D_i/D_{i+1}, and let $D_i/D_{i-1} =: c_i \prod (as - br)^{\ell_i(b/a)}$, where the product is taken over all pairs $(1,b)$ and $(0,1)$, and $1/0$ is denoted by ∞. The factors $(as-br)^{\ell_i(b/a)}$, with $\ell_i(b/a) \neq 0$ are called the elementary divisors of $A(s)$, and the integers ℓ_i the elementary exponents of $A(s)$.*

It is easily seen that the pairs $(0,1)$ corresponds to the i.e.d. while the remaining pairs to the f.e.d. .

Example 5 *Consider the polynomial matrix $A(s)$ defined in Example 4 i.e.*

$$A(s) = \begin{bmatrix} 1 & s^2 \\ 0 & s+1 \end{bmatrix}$$

Define also the homogeneous polynomial matrix

$$A^H(s,r) = \begin{bmatrix} r^2 & s^2 \\ 0 & sr+r^2 \end{bmatrix}$$

Then

$$D_0 = 1, D_1 = 1, D_2 = r^3 (s + r)$$

and therefore the Smith form of $A^H(s,r)$ over $R[s,r]$ is given by

$$S^C_{A^H(s,r)}(s,r) = \begin{bmatrix} 1 & 0 \\ 0 & r^3 (s+r) \end{bmatrix}$$

where we have the following pairs of (a,b) : $(1,1)$ with exponent 1 and $(0,1)$ with exponent 3. The first pair corresponds to the f.e.d. $(s+1)^1$, while the second pair corresponds to the i.e.d. r^3.

An extension of the e.u.e. into the 2-D setting is given by two transformations, factor and zero coprime equivalence [11]. While both of them preserve the invariant polynomials of the equivalent matrices, the second ones has the additionally property to preserve the ideals of a polynomial matrix and therefore is more restrictive. Since we are interested only in the invariant polynomials of the homogeneous polynomial matrices and not in their corresponding ideals we present and use only the first of the above two transformations.

Definition 7 *$A_1(s,r), A_2(s,r) \in P(m,l)$ are said to be* ***factor coprime equivalent*** *(f.c.e.) if there exists polynomial matrices $M(s,r), N(s,r)$ such that*

$$\begin{bmatrix} M(s,r) & A_2(s,r) \end{bmatrix} \begin{bmatrix} A_1(s,r) \\ -N(s,r) \end{bmatrix} = 0 \quad (11)$$

where the compound matrices

$$\begin{bmatrix} M(s,r) & A_2(s,r) \end{bmatrix} \quad ; \quad \begin{bmatrix} A_1(s,r) \\ -N(s,r) \end{bmatrix} \quad (12)$$

are factor coprime i.e. if all the $(r+m) \times (r+m)$ (resp. $(r+l) \times (r+l)$) minors of $\begin{bmatrix} M(s,r) & A_2(s,r) \end{bmatrix}$ (resp. $\begin{bmatrix} A_1(s,r) \\ -N(s,r) \end{bmatrix}$) have no polynomial factor.

Some nice properties of the above transformation are given by the following Theorem.

Theorem 4 *[12], [11]*

1. F.c.e. is only reflexive and transitive and therefore is not an equivalence relation. F.c.e. is an equivalence relation on $P(m,m)$ the set of square and nonsingular polynomial matrices.

2. F.c.e. leaves invariant the invariant polynomials of the f.c.e. polynomial matrices.

Since a) the above transformation leaves invariant the invariant polynomials of the equivalent polynomial matrices and b) the fact that the elementary divisor structure of a polynomial matrix is completely characterized by the invariant polynomials of its homogeneous polynomial matrix, it seems quite natural to reduce the problem of equivalence between two 1-d polynomial matrices to the problem of equivalence between its respective homogeneous polynomial matrices.

Definition 8 *$A_1(s), A_2(s) \in P(m,l)$ are defined to be* ***factor equivalent*** *if their respective homogeneous polynomial matrices $A_1^H(s,r), A_2^H(s,r)$ are factor coprime equivalent.*

Due to the properties of the factor coprime equivalence, it is easily to prove the following.

Theorem 5 *i) F.e. is only reflexive and transitive and therefore is not an equivalence relation. F.e. is an equivalence relation on $P(m,m)$ the set of square and nonsingular polynomial matrices.*

ii) F.e. leaves invariant the finite and infinite elementary divisors of the equivalent polynomial matrices.

Example 6 *Consider the polynomial matrices $A_1(s), A_2(s)$ defined in Example 4,and their respective homogeneous polynomial matrices*

$$A_1^H(s,r) = \begin{bmatrix} r^2 & s^2 \\ 0 & sr + r^2 \end{bmatrix}$$

$$A_2^H(s,r) = \begin{bmatrix} s & 0 & -r & 0 \\ 0 & s & 0 & -r \\ r & 0 & 0 & s \\ 0 & r & 0 & r \end{bmatrix}$$

Then we can find polynomial matrices $M(s,r), N(s,r)$ such that

$$\underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{M(s,r)} \underbrace{\begin{bmatrix} r^2 & s^2 \\ 0 & sr + r^2 \end{bmatrix}}_{A_1^H(s,r)} =$$

$$= \underbrace{\begin{bmatrix} s & 0 & -r & 0 \\ 0 & s & 0 & -r \\ r & 0 & 0 & s \\ 0 & r & 0 & r \end{bmatrix}}_{A_2^H(s,r)} \underbrace{\begin{bmatrix} r & 0 \\ 0 & r \\ s & 0 \\ 0 & s \end{bmatrix}}_{N(s,r)}$$

where

$$S^C_{\begin{bmatrix} M & A_1^H \end{bmatrix}}(s,r) = \begin{bmatrix} I_4 & 0_{4\times 2} \end{bmatrix}$$

$$S^C_{\begin{bmatrix} A_2^H \\ -N \end{bmatrix}}(s,r) = \begin{bmatrix} I_2 \\ 0_{4\times 2} \end{bmatrix}$$

Therefore, $A_1(s), A_2(s)$ are f.e. and thus, according to Theorem 5 they possess the same f.e.d and i.e.d.. However, it is easily seen that the compound matrix $\begin{bmatrix} M & A_1 \end{bmatrix}$ has singularities at $s = r = 0$ for any matrix $M(s,r)$ and therefore, $A_1^H(s,r), A_2^H(s,r)$ are not zero coprime equivalent ([12], [11]), although they possess the same invariant polynomials. Therefore, it is seen that zero coprime equivalence would be quite restrictive for our purpose. This is easily checked out in case where $A_1(s), A_2(s)$ are of different dimensions. Then there is no zero coprime equivalence transformation between $A_1^H(s,r), A_2^H(s,r)$.

For 1-D systems, [13] has presented an algorithm that reduces a general arbitrary polynomial matrix $A(s)$ to an equivalent matrix pencil. More specifically, given the polynomial matrix $A(s)$ in (3) and the matrix pencil

$$sE-A=\begin{bmatrix} sI_m & -I_m & 0 & \cdots & 0 \\ 0 & sI_m & -I_m & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -I_m \\ A_0 & A_1 & A_2 & \cdots & A_q s + A_{q-1} \end{bmatrix} \tag{13}$$

the following holds.

Theorem 6 *The polynomial matrix $A(s)$ defined in (3) and the matrix pencil $sE-A$ defined in (13) are f.e..*

Proof. Consider the transformation

$$\underbrace{\begin{bmatrix} 0_{(q-1)m,p} \\ I_p \end{bmatrix}}_{M(s,r)} A^H(s,r) = [sE-rA] \underbrace{\begin{bmatrix} r^{q-1}I_m \\ r^{q-2}sI_m \\ \vdots \\ rs^{q-2}I_m \\ s^{q-1}I_m \end{bmatrix}}_{N(s,r)}$$

Then, the compound matrix $\begin{bmatrix} M(s,r) & sE-rA \end{bmatrix}$ has two $qm \times qm$ minors equal to $s^{(q-1)m}$ and $(-r)^{(q-1)m}$ respectively and thus the matrices are factor coprime These minors are .

$$\det \begin{bmatrix} sI_m & -rI_m & 0 & \cdots & 0 & 0 \\ 0 & sI_m & -rI_m & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & sI_m & 0 \\ A_0 r & A_1 r & A_2 r & \cdots & A_{q-2} r & I_p \end{bmatrix}$$

$$\det \begin{bmatrix} -rI_m & 0 & \cdots & 0 & 0 \\ sI_m & -rI_m & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -rI_m & 0 \\ A_1 r & A_2 r & \cdots & A_q s + A_{q-1} r & I_m \end{bmatrix}$$

and are equal to $s^{(q-1)m}$ and $(-r)^{(q-1)m}$ respectively. Similarly the the compound matrix $\begin{bmatrix} A^H(s,r) \\ -N(s,r) \end{bmatrix}$ has two coprime $m \times m$ minors, $s^{(q-1)m}$ and $r^{(q-1)m}$, and thus is factor coprime i.e.

$$\det \left[r^{q-1} I_m \right] = r^{(q-1)m} \;;\; \det \left[s^{q-1} I_m \right] = s^{(q-1)m}$$

Therefore, the matrices $\begin{bmatrix} M(s,r) & sE-rA \end{bmatrix}$ and $\begin{bmatrix} A^H(s,r) \\ -N(s,r) \end{bmatrix}$ are factor coprime, $A^H(s,r)$ and $sE-rA$ are factor coprime equivalent and $A(s), sE-A$ are factor equivalent. ■

An illustrative example of the above theorem has already been given in example 6. A direct consequence of the above theorem is given by the following lemma.

Lemma 7 *$A(s)$ and $sE-A$ possess the same finite and infinite elementary divisor structure.*

Proof. $A(s)$ and $sE-A$ are f.e. from Theorem 6 and thus according to Theorem 5 possess the same finite and infinite elementary divisor structure. ■

A completely different and more transparent approach to the problem of equivalence between 1-D polynomial matrices, without using the theory of 2-D polynomial matrices, is given in the next section.

4 The polynomial matrix approach

Although e.u.e. preserves the finite elementary divisors, it does not preserve the infinite elementary divisors, as we can see in the following example.

Example 7 *Consider the following e.u.e. transformation*

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{M(s)} \underbrace{\begin{bmatrix} 1 & s^2 \\ 0 & s+1 \end{bmatrix}}_{A_1(s)} = \underbrace{\begin{bmatrix} 1 & s^3 \\ 0 & s+1 \end{bmatrix}}_{A_2(s)} \underbrace{\begin{bmatrix} 1 & s^2-s^3 \\ 0 & 1 \end{bmatrix}}_{N(s)}$$

Although $A_1(s), A_2(s)$ have the same finite elementary divisors, i.e.

$$S^C_{A_1(s)}(s) = \begin{bmatrix} 1 & 0 \\ 0 & s+1 \end{bmatrix} = S^C_{A_2(s)}(s)$$

they have different infinite elementary divisors i.e.

$$S^0_{\tilde{A}_1(s)}(s) = S^0 \begin{bmatrix} s^2 & 1 \\ 0 & s+s^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & s^3 \end{bmatrix}$$

$$S^0_{\tilde{A}_2(s)}(s) = S^0 \begin{bmatrix} s^3 & 1 \\ 0 & s^2+s^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & s^5 \end{bmatrix}$$

The above example indicates that further restrictions must be placed on the compound matrices (9), in order to ensure that the associated transformation will leave invariant both the finite and infinite elementary divisors. A new transformation between polynomial matrices of the same set $P(m,l)$ is given in the following definition.

Definition 9 *Two matrices $A_1(s), A_2(s) \in P(m,l)$ are said to be **divisor equivalent** (d.e.) if there exist polynomial matrices $M(s), N(s)$ of appropriate dimensions, such that (8) is satisfied, where*

(i) the compound matrices in (9) are left prime and right prime matrices, respectively,

(ii) the compound matrices in (9) have no infinite elementary divisors,

(iii) the following degree conditions are satisfied

$$\begin{aligned} d\begin{bmatrix} M(s) & A_2(s) \end{bmatrix} &\leq d\left[A_2(s)\right] \\ d\begin{bmatrix} A_1(s) \\ -N(s) \end{bmatrix} &\leq d\left[A_1(s)\right] \end{aligned} \tag{14}$$

where $d[P]$ denotes the degree of $P(s)$ seen as a polynomial with nonzero matrix coefficients.

An important property of the above transformation is given by the following Theorem.

Theorem 8 *If $A_1(s), A_2(s) \in P(m,l)$ are divisor equivalent then they have the same finite and infinite elementary divisors.*

Proof. According to condition (i) of "divisor equivalence", $A_1(s)$ and $A_2(s)$ are also e.u.e. and thus have the same finite elementary divisors.

(8) may be rewritten by setting $s = \frac{1}{w}$, as

$$\left[\begin{array}{cc} M(\frac{1}{w}) & A_2(\frac{1}{w}) \end{array}\right] \left[\begin{array}{c} A_1(\frac{1}{w}) \\ -N(\frac{1}{w}) \end{array}\right] = 0$$

and then premultiplying and postmultiplying by $w^{d\left[\begin{array}{cc} M(s) & A_2(s) \end{array}\right]}$ and $w^{d\left[\begin{array}{c} A_1(s) \\ -N(s) \end{array}\right]}$ respectively as

$$w^{d\left[\begin{array}{cc} M(s) & A_2(s) \end{array}\right]} \left[\begin{array}{cc} M(\frac{1}{w}) & A_2(\frac{1}{w}) \end{array}\right] \times \tag{15}$$

$$\times \left[\begin{array}{c} A_1(\frac{1}{w}) \\ -N(\frac{1}{w}) \end{array}\right] w^{d\left[\begin{array}{c} A_1(s) \\ -N(s) \end{array}\right]} = 0 \Longleftrightarrow$$

$$\left[\begin{array}{c} \widetilde{M(w) \quad A_2(w)} \end{array}\right] \left[\begin{array}{c} \widetilde{A_1(w)} \\ -N(w) \end{array}\right] = 0$$

where ~ denotes the dual matrix. Now since $d\left[\begin{array}{cc} M(s) & A_2(s) \end{array}\right] \leq d\left[A_2(s)\right]$ and $d\left[\begin{array}{c} A_1(s) \\ -N(s) \end{array}\right] \leq d\left[A_1(s)\right]$ equation (15) may be rewritten as

$$\left[\begin{array}{cc} M'(w) & \tilde{A}_2(w) \end{array}\right] \left[\begin{array}{c} \tilde{A}_1(w) \\ -N'(w) \end{array}\right] = 0 \tag{16}$$

The compound matrix $\left[\begin{array}{cc} M(s) & A_2(s) \end{array}\right]$ (respectively $\left[\begin{array}{c} A_1(s) \\ -N(s) \end{array}\right]$) has no infinite elementary divisors and therefore its dual $\left[\begin{array}{cc} M'(w) & \tilde{A}_2(w) \end{array}\right]$ (respectively $\left[\begin{array}{c} \tilde{A}_1(w) \\ -N'(w) \end{array}\right]$) has no finite zeros at $w = 0$. Therefore, the relation (15) is an {0}-equivalence relation which preserves the finite elementary divisors of $\tilde{A}_1(w), \tilde{A}_2(w)$ at $w = 0$ or otherwise the infinite elementary divisors of $A_1(s), A_2(s)$. ■

Example 8 *Consider the polynomial matrices $A_1(s), A_2(s)$ defined in Example 4. Then we can find polynomial matrices $M(s), N(s)$ such that*

$$\underbrace{\left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ s-2 & 0 \\ 0 & s-2 \end{array}\right]}_{M(s)} \underbrace{\left[\begin{array}{cc} 1 & s^2 \\ 0 & s+1 \end{array}\right]}_{A_2(s)} =$$

$$= \underbrace{\left[\begin{array}{cccc} s & 0 & -1 & 0 \\ 0 & s & 0 & -1 \\ 1 & 0 & 0 & s \\ 0 & 1 & 0 & 1 \end{array}\right]}_{A_1(s)} \underbrace{\left[\begin{array}{cc} s-2 & 0 \\ 0 & s-2 \\ s(s-2) & 0 \\ 0 & s(s-2) \end{array}\right]}_{N(s)}$$

is a divisor equivalence transformation i.e.

$$S^C_{\left[\begin{array}{cc} M & A_1 \end{array}\right]}(s) = S^0_{\left[\begin{array}{cc} \tilde{M} & \tilde{A}_1 \end{array}\right]}(s) = \left[\begin{array}{cc} I_4 & 0_{4\times 2} \end{array}\right]$$

$$S^C_{\left[\begin{array}{c} A_2 \\ -N \end{array}\right]}(s) = S^0_{\left[\begin{array}{c} \tilde{A}_2 \\ -\tilde{N} \end{array}\right]}(s) = \left[\begin{array}{c} I_2 \\ 0_{4\times 2} \end{array}\right]$$

and

$$d\left[\begin{array}{cc} M & A_1 \end{array}\right] = 1 = d\left[A_1\right]$$

$$d\left[\begin{array}{c} A_2 \\ -N \end{array}\right] = 2 = d\left[A_2\right]$$

Therefore, $A_1(s), A_2(s)$ are divisor equivalent and thus, according to Theorem 8 possess the same finite and infinite elementary divisors.

Although d.e. preserves both the f.e.d. and i.e.d., it is not known if d.e. a) is an equivalence relation on $P(p,m)$ and b) provides us with necessary and sufficient conditions for two polynomial matrices to possess the same f.e.d. and i.e.d. Also the exact geometrical meaning of the degree conditions appearing in the definition of d.e. is under research. Now consider the following set of polynomial matrices

$$R_c[s] := \left\{ \begin{array}{c} A(s) = A_0 + A_1 s + \cdots + A_q s^q \in R[s]^{r\times r} \\ \det|A(s)| \neq 0 \text{ and } c = rq,\ r \geq 2 \end{array} \right\} \tag{17}$$

Example 9 *The polynomial matrices $A_1(s), A_2(s)$ defined in example 4 belong to $R_4[s]$ since $r_1 q_1 = 2\times 2 = 1\times 4 = r_2 q_2$.*

The degree conditions of d.e. in $R_c[s]$ are redundant as we can see in the following Lemma.

Lemma 9 *[14]*

(a) Let $A_1(s)$ and $A_2(s) \in \mathbb{R}_c[s]$ with dimensions $m\times m$ and $(m+r)\times(m+r)$ respectively where $r \neq 0$. Then the first two conditions of d.e. implies the degree conditions of d.e. i.e. $\deg M(s) \leq \deg A_2(s)$ and $\deg N(s) \leq \deg A_1(s)$.

(b) Let $A_1(s)$ and $A_2(s) \in \mathbb{R}_c[s]$ having the same dimensions $m\times m$ and therefore the same degree d. If $A_1(s), A_2(s)$ satisfies (8) and the first two conditions of d.e. then $\deg M(s) = \deg N(s)$.

Therefore, in this special case we are able to restate the definition of d.e. on $R_c[s]$ with only two conditions.

Definition 10 *Two matrices $A_1(s), A_2(s) \in \mathbb{R}_c[s]$ are called **divisor equivalent** (d.e.) if there exist polynomial matrices $M(s), N(s)$ of appropriate dimensions, such that equation (8) is satisfied where the compound matrices in (9) have full rank and no f.e.d. nor i.e.d..*

Some properties of d.e. are given in the following Theorem.

Theorem 10 *[14]*
(a) $A_1(s), A_2(s) \in \mathbb{R}_c[s]$ are d.e. iff they have the same f.e.d. and i.e.d..
(b) D.e. is an equivalence relation on $\mathbb{R}_c[s]$.

A different approach, concerning the equivalence between two polynomial matrices on $R_c[s]$ is presented in [15].

Definition 11 *[15] $A_1(s)$ and $A_2(s) \in \mathbb{R}_c[s]$ are called* ***strictly equivalent*** *iff their equivalent matrix pencils $sE_1 - A_1 \in \mathbb{R}^{c\times c}$ and $sE_2 - A_2 \in \mathbb{R}^{c\times c}$ proposed in (13), are strictly equivalent in the sense of [16].*

D.e. and s.e. define the same equivalence class on $R_c[s]$.

Theorem 11 *[14] Strict equivalence (Definition 11) belongs to the same equivalence class as d.e..*

A geometrical meaning of d.e. is given in the sequel.

Definition 12 *[15] Two AR-representations*

$$A_i(\sigma)\,\xi_k^i = 0, k = 0, 1, 2, ..., N$$

where σ is the shift operator, $A_i(\sigma) \in R_c[\sigma]^{r_i\times r_i}$, $i = 1, 2$ will be called ***fundamentally equivalent*** *(f.e.) over the finite time interval $k = 0, 1, 2, ..., N$ iff there exists a bijective polynomial map between their respective behaviors $\mathcal{B}^N_{A_1(\sigma)}, \mathcal{B}^N_{A_2(\sigma)}$.*

Theorem 12 *D.e. implies f.e..*

Proof. From (8) we have

$$M(\sigma)A_1(\sigma) = A_2(\sigma)N(\sigma) \quad (18)$$

By multiplying (18) on the right by ξ_k^1 we get

$$\begin{gathered} M(\sigma)A_1(\sigma)\xi_k^1 = A_2(\sigma)N(\sigma)\xi_k^1 \Longrightarrow \\ 0 = A_2(\sigma)N(\sigma)\xi_k^1 \Longrightarrow \\ \exists \xi_k^2 \in \mathcal{B}_{A_2(\sigma)} \text{ s.t. } \xi_k^2 = N(\sigma)\xi_k^1 \end{gathered} \quad (19)$$

According to the conditions of d.e., $\left[\begin{array}{cc} A_1(\sigma)^T & -N(\sigma)^T \end{array}\right]^T$ has full rank and no f.e.d. or i.e.d.. This implies [8] that $\xi_k^1 = 0$. Therefore the map defined by the polynomial matrix $N(\sigma) : \mathcal{B}^N_{A_1(s)} \to \mathcal{B}^N_{A_2(s)} \mid \xi_k^1 \mapsto \xi_k^2$ is injective. Furthermore, $\dim \mathcal{B}^N_{A_1(s)} = c = \dim \mathcal{B}^N_{A_2(s)}$, since $A_i(\sigma) \in R_c[\sigma]^{r_i\times r_i}$, and thus $N(\sigma)$ is a bijection between $\mathcal{B}^N_{A_1(\sigma)}, \mathcal{B}^N_{A_2(\sigma)}$. ■

However, certain questions remain open concerning the converse of the above theorem.

5 Conclusions

The forward and backward behaviour of a discrete time AR-representation over a closed time interval is connected with the finite and infinite elementary divisor structure of the polynomial matrix involved in the AR-representation. Furthermore, it is known that a polynomial matrix description can always be written as an AR-representation, and many problems arising from the Rosenbrock system theory can be reduced to problems based on AR-representation theory. This was the motivation of this work that presents three new polynomial matrix transformations, strong equivalence, factor equivalence and divisor equivalence that preserve both finite and infinite elementary divisor structure of polynomial matrices. More specifically, it is shown that strong equivalence is an equivalence relation and provides necessary and sufficient conditions for two polynomial matrices to possess the same elementary divisor structure. However, its main disadvantage is that it consists of two separate transformations. We have shown that we can overcome this problem using the homogeneous polynomial matrix form of the one variable polynomial matrices and then using the known transformations from the 2-D systems theory. Following this reasoning, we have introduced the factor equivalence transformation. Although factor equivalence is simpler in the sense that it uses only one pair of transformation matrices instead of two (strong equivalence), it suffers since an extra step (homogenization) is needed. A solution to this problem is given by adding extra conditions to extended unimodular equivalence transformation giving birth to divisor equivalence. We have shown that both factor equivalence and divisor equivalence provide necessary conditions for two polynomial matrices to possess the same elementary divisor structure. The conditions become necessary and sufficient, in the case of regular (square and nonsingular) matrices. In this special set of matrices, both transformations are equivalence relations sharing the same equivalence class. A geometrical interpretation of d.e. in terms of maps between the solution spaces of AR-representations, is given in the special case of regular polynomial matrices.

Finally, certain questions remain open concerning the sufficiency of divisor equivalence for nonsquare polynomial matrices, or square polynomial matrices with zero determinant. [15] has proposed a new notion of equivalence, named fundamental equivalence, in terms of mappings between discrete time AR-representations described by square and nonsingular polynomial matrices. Further research is now focused on: a) how can fundamental equivalence be extended to nonsquare polynomial matrices, b) what are its invariants, and c) which is the connection between the transformations presented in this work and f.e. transformation. An extension of these results to the Rosenbrock system matrix theory is also under research.

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